A Puzzle

- The option value is independent of the stock's expected return $\mu - q$.
- So replace it by r.
- Then why not use

$$\frac{e^{r\Delta t} - d}{u - d}$$

as the risk-neutral probability?

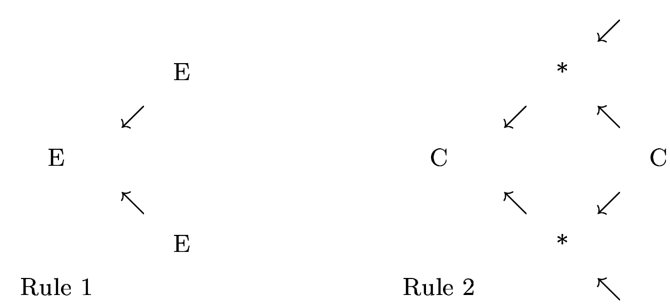
Traversal Sequence

- Can the standard quadratic-time binomial tree algorithm for American options be improved?
  - By an order.
  - By a constant factor.
- It helps to skip nodes.
- Note the traversal sequence of backward induction on the tree.
  - By time.

Diagonal Traversal of the Tree (Curran, 1995)

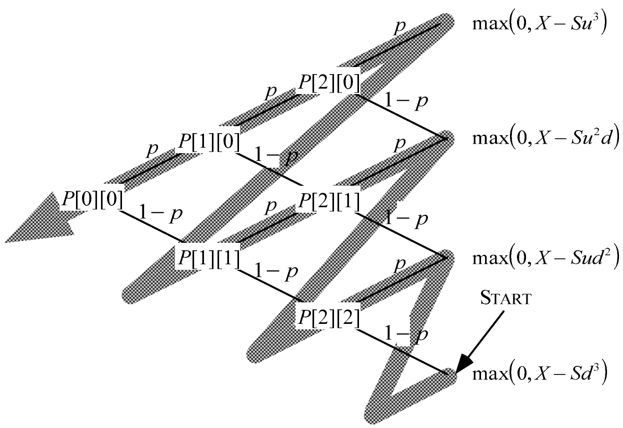
Properties of the propagation of early exercise nodes (E) and non-early-exercise nodes (C) during backward induction.

1. A node is an early-exercise node if both its successor nodes are exercised early.
   - A terminal node that is in-the-money is considered an early exercise node.
   - A terminal node that is out-the-money is considered a non-early-exercise node.
2. If a node is a non-early-exercise node, then all the earlier nodes at the same horizontal level are also non-early-exercise nodes (assume $ud = 1$).



Diagonal Traversal of the Tree (continued)

- An early-exercise node is trivial to evaluate.
  - The difference of the strike price and the stock price.
- A non-early-exercise node must be evaluated by backward induction.
- Suppose we traverse the tree diagonally.
- Convince yourself that this procedure is well-defined.

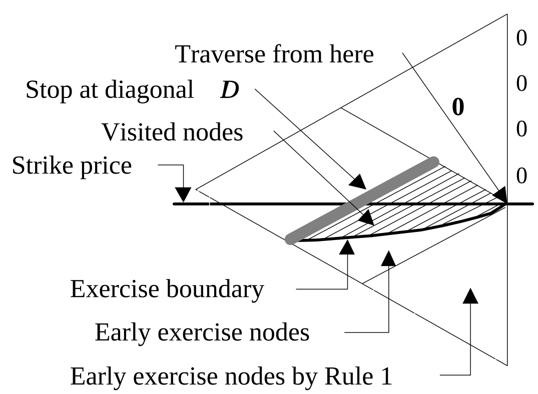


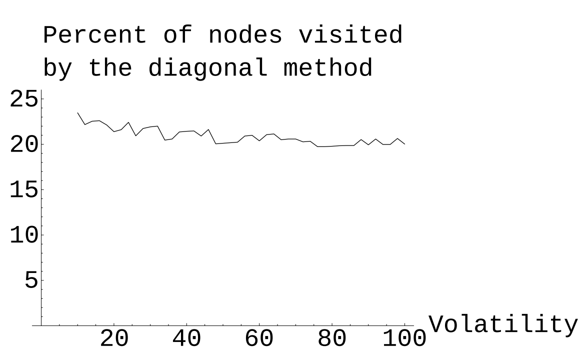
Diagonal Traversal of the Tree (continued)

- Nothing is achieved if the whole tree needs to be explored.
- Need a stopping rule.
- The process stops when a diagonal D consisting *entirely* of non-early-exercise nodes has been encountered.
  - By Rule 2, all early-exercise nodes have been found.

