Backward Induction (Zermelo)

- The above expression calculates C from the two successor nodes C_u and C_d and none beyond.
- The same computation happens at C_u and C_d, too, as demonstrated in Eq. (48) on p. 222.
- This recursive procedure is called backward induction.
- Now, C equals

$$\begin{aligned} & [p^2 C_{uu} + 2p(1-p)\, C_{ud} + (1-p)^2 C_{dd}](1/R^2) \\ = & [p^2 \cdot \max\left(0, Su^2 - X\right) + 2p(1-p) \cdot \max\left(0, Sud - X\right) \\ & +(1-p)^2 \cdot \max\left(0, Sd^2 - X\right)](1/R^2). \end{aligned}$$

Backward Induction (continued)

- In the n-period case,

$$C = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, Su^j d^{n-j} - X\right)}{R^n}. \tag{49}$$

  – The value of a call on a non-dividend-paying stock is the expected discounted payoff at expiration in a risk-neutral economy.
- The value of a European put is

$$P = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, X - Su^j d^{n-j}\right)}{R^n}.$$

Risk-Neutral Pricing Methodology

- Every derivative can be priced as if the economy were risk-neutral.
- For a European-style derivative with the terminal payoff function $\mathcal{D}$, its value is

$$e^{-\hat{r}n} E^{\pi}[\,\mathcal{D}\,].$$

  – E^{π} means the expectation is taken under the risk-neutral probability.
- The "equivalence" between arbitrage freedom in a model and the existence of a risk-neutral probability is called the (first) fundamental theorem of asset pricing.

Self-Financing

- Delta changes over time.
- The maintenance of an equivalent portfolio is dynamic.
- The maintaining of an equivalent portfolio does not depend on our correctly predicting future stock prices.
- The portfolio's value at the end of the current period is precisely the amount needed to set up the next portfolio.
- The trading strategy is self-financing because there is neither injection nor withdrawal of funds throughout.
  – Changes in value are due entirely to capital gains.

The Binomial Option Pricing Formula

- Let a be the minimum number of upward price moves for the call to finish in the money.
- a is the smallest nonnegative integer such that $Su^a d^{n-a} \geq X$, or

$$a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil. \tag{50}$$

The Binomial Option Pricing Formula (concluded)

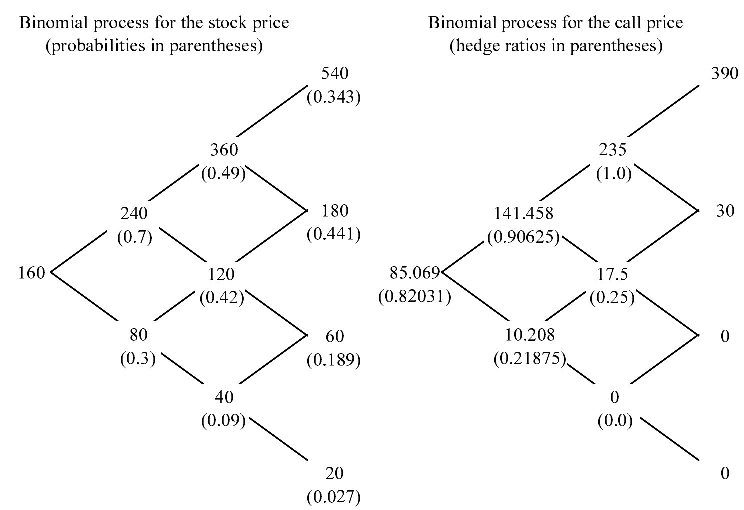
Hence,

$$\begin{aligned} & C \\ = & \frac{\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \left(Su^j d^{n-j} - X \right)}{R^n} \qquad (51) \\ = & S \sum_{j=a}^{n} \binom{n}{j} \frac{(pu)^j ((1-p)\, d)^{n-j}}{R^n} - \frac{X}{R^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \\ = & S \sum_{j=a}^{n} b\left(j; n, pue^{-\hat{r}}\right) - Xe^{-\hat{r}n} \sum_{j=a}^{n} b(j; n, p). \end{aligned}$$

Example

- A non-dividend-paying stock is selling for \$160.
- $u = 1.5$ and $d = 0.5$.
- $r = 18.232\%$ per period.
- Consider a European call on this stock with $X = 150$ and $n = 3$.
- The call value is \$85.069 by backward induction.
- Also the PV of the expected payoff at expiration,

$$\frac{390 \times 0.343 + 30 \times 0.441}{(1.2)^3} = 85.069.$$



Example (continued)

- Mispricing leads to arbitrage profits.
- Suppose the option is selling for \$90 instead.
- Sell the call for \$90 and invest \$85.069 in the replicating portfolio with 0.82031 shares of stock required by delta.
- Borrow $0.82031 \times 160 - 85.069 = 46.1806$ dollars.
- The fund that remains, $90 - 85.069 = 4.931$ dollars, is the arbitrage profit as we will see.

