Pricing Some Path-Dependent Options

- Not all path-dependent derivatives are hard to price.
- Barrier options are easy to price.
- When averaging is done *geometrically*, the option payoffs are

$$\max\left((S_0 S_1 \cdots S_n)^{1/(n+1)} - X, 0\right),$$
$$\max\left(X - (S_0 S_1 \cdots S_n)^{1/(n+1)}, 0\right).$$

Pricing Some Path-Dependent Options (concluded)

- The limiting analytical solutions are the Black-Scholes formulas with the volatility set to $\sigma_a \equiv \sigma/\sqrt{3}$ and the dividend yield set to $q_a \equiv (r + q + \sigma^2/6)/2$:

$$C = Se^{-q_a \tau} N(x) - Xe^{-r\tau} N(x - \sigma_a \sqrt{\tau}), \tag{61}$$
$$P = Xe^{-r\tau} N(-x + \sigma_a \sqrt{\tau}) - Se^{-q_a \tau} N(-x). \tag{61'}$$

 – $x \equiv \frac{\ln(S/X) + \left(r - q_a + \sigma_a^2/2\right)\tau}{\sigma_a \sqrt{\tau}}$.

Approximation Algorithm for Asian Options

- Based on the BOPM.
- Consider a node at time j with the underlying asset price equal to $S_0 u^{j-i} d^i$.
- Name such a node $N(j, i)$.
- The running sum $\sum_{m=0}^{j} S_m$ at this node has a maximum value of

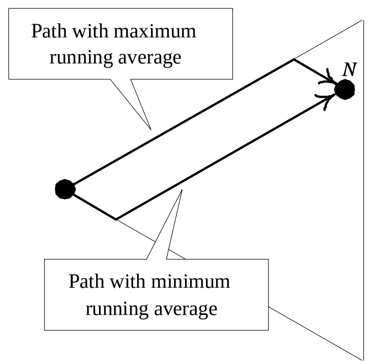
$$S_0(1 + \overbrace{u + u^2 + \cdots + u^{j-i} + u^{j-i}d + \cdots + u^{j-i}d^i}^{j})$$
$$= S_0 \frac{1 - u^{j-i+1}}{1 - u} + S_0 u^{j-i} d \frac{1 - d^i}{1 - d}.$$

Approximation Algorithm for Asian Options (continued)

- Divide this value by $j + 1$ and call it $A_{\max}(j, i)$.
- Similarly, the running sum has a minimum value of

$$S_0(1 + \overbrace{d + d^2 + \cdots + d^i + d^i u + \cdots + d^i u^{j-i}}^{j})$$
$$= S_0 \frac{1 - d^{i+1}}{1 - d} + S_0 d^i u \frac{1 - u^{j-i}}{1 - u}.$$

- Divide this value by $j + 1$ and call it $A_{\min}(j, i)$.
- $A_{\min}$ and $A_{\max}$ are running averages.



Approximation Algorithm for Asian Options (continued)

- The possible running averages at $N(j,i)$ are far too many: $\binom{j}{i}$.
- But all lie between $A_{\min}(j,i)$ and $A_{\max}(j,i)$.
- Pick $k+1$ equally spaced values in this range and treat them as the true and only running averages:

$$A_m(j,i) \equiv \left(\frac{k-m}{k}\right) A_{\min}(j,i) + \left(\frac{m}{k}\right) A_{\max}(j,i)$$

for $m = 0, 1, \ldots, k$.

Approximation Algorithm for Asian Options (continued)

- Such "bucketing" introduces errors, but it works well in practice (Hull and White, 1993).
- An alternative is to pick values whose logarithms are equally spaced.
- Still other alternatives are also possible.

Approximation Algorithm for Asian Options (continued)

- Backward induction calculates the option values at each node for the $k+1$ running averages.
- Suppose the current node is $N(j,i)$ and the running average is a.
- Assume the next node is $N(j+1,i)$, after an up move.
- As the asset price there is $S_0 u^{j+1-i} d^i$, we seek the option value corresponding to the running average

$$A_{\text{u}} \equiv \frac{(j+1)\,a + S_0 u^{j+1-i} d^i}{j+2}.$$

Approximation Algorithm for Asian Options (continued)