Diagonal Traversal of the Tree (continued)

- When the algorithm finds an early exercise node in traversing a diagonal, it can stop immediately and move on to the next diagonal.
  - By Rule 1 and the sequence by which the nodes on the diagonals are traversed, the rest of the nodes on the current diagonal must all be early-exercise nodes.
  - They are hence computable on the fly when needed.
- Also by Rule 1, the traversal can start from the zero-valued terminal node just above the strike price.
- The upper triangle above the strike price can be skipped since its nodes are all zero valued.





Diagonal Traversal of the Tree (continued)

- It remains to calculate the option value.
- It is the sum of the discounted option values of the nodes on D, each multiplied by the probability that the stock price hits the diagonal for the *first* time at that node.
  – How do the payoff influence the root?
  – Cannot go from the root to a node at which the option will be exercised without passing through D.

Diagonal Traversal of the Tree (continued)

- For a node on D which is the result of i up moves and j down moves, the said probability is $\binom{i+j-1}{i} p^i (1-p)^j$.
  – A valid path must pass through the node which is the result of i up moves and $j-1$ down moves.
- Call the option value on this node P_i.
- The desired option value then equals

$$\sum_{i=0}^{a-1} \binom{i+j-1}{i} p^i (1-p)^j P_i e^{-(i+j)\, r\Delta t}.$$

Diagonal Traversal of the Tree (continued)

- Since each node on D has been evaluated by that time, this part of the computation consumes $O(n)$ time.
- The space requirement is also linear in n since only the diagonal has to be allocated space.
- This idea can save computation time when D does not take long to find.

Diagonal Traversal of the Tree (continued)

- Rule 2 is true with or without dividends.
- Suppose now that the stock pays a continuous dividend yield $q \leq r$ (or $r \leq q$ for calls by parity).
- Recall $p = \frac{e^{(r-q)\Delta t} - d}{u-d}$.
- Rule 1 continues to hold since, for a current stock price of $Su^i d^j$:

Diagonal Traversal of the Tree (concluded)

$$
\begin{aligned}
& (pP_u + (1-p)\, P_d)\, e^{-r\Delta t} \\
= \; & \left[p \left(X - Su^{i+1} d^j \right) + (1-p) \left(X - Su^i d^{j+1} \right) \right] e^{-r\Delta t} \\
= \; & Xe^{-r\Delta t} - Su^i d^j (pu + (1-p)\, d)\, e^{-r\Delta t} \\
= \; & Xe^{-r\Delta t} - Su^i d^j e^{-q\Delta t} \\
\leq \; & Xe^{-r\Delta t} - Su^i d^j e^{-r\Delta t} \\
\leq \; & X - Su^i d^j .
\end{aligned}
$$

Sensitivity Analysis of Options

Sensitivity Measures ("The Greeks")

- Understanding how the value of a security changes relative to changes in a given parameter is key to hedging.
  - Duration, for instance.
- We now ask same questions of options.
- Let $x \equiv \frac{\ln(S/X)+(r+\sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$ (recall p. 253).
- $N'(y) = (1/\sqrt{2\pi}\,)\, e^{-y^2/2} > 0$, the density function of standard normal distribution.

Delta

- Defined as $\Delta \equiv \partial f/\partial S$.
  - f is the price of the derivative.
  - S is the price of the underlying asset.
- The delta of a portfolio of derivatives on the same underlying asset is the sum of their individual deltas.
- The delta used in the BOPM is the discrete analog.

Delta (concluded)

- The delta of a European call on a non-dividend-paying stock equals

$$\frac{\partial C}{\partial S} = N(x) > 0. \tag{57}$$

- The delta of a European put equals

$$\frac{\partial P}{\partial S} = N(x) - 1 < 0.$$

- The delta of a long stock is 1.