Example (continued)

Time 1:

- Suppose the stock price moves to \$240.
- The new delta is 0.90625.
- Buy $0.90625 - 0.82031 = 0.08594$ more shares at the cost of $0.08594 \times 240 = 20.6256$ dollars financed by borrowing.
- Debt now totals $20.6256 + 46.1806 \times 1.2 = 76.04232$ dollars.

Example (continued)

Time 2:

- Suppose the stock price plunges to \$120.
- The new delta is 0.25.
- Sell $0.90625 - 0.25 = 0.65625$ shares for an income of $0.65625 \times 120 = 78.75$ dollars.
- Use this income to reduce the debt to $76.04232 \times 1.2 - 78.75 = 12.5$ dollars.

Example (continued)

Time 3 (the case of rising price):

- The stock price moves to \$180.
- The call we wrote finishes in the money.
- For a loss of $180 - 150 = 30$ dollars, close out the position by either buying back the call or buying a share of stock for delivery.
- Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + 30 = 45$ dollars.
- It is repaid by selling the 0.25 shares of stock for $0.25 \times 180 = 45$ dollars.

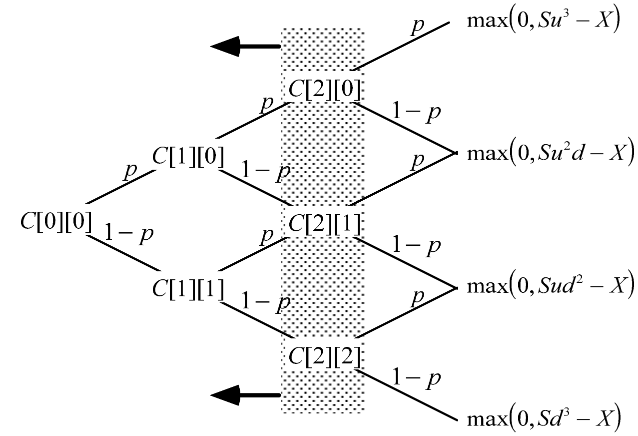
Example (concluded)

Time 3 (the case of declining price):

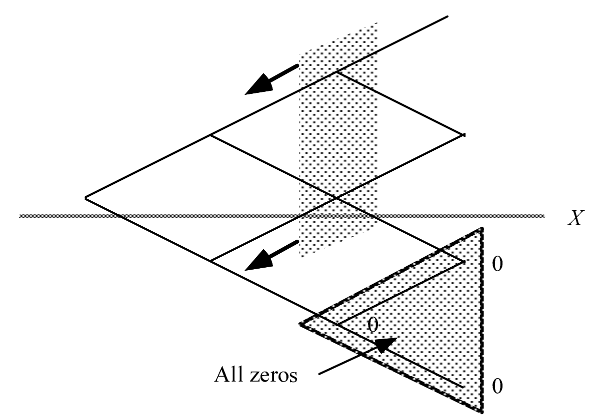
- The stock price moves to \$60.
- The call we wrote is worthless.
- Sell the 0.25 shares of stock for a total of $0.25 \times 60 = 15$ dollars.
- Use it to repay the debt of $12.5 \times 12 = 15$ dollars.

Binomial Tree Algorithms for European Options

- The BOPM implies the binomial tree algorithm that applies backward induction.
- The total running time is $O(n^2)$.
- The memory requirement is $O(n^2)$.
  - Can be further reduced to $O(n)$ by reusing space
- To price European puts, simply replace the payoff.



Further Improvement for Calls



Optimal Algorithm

- Can reduce the running time to $O(n)$ and the memory requirement to $O(1)$.
- Note that

$$b(j;n,p) = \frac{p(n-j+1)}{(1-p)\,j}\, b(j-1;n,p).$$

- The following program computes $b(j;n,p)$ in $b[\,j\,]$,

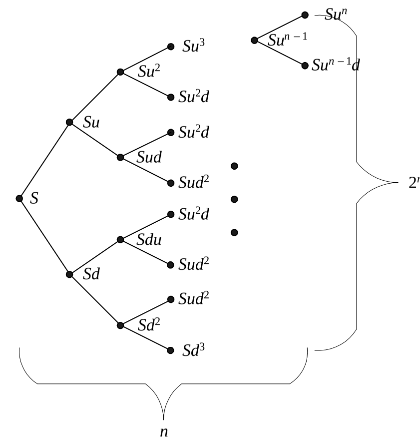
$b[\,a\,] := \binom{n}{a} p^a (1-p)^{n-a}$;
for $(j = a+1$ **to** $n)$
$\quad b[\,j\,] := b[\,j-1\,] \times p \times (n-j+1)/((1-p) \times j)$;

- It clearly runs in $O(n)$ steps.

Optimal Algorithm (continued)

- With the $b(j;n,p)$ available, the risk-neutral valuation formula (51) on p. 230 is trivial to compute.
- We only need a single variable to store the $b(j;n,p)$s as they are being sequentially computed.
- This linear-time algorithm computes the discounted expected value of $\max(S_n - X, 0)$.
- The above technique cannot be applied to American options because of early exercise.
- So algorithms for American options usually run in $O(n^2)$ time.