- But A_{u} is not likely to be one of the $k+1$ running averages at $N(j+1,i)$!
- Find the running averages that bracket it, that is,

$$A_\ell(j+1,i) \leq A_{\mathrm{u}} \leq A_{\ell+1}(j+1,i).$$

- Express A_{u} as a linearly interpolated value of the two running averages,

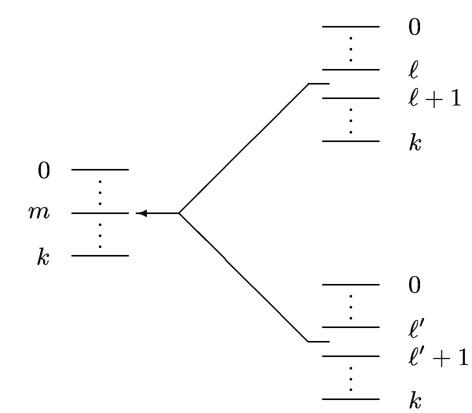
$$A_{\mathrm{u}} = x A_\ell(j+1,i) + (1-x)\, A_{\ell+1}(j+1,i), \quad 0 \leq x \leq 1.$$

Approximation Algorithm for Asian Options (continued)

- Obtain the approximate option value given the running average A_{u} via

$$C_{\mathrm{u}} \equiv x C_\ell(j+1,i) + (1-x)\, C_{\ell+1}(j+1,i).$$

  - $C_\ell(t,s)$ denotes the option value at node $N(t,s)$ with running average $A_\ell(t,s)$.
- This interpolation introduces the second source of error.



Approximation Algorithm for Asian Options (continued)

- The same steps are repeated for the down node $N(j+1,i+1)$ to obtain another approximate option value C_{d}.
- Finally obtain the option value as

$$(pC_{\mathrm{u}} + (1-p)\, C_{\mathrm{d}})\, e^{-r\Delta t}.$$

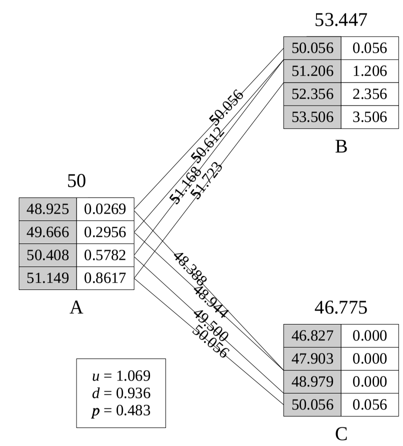
- The running time is $O(kn^2)$.
  - There are $O(n^2)$ nodes.
  - Each node has $O(k)$ buckets.

Approximation Algorithm for Asian Options (concluded)

- Arithmetic average-rate options were assumed to be newly issued: There was no historical average to deal with.
- This problem can be easily dealt with (see text).

A Numerical Example

- Consider a European arithmetic average-rate call with strike price 50.
- Assume zero interest rate in order to dispense with discounting.
- The minimum running average at node A in the figure on p. 345 is 48.925.
- The maximum running average at node A in the same figure is 51.149.



A Numerical Example (continued)

- Each node picks $k = 3$ for 4 equally spaced running averages.
- The same calculations are done for node A's successor nodes B and C.
- Suppose node A is 2 periods from the root node.
- Consider the up move from node A with running average 49.666.

A Numerical Example (continued)

- Because the stock price at node B is 53.447, the new running average will be

$$\frac{3 \times 49.666 + 53.447}{4} \approx 50.612.$$

- With 50.612 lying between 50.056 and 51.206 at node B, we solve

$$50.612 = x \times 50.056 + (1 - x) \times 51.206$$

to obtain $x \approx 0.517$.

A Numerical Example (continued)

- The option values corresponding to running averages 50.056 and 51.206 at node B are 0.056 and 1.206, respectively.
- Their contribution to the option value corresponding to running average 49.666 at node A is weighted linearly as

$$x \times 0.056 + (1 - x) \times 1.206 \approx 0.611.$$

A Numerical Example (continued)

- Now consider the down move from node A with running average 49.666.
- Because the stock price at node C is 46.775, the new running average will be

$$\frac{3 \times 49.666 + 46.775}{4} \approx 48.944.$$

- With 48.944 lying between 47.903 and 48.979 at node C, we solve

$$48.944 = x \times 47.903 + (1 - x) \times 48.979$$

to obtain $x \approx 0.033$.

A Numerical Example (concluded)

- The option values corresponding to running averages 47.903 and 48.979 at node C are both 0.0.
- Their contribution to the option value corresponding to running average 49.666 at node A is 0.0.
- Finally, the option value corresponding to running average 49.666 at node A equals

$$p \times 0.611 + (1 - p) \times 0.0 \approx 0.2956,$$

where $p = 0.483$.

- The remaining three option values at node A can be computed similarly.