Delta Neutrality

- A position with a total delta equal to 0 is delta-neutral.
- A delta-neutral portfolio is immune to small price changes in the underlying asset.
- Creating one serves for hedging purposes.
  - A portfolio consisting of a call and $-\Delta$ shares of stock is delta-neutral.
  - Short Δ shares of stock to hedge a long call.
- Hedge a position in a security with a delta of Δ_1 by shorting Δ_1/Δ_2 units of a security with a delta of Δ_2.

Theta (Time Decay)

- Defined as the rate of change of a security's value with respect to time, or $\Theta \equiv -\partial f / \partial \tau$.
- For a European call on a non-dividend-paying stock,

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N(x - \sigma\sqrt{\tau}) < 0.$$

  - The call loses value with the passage of time.
- For a European put,

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} + rXe^{-r\tau}N(-x + \sigma\sqrt{\tau}).$$

  - Can be negative or positive.

Gamma

- Defined as the rate of change of its delta with respect to the price of the underlying asset, or $\Gamma \equiv \partial^2 \Pi / \partial S^2$.
- Measures how sensitive delta is to changes in the price of the underlying asset.
- A portfolio with a high gamma needs in practice be rebalanced more often to maintain delta neutrality.
- Delta $\sim$ duration, gamma $\sim$ convexity.
- The gamma of a European call or put on a non-dividend-paying stock is $N'(x)/(S\sigma\sqrt{\tau}) > 0$.

Vega[a] (Lambda, Kappa, Sigma)

- Defined as the rate of change of its value with respect to the volatility of the underlying asset $\Lambda \equiv \partial \Pi / \partial \sigma$.
- Volatility often changes over time.
- A security with a high vega is very sensitive to small changes in volatility.
- The vega of a European call or put on a non-dividend-paying stock is $S\sqrt{\tau}\,N'(x) > 0$.
  - Higher volatility increases option value.

[a] Vega is not Greek.

Rho

- Defined as the rate of change in its value with respect to interest rates $\rho \equiv \partial \Pi / \partial r$.
- The rho of a European call on a non-dividend-paying stock is

$$X\tau e^{-r\tau}N(x - \sigma\sqrt{\tau}) > 0.$$

- The rho of a European put on a non-dividend-paying stock is

$$-X\tau e^{-r\tau}N(-x + \sigma\sqrt{\tau}) < 0.$$

Numerical Greeks

- Needed when closed-form formulas do not exist.
- Take delta as an example.
- A standard method computes the finite difference,

$$\frac{f(S+\Delta S)-f(S-\Delta S)}{2\Delta S}.$$

- The computation time roughly doubles that for evaluating the derivative security itself.

An Alternative Numerical Delta

- Use the intermediate results of the binomial tree algorithm (Pelsser and Vorst, 1994).
- When the algorithm reaches the end of the first period, f_u and f_d are computed.
- These values correspond to derivative values at stock prices Su and Sd, respectively.
- Delta is approximated by

$$\frac{f_u - f_d}{Su - Sd}.$$

- Almost zero extra computational effort.

Numerical Gamma

- At the stock price $(Suu + Sud)/2$, delta is approximately $(f_{uu} - f_{ud})/(Suu - Sud)$.
- At the stock price $(Sud + Sdd)/2$, delta is approximately $(f_{ud} - f_{dd})/(Sud - Sdd)$.
- Gamma is the rate of change in deltas between $(Suu + Sud)/2$ and $(Sud + Sdd)/2$, that is,

$$\frac{\frac{f_{uu}-f_{ud}}{Suu-Sud} - \frac{f_{ud}-f_{dd}}{Sud-Sdd}}{(Suu - Sdd)/2}. \tag{58}$$

Finite Difference Fails for Numerical Gamma

- Numerical differentiation gives

$$\frac{f(S+\Delta S) - 2f(S) + f(S-\Delta S)}{(\Delta S)^2}.$$

- Why doesn't it work?
- Why did the binomial tree version work?

Other Numerical Greeks

- The theta can be computed as

$$\frac{f_{ud} - f}{2(\tau/n)}.$$

  - In fact, the theta of a European option will be shown to be computable from delta and gamma.
- For vega and rho, there is no alternative but to run the binomial tree algorithm twice.