Remarks on Asian Option Pricing

- Asian option pricing is an active research area.
- The above algorithm overestimates the "true" value (Dai, Huang, and Lyuu, 2000).
- To guarantee convergence, k needs to grow with n.
- Analytical approximations for European Asian options exist.
- There is a convergent approximation algorithm that does away with interpolation (Dai and Lyuu, 2002).

Remarks on Asian Option Pricing (concluded)

- There is an $O(kn^2)$-time algorithm with an error bound of $O(Xn/k)$ from the naive $O(2^n)$-time binomial tree algorithm in the case of European Asian options (Aingworth, Motwani, and Oldham, 2000).
  - k can be varied for trade-off between time and accuracy.
- Another approximation algorithm reduces the error to $O(X\sqrt{n}/k)$ (Dai, Huang, and Lyuu, 2000).
  - If we pick k proportional to n, then the error decreases at least as $n^{0.5}$.

Forwards, Futures, Futures Options, Swaps

Terms

- r will denote the riskless interest rate.
- The current time is t.
- The maturity date is T.
- The remaining time to maturity is $\tau \equiv T - t$ (all measured in years).
- The spot price S, the spot price at maturity is S_T.
- The delivery price is X.

Terms (concluded)

- The forward or futures price is F for a newly written contract.
- The value of the contract is f.
- A price with a subscript t usually refers to the price at time t.
- Continuous compounding will be assumed throughout this chapter.

Forward Contracts

- Forward contracts are for the delivery of the underlying asset for a certain delivery price on a specific time.
  - Foreign currencies, bonds, corn, etc.
- Ideal for hedging purposes.
- A farmer enters into a forward contract with a food processor to deliver 100,000 bushels of corn for $2.5 per bushel on September 27, 1995.
- The farmer is assured of a buyer at an acceptable price.
- The processor knows the cost of corn in advance.

Forward Contracts (concluded)

- A forward agreement limits both risk and rewards.
  - If the spot price of corn rises on the delivery date, the farmer will miss the opportunity of extra profits.
  - If the price declines, the processor will be paying more than it would.
- Either side has an incentive to default.
- Other problems: The food processor may go bankrupt, the farmer can go bust, the farmer might not be able to harvest 100,000 bushels of corn because of bad weather, the cost of growing corn may skyrocket, etc.

Spot and Forward Exchange Rates

- Let S denote the spot exchange rate.
- Let F denote the forward exchange rate one year from now (both in domestic/foreign terms).
- r_{f} denotes the annual interest rates of the foreign currency.
- r_{ℓ} denotes the annual interest rates of the local currency.
- Arbitrage opportunities will arise unless these four numbers satisfy an equation.

Interest Rate Parity (Keynes,[a] 1923)

$$\frac{F}{S} = e^{r_\ell - r_f}. \tag{62}$$

- A holder of the local currency can do either of:
  - Lend the money in the domestic market to receive e^{r_ℓ} one year from now.
  - Convert the local currency for the foreign currency, lend for one year in the foreign market, and convert the foreign currency into the local currency at the fixed forward exchange rate in the future, F, by selling forward the foreign currency now.

[a]Keynes (1883–1946) was one of the greatest economists in history.

Interest Rate Parity (concluded)

- No money changes hand in entering into a forward contract.
- One unit of local currency will hence become Fe^{r_f}/S one year from now in the 2nd case.
- If $Fe^{r_f}/S > e^{r_\ell}$, an arbitrage profit can result from borrowing money in the domestic market and lending it in the foreign market.
- If $Fe^{r_f}/S < e^{r_\ell}$, an arbitrage profit can result from borrowing money in the foreign market and lending it in the domestic market.

Forward Price

- The payoff of a forward contract at maturity is

$$S_T - X.$$

- Forward contracts do not involve any initial cash flow.
- The forward price is the delivery price which makes the forward contract zero valued.
  - That is, $f = 0$ when $F = X$.

Forward Price (concluded)

- The delivery price cannot change because it is written in the contract.
- But the forward price may change after the contract comes into existence.
  - The value of a forward contract, f, is 0 at the outset.
  - It will fluctuate with the spot price thereafter.
  - This value is enhanced when the spot price climbs and depressed when the spot price declines.
- The forward price also varies with the maturity of the contract.

Forward Price: Underlying Pays No Income

Lemma 13 *For a forward contract on an underlying asset providing no income,*

$$F = Se^{r\tau}. \tag{63}$$

- If $F > Se^{r\tau}$, borrow S dollars for τ years, buy the underlying asset, and short the forward contract with delivery price F.
- At maturity, sell the asset for F and use $Se^{r\tau}$ to repay the loan, leaving an arbitrage profit of $F - Se^{r\tau} > 0$.
- If $F < Se^{r\tau}$, do the reverse.