Extensions of Options Theory

Pricing Corporate Securities (Black and Scholes, 1973)

- Interpret the underlying asset interpretated as the total value of the firm.
- The option pricing methodology can be applied to pricing corporate securities.
- Assume:
  - A firm can finance payouts by the sale of assets.
  - If a promised payment to an obligation other than stock is missed, the claim holders take ownership of the firm and the stockholders get nothing.

Risky Zero-Coupon Bonds and Stock

- Consider XYZ.com.
- Capital structure:
  - n shares of its own common stock, S.
  - Zero-coupon bonds with an aggregate par value of X.
- What is the value of the bonds, B?
- What is the value of the XYZ.com stock?

Risky Zero-Coupon Bonds and Stock (continued)

- On the bonds' maturity date, if the total value of the firm V^* is less than the bondholders' claim X, the firm declares bankruptcy and the stock becomes worthless.
- If $V^* > X$, then the bondholders obtain X and the stockholders $V^* - X$.

	$V^* \leq X$	$V^* > X$
Bonds	V^*	X
Stock	0	$V^* - X$

Risky Zero-Coupon Bonds and Stock (continued)

- The stock is a call on the total value of the firm with a strike price of X and an expiration date equal to the bonds'.
  - This call provides the limited liability for the stockholders.
- The bonds are a covered call on the total value of the firm.
- C stands for this call and V the total value of the firm.

Risky Zero-Coupon Bonds and Stock (continued)

- Thus $nS = C$ and $B = V - C$.
- Knowing C amounts to knowing how the value of the firm is divided between stockholders and bondholders.
- Whatever the value of C, the total value of the stock and bonds at maturity remains V^*.
- The relative size of debt and equity is irrelevant to the firm's current value V.

Risky Zero-Coupon Bonds and Stock (continued)

- From Theorem 12 (p. 253) and the put-call parity,

$$\begin{aligned} nS &= VN(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \\ B &= VN(-x) + Xe^{-r\tau} N(x - \sigma\sqrt{\tau}). \end{aligned}$$

  - $x \equiv \frac{\ln(V/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$.
- The continuously compounded yield to maturity of the firm's bond is $(1/\tau)\ln(X/B)$.

Risky Zero-Coupon Bonds and Stock (concluded)

- Define default premium as the yield difference between risky and riskless bonds,

$$\begin{aligned}&(1/\tau)\ln(X/B) - r\\ =\ & -\frac{1}{\tau}\ln\left(N(-z) + \frac{1}{\omega}N(z - \sigma\sqrt{\tau})\right).\end{aligned}$$

  - $\omega \equiv Xe^{-r\tau}/V$.
  - $z \equiv (\ln\omega)/(\sigma\sqrt{\tau}) + (1/2)\,\sigma\sqrt{\tau} = -x + \sigma\sqrt{\tau}$.
  - Note that ω is the debt-to-total-value ratio.

A Numerical Example

- XYZ.com's assets consist of 1,000 shares of Merck as of March 20, 1995.
  - Merck's market value per share is \$44.5.
- XYZ.com's securities consist of 1,000 shares of common stock and 30 zero-coupon bonds maturing on July 21, 1995.
- Each bond promises to pay \$1,000 at maturity.
- $n = 1000$, $V = 44.5 \times n = 44500$, and $X = 30 \times n = 30000$.

			—Call—		—Put—	
Option	Strike	Exp.	Vol.	Last	Vol.	Last
Merck	30	Jul	328	15 1/4	...	...
44 1/2	35	Jul	150	9 1/2	10	1/16
44 1/2	40	Apr	887	4 3/4	136	1/16
44 1/2	40	Jul	220	5 1/2	297	1/4
44 1/2	40	Oct	58	6	10	1/2
44 1/2	45	Apr	3050	7/8	100	1 1/8
44 1/2	45	May	462	1 3/8	50	1 3/8
44 1/2	45	Jul	883	1 15/16	147	1 3/4
44 1/2	45	Oct	367	2 3/4	188	2 1/16

A Numerical Example (continued)

- The Merck option relevant for pricing is the July call with a strike price of $X/n = 30$ dollars.
- Such an is selling for \$15.25.
- So XYZ.com's stock is worth $15.25 \times n = 15250$ dollars.
- The entire bond issue is worth $B = 44500 - 15250 = 29250$ dollars—\$975 per bond.