Example

- r is the annualized 3-month riskless interest rate.
- S is the spot price of the 6-month zero-coupon bond.
- A new 3-month forward contract on a 6-month zero-coupon bond should command a delivery price of $Se^{r/4}$.
- So if $r = 6\%$ and $S = 970.87$, then the delivery price is

$$970.87 \times e^{0.06/4} = 985.54.$$

Contract Value: The Underlying Pays No Income

The value of a forward contract is

$$f = S - Xe^{-r\tau}.$$

- Consider a portfolio of one long forward contract, cash amount $Xe^{-r\tau}$, and one short position in the underlying asset.
- The cash will grow to X at maturity, which can be used to take delivery of the forward contract.
- The delivered asset will then close out the short position.
- Since the value of the portfolio is zero at maturity, its PV must be zero.

Forward Price: Underlying Pays Predictable Income

Lemma 14 *For a forward contract on an underlying asset providing a predictable income with a PV of I,*

$$F = (S - I)\, e^{r\tau}. \tag{64}$$

- If $F > (S - I)\, e^{r\tau}$, borrow S dollars for τ years, buy the underlying asset, and short the forward contract with delivery price F.
- At maturity, the asset is sold for F, and $(S - I)\, e^{r\tau}$ is used to repay the loan, leaving an arbitrage profit of $F - (S - I)\, e^{r\tau} > 0$.
- If $F < (S - I)\, e^{r\tau}$, reverse the above.

Example

- Consider a 10-month forward contract on a \$50 stock.
- The stock pays a dividend of \$1 every 3 months.
- The forward price is

$$\left(50 - e^{-r_3/4} - e^{-r_6/2} - e^{-3 \times r_9/4}\right) e^{r_{10} \times (10/12)}.$$

  - r_i is the annualized i-month interest rate.

Underlying Pays a Continuous Dividend Yield of q

The value of a forward contract at any time prior to maturity is

$$f = Se^{-q\tau} - Xe^{-r\tau}. \quad (65)$$

- Consider a portfolio of one long forward contract, cash amount $Xe^{-r\tau}$, and a short position in $e^{-q\tau}$ units of the underlying asset.
- All dividends are paid for by shorting additional units of the underlying asset.
- The cash will grow to X at maturity.
- The short position will grow to exactly one unit of the underlying asset.

Underlying Pays a Continuous Dividend Yield (concluded)

- There is sufficient fund to take delivery of the forward contract.
- This offsets the short position.
- Since the value of the portfolio is zero at maturity, its PV must be zero.
- One consequence of Eq. (65) is that the forward price is

$$F = Se^{(r-q)\tau}. \quad (66)$$

Futures Contracts vs. Forward Contracts

- They are traded on a central exchange.
- A clearinghouse.
  - Credit risk is minimized.
- Futures contracts are standardized instruments.
- Gains and losses are marked to market daily.
  - Adjusted at the end of each trading day based on the settlement price.
  - The settlement price is some kind of average traded price immediately before the end of trading.

Size of a Futures Contract

- The amount of the underlying asset to be delivered under the contract.
  - 5,000 bushels for the corn futures on the CBT.
  - One million U.S. dollars for the Eurodollar futures on the CME.
- A position can be closed out (or offset) by entering into a reversing trade to the original one.
- Most futures contracts are closed out in this way rather than have the underlying asset delivered.
  - Forward contracts are meant for delivery.

Daily Settlements

- Price changes in the futures contract are settled daily.
- Hence the spot price rather than the initial futures price is paid on the delivery date.
- Marking to market nullifies any financial incentive for not making delivery.
  - A farmer enters into a forward contract to sell a food processor 100,000 bushels of corn at $2.00 per bushel in November.
  - Suppose the price of corn rises to $2.5 by November.

Daily Settlements (concluded)

- (continued)
  - The farmer has incentive to sell his harvest in the spot market at $2.5.
  - With marking to market, the farmer has transferred $0.5 per bushel from his futures account to that of the food processor by November.
  - When the farmer makes delivery, he is paid the spot price, $2.5 per bushel.
  - The farmer has little incentive to default.
  - The net price remains $2.00 per bushel, the original delivery price.

Delivery and Hedging

- Delivery ties the futures price to the spot price.
- On the delivery date, the settlement price of the futures contract is determined by the spot price.
- Hence, when the delivery period is reached, the futures price should be very close to the spot price.
- Changes in futures prices usually track those in spot prices.
- This makes hedging possible.
- Before the delivery date, the futures price could be above or below the spot price.