A Numerical Example (continued)

- The XYZ.com bonds are equivalent to a default-free zero-coupon bond with $\$X$ par value plus n written European puts on Merck at a strike price of $30.
  - By the put-call parity.
- The difference between B and the price of the default-free bond is the value of these puts.
- The table next page shows the total market values of the XYZ.com stock and bonds under various debt amounts X.

Promised payment to bondholders X	Current market value of bonds B	Current market value of stock nS	Current total value of firm V
30,000	29,250.0	15,250.0	44,500
35,000	35,000.0	9,500.0	44,500
40,000	39,000.0	5,500.0	44,500
45,000	42,562.5	1,937.5	44,500

A Numerical Example (continued)

- If the promised payment to bondholders is $45,000, the relevant option is the July call with a strike price of $45000/n = 45$ dollars.
- Since that option is selling for $1 15/16, the market value of the XYZ.com stock is $(1 + 15/16) \times n = 1937.5$ dollars.
- The market value of the stock decreases as the debt-equity ratio increases.

A Numerical Example (continued)

- There are conflicts between stockholders and bondholders.
- An option's terms cannot be changed after issuance.
- But a firm can change its capital structure.
- There lies one difference between options and corporate securities.
- So parameters such volatility, dividend, and strike price are under partial control of the stockholders.

A Numerical Example (continued)

- Suppose XYZ.com issues 15 more bonds with the same terms to buy back stock.
- The total debt is now $X = 45{,}000$ dollars.
- Table on p. 309 says the total market value of the bonds should be \$42,562.5.
- The *new* bondholders pay $42562.5 \times (15/45) = 14187.5$ dollars.
- The remaining stock is worth \$1,937.5.

A Numerical Example (continued)

- The stockholders therefore gain

$$14187.5 + 1937.5 - 15250 = 875$$

dollars.

- The *original* bondholders lose an equal amount,

$$29250 - \frac{30}{45} \times 42562.5 = 875. \qquad (59)$$

A Numerical Example (continued)

- Suppose the stockholders distribute \$14,833.3 cash dividends by selling $(1/3) \times n$ Merck shares.
- They now have \$14,833.3 in cash plus a call on $(2/3) \times n$ Merck shares.
- The strike price remains $X = 30000$.
- This is equivalent to owning two-thirds of a call on n Merck shares with a total strike price of \$45,000.
- Since n such calls are worth \$1,937.5 from table on p. 306, the total market value of the XYZ.com stock is $(2/3) \times 1937.5 = 1291.67$ dollars.

A Numerical Example (concluded)

- The market value of the XYZ.com bonds is hence $(2/3) \times n \times 44.5 - 1291.67 = 28375$ dollars.
- Hence the stockholders gain

$$14833.3 + 1291.67 - 15250 \approx 875$$

dollars.

- The bondholders watch their value drop from \$29,250 to \$28,375, a loss of \$875.

Other Examples

- Subordinated debts as bull call spreads.
- Warrants as calls.
- Callable bonds as American calls with 2 strike prices.
- Convertible bonds.

Barrier Options

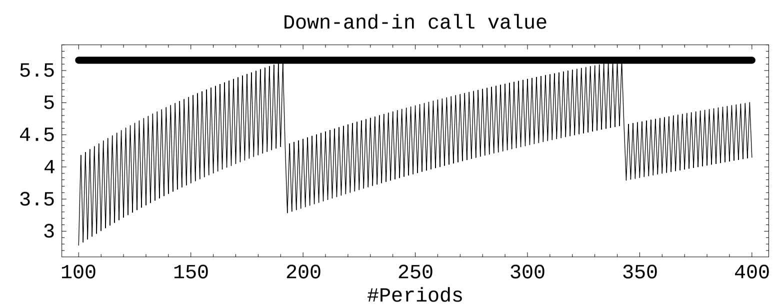
- Their payoff depends on whether the underlying asset's price reaches a certain price level H.
- A knock-out option is an ordinary European option which ceases to exist if the barrier H is reached by the price of its underlying asset.
- A call knock-out option is sometimes called a down-and-out option if $H < S$.
- A put knock-out option is sometimes called an up-and-out option when $H > S$.