Daily Cash Flows

- Let F_i denote the futures price at the end of day i.
- The contract's cash flow on day i is $F_i - F_{i-1}$.
- The net cash flow over the life of the contract is

$$(F_1 - F_0) + (F_2 - F_1) + \cdots + (F_n - F_{n-1}) = F_n - F_0 = S_T - F_0. \quad (67)$$

- A futures contract has the same accumulated payoff $S_T - F_0$ as a forward contract.
- The actual payoff may differ because of the reinvestment of daily cash flows and how $S_T - F_0$ is distributed.

Forward and Futures Prices

Futures price equals forward price if interest rates are nonstochastic[a] (Cox, Ingersoll, and Ross, 1981)!

- Consider forward and futures contracts on the same underlying asset with n days to maturity.
- The interest rate for day i is r_i.
- One dollar at the beginning of day i grows to $R_i \equiv e^{r_i}$ by day's end.

[a] This "justifies" treating a futures contract as if it were a forward contract, ignoring its marking-to-market feature.

The Proof (continued)

- Let F_i be the futures price at the end of day i.
- So one dollar invested in the n-day discount bond at the end of day zero will be worth

$$R \equiv \prod_{j=1}^{n} R_j$$

by the end of day n.

- Maintain $\prod_{j=1}^{i} R_j$ long futures positions at the end of day $i-1$ and invest the cash flow at the end of day i in riskless bonds maturing on delivery day n.

The Proof (continued)

- The cash flow from the position on day i is $(F_i - F_{i-1}) \prod_{j=1}^{i} R_j$.
  - Day i starts with $\prod_{j=1}^{i} R_j$ contracts.
- This amount is compounded until end of day n to become

$$(F_i - F_{i-1}) \prod_{j=1}^{i} R_j \prod_{j=i+1}^{n} R_j = (F_i - F_{i-1}) \prod_{j=1}^{n} R_j = (F_i - F_{i-1})\, R.$$

The Proof (continued)

- The value at the end of day n is hence

$$\sum_{i=1}^{n} (F_i - F_{i-1})\, R = (F_n - F_0)\, R = (S_T - F_0)\, R.$$

- Observe that no investment is required for the strategy.
- Suppose the forward price f_0 exceeds the futures price F_0.
- Short R forward contracts, borrow $f_0 - F_0$, and carry out the above strategy.

The Proof (concluded)

- The initial cash flow is $f_0 - F_0 > 0$.
- At the end of day n, the debt grows to $(f_0 - F_0)\, R$, and the net value is

$$f_0 R - S_T R - (f_0 - F_0)\, R + (S_T - F_0)\, R = 0.$$

- So $f_0 - F_0$ is a pure arbitrage profit.
- The case of $f_0 < F_0$ is symmetrical.

Remarks

- When interest rates are stochastic, forward and futures prices are no longer theoretically identical.
  - Suppose interest rates are uncertain and futures prices move in the same direction as interest rates.
  - Then futures prices will exceed forward prices.
- For short-term contracts, the differences tend to be small.
- Unless stated otherwise, assume forward and futures prices are identical.

Futures Options

- The underlying of a futures option is a futures contract.
- Upon exercise, the option holder takes a position in the futures contract with a futures price equal to the option's strike price.
  - A call holder acquires a long futures position.
  - A put holder acquires a short futures position.
- The futures contract is then marked to market, and the futures position of the two parties will be at the prevailing futures price.

Futures Options (concluded)

- It works as if the call writer delivered a futures contract to the option holder and paid the holder the prevailing futures price minus the strike price.
- It works as if the put writer took delivery a futures contract from the option holder and paid the holder the strike price minus the prevailing futures price.
- The amount of money that changes hands upon exercise is the difference between the strike price and the prevailing futures price.

Forward Options

- Similar to futures options except that what is delivered is a forward contract with a delivery price equal to the option's strike price.
  - Exercising a call forward option results in a long position in a forward contract.
  - Exercising a put forward option results in a short position in a forward contract.
- Exercising a forward option incurs no immediate cash flows.

Example

- Consider a call with strike \$100 and an expiration date in September.
- The underlying asset is a forward contract with a delivery date in December.
- Suppose the forward price in July is \$110.
- Upon exercise, the call holder receives a forward contract with a delivery price of \$100.
- If an offsetting position is then taken in the forward market, a \$10 profit in *September* will be assured.
- A call on the futures would realize the \$10 profit in *July*.