Barrier Options (concluded)

- A knock-in option comes into existence if a certain barrier is reached.
- A down-and-in option is a call knock-in option that comes into existence only when the barrier is reached and $H < S$.
- An up-and-in is a put knock-in option that comes into existence only when the barrier is reached and $H > S$.
- Formulas exist for all kinds of barrier options.

Binomial Tree Algorithms

- Barrier options can be priced by binomial tree algorithms.
- Convergence is erratic because H is not on a price level.
- Hence the algorithms are useless.
- Solutions will be presented later.



Foreign Currencies

- S denotes the spot exchange rate in domestic/foreign terms.
- σ denotes the volatility of the exchange rate.
- r denotes the domestic interest rate.
- $\hat{r}$ denotes the foreign interest rate.
- A foreign currency is analogous to a stock paying a known dividend yield.
  - Foreign currencies pay a "continuous dividend yield" equal to $\hat{r}$ in the foreign currency.

Foreign Exchange Options

- Foreign exchange options are settled via delivery of the underlying currency.
- A primary use of foreign exchange (or forex) options is to hedge currency risk.
- Consider a U.S. company expecting to receive 100 million Japanese yen in March 2000.
- Those 100 million Japanese yen will be exchanged for U.S. dollars.

Foreign Exchange Options (continued)

- The contract size for the Japanese yen option is JPY6,250,000.
- The company purchases $100{,}000{,}000/6{,}250{,}000 = 16$ puts on the Japanese yen with a strike price of \$.0088 and an exercise month in March 2000.
- This gives the company the right to sell 100,000,000 Japanese yen for $100{,}000{,}000 \times .0088 = 880{,}000$ U.S. dollars.

Foreign Exchange Options (concluded)

- The formulas derived for stock index options in Eqs. (55) on p. 266 apply with the dividend yield equal to $\hat{r}$:

$$C = Se^{-\hat{r}\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \quad (60)$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-\hat{r}\tau} N(-x). \quad (60')$$

  – $x \equiv \frac{\ln(S/X) + (r - \hat{r} + \sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$.

Path-Dependent Derivatives

- Let $S_0, S_1, \ldots, S_n$ denote the prices of the underlying asset over the life of the option.
- S_0 is the known price at time zero.
- S_n is the price at expiration.
- The standard European call has a terminal value depending only on the last price, $\max(S_n - X, 0)$.
- Its value thus depends only on the underlying asset's terminal price regardless of how it gets there.

Path-Dependent Derivatives (continued)

- In contrast, some derivatives are path-dependent in that their terminal payoffs depend "critically" on the paths.
- The (arithmetic) average-rate call has a terminal value given by

$$\max\left(\frac{1}{n+1}\sum_{i=0}^{n} S_i - X, 0\right).$$

- The average-rate put's terminal value is given by

$$\max\left(X - \frac{1}{n+1}\sum_{i=0}^{n} S_i, 0\right).$$

Path-Dependent Derivatives (continued)

- Average-rate options are also called Asian options.
- They are useful hedging tools for firms that will make a stream of purchases over a time period because the costs are likely to be linked to the average price.
- They are mostly European.

Path-Dependent Derivatives (concluded)

- A lookback call option on the minimum has a terminal payoff of $S_n - \min_{0 \le i \le n} S_i$.
- A lookback put option on the maximum has a terminal payoff of $\max_{0 \le i \le n} S_i - S_n$.
- The fixed-strike lookback option provides a payoff of $\max(\max_{0 \le i \le n} S_i - X, 0)$ for the call and $\max(X - \min_{0 \le i \le n} S_i, 0)$ for the put.
- Lookback call and put options on the average are called average-strike options.

Average-Rate Options

- Average-rate options are notoriously hard to price.
- The binomial tree for the averages does not combine.
- A straightforward algorithm is to enumerate the 2^n price paths for an n-period binomial tree and then average the payoffs.
- But the exponential complexity makes this naive algorithm impractical.
- As a result, the Monte Carlo method and approximation algorithms are some of the alternatives left.