The Full Uncombining Tree



Toward the Black-Scholes Formula

- The binomial model suffers from two unrealistic assumptions.
  - The stock price takes on only two values in a period.
  - Trading occurs at discrete points in time.
- As the number of periods increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously.
- A proper calibration of the model parameters makes the BOPM converge to the continuous-time model.
- We will skim through the proof.

Toward the Black-Scholes Formula (continued)

- Let τ denote the time to expiration of the option measured in years.
- Let r be the continuously compounded annual rate.
- With n periods during the option's life, each period represents a time interval of τ/n.
- Need to adjust the period-based u, d, and interest rate $\hat{r}$ to match the empirical results as n goes to infinity.
  - $\hat{r} = r\tau/n$.
  - The period gross return $R = e^{\hat{r}}$.

Toward the Black-Scholes Formula (continued)

- Use

$$\widehat{\mu} \equiv \frac{1}{n} E\left[\ln \frac{S_\tau}{S}\right] \quad \text{and} \quad \widehat{\sigma}^2 \equiv \frac{1}{n} \text{Var}\left[\ln \frac{S_\tau}{S}\right]$$

to denote, resp., the expected value and variance of the period continuously compounded rate of return.

- Under the BOPM, it is not hard to show that

$$\begin{aligned} \widehat{\mu} &= q\ln(u/d) + \ln d, \\ \widehat{\sigma}^2 &= q(1-q)\ln^2(u/d). \end{aligned}$$

Toward the Black-Scholes Formula (continued)

- Assume the stock's true continuously compounded rate of return over τ years has mean $\mu\tau$ and variance $\sigma^2\tau$.
  - Call σ the stock's (annualized) volatility.
- The BOPM to converge to the distribution only if

$$\begin{aligned} n\widehat{\mu} &= n(q\ln(u/d) + \ln d) \to \mu\tau, & (52) \\ n\widehat{\sigma}^2 &= nq(1-q)\ln^2(u/d) \to \sigma^2\tau. & (53) \end{aligned}$$

- Impose $ud = 1$ to make nodes at the same horizontal level of the tree have identical price (review p. 240).

Toward the Black-Scholes Formula (continued)

- The above requirements can be satisfied by

$$u = e^{\sigma\sqrt{\tau/n}}, \quad d = e^{-\sigma\sqrt{\tau/n}}, \quad q = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{\tau}{n}}. \qquad (54)$$

  - With Eqs. (54),

$$\begin{aligned} n\widehat{\mu} &= \mu\tau \\ n\widehat{\sigma}^2 &= \left(1 - \left(\frac{\mu}{\sigma}\right)^2 \frac{\tau}{n}\right)\sigma^2\tau \to \sigma^2\tau \end{aligned}$$

- Other choices are possible (see text).

Toward the Black-Scholes Formula (continued)

- The no-arbitrage inequalities $u > R > d$ may not hold under Eqs. (54).
- The risk-neutral probability may lie outside $[0, 1]$.
- The problems disappear when n satisfies

$$e^{\sigma\sqrt{\tau/n}} > e^{r\tau/n},$$

in other words, when $n > r^2\tau/\sigma^2$.

  - So they go away if n is large enough.
  - Other solutions will be presented later.

Toward the Black-Scholes Formula (continued)

- What is the limiting probabilistic distribution of the continuously compounded rate of return $\ln(S_\tau/S)$?
- The central limit theorem makes $\ln(S_\tau/S)$ converge to the normal distribution with mean $\mu\tau$ and variance $\sigma^2\tau$.
- So $\ln S_\tau$ approaches the normal distribution with mean $\mu\tau + \ln S$ and variance $\sigma^2\tau$.
- S_τ has a lognormal distribution in the limit.

Toward the Black-Scholes Formula (continued)

Lemma 11 *The continuously compounded rate of return* $\ln(S_\tau/S)$ *approaches the normal distribution with mean* $(r - \sigma^2/2)\,\tau$ *and variance* $\sigma^2\tau$ *in a risk-neutral economy.*

- Let q equal the risk-neutral probability $p \equiv (e^{r\tau/n} - d)/(u - d)$.
- Let $n \to \infty$.

Toward the Black-Scholes Formula (continued)

- Lemma 11 and Eq. (41) on p. 151 imply the expected stock price at expiration in a risk-neutral economy is $Se^{r\tau}$.
- The stock's expected annual rate of return[a] is thus the riskless rate r.

[a] In the sense of $(1/\tau)\ln E[\,S_\tau/S\,]$ not $(1/\tau)E[\,\ln(S_{\tau/S})\,]$.

Toward the Black-Scholes Formula (concluded)

Theorem 12 (The Black-Scholes Formula)

$$C = SN(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau}),$$
$$P = Xe^{-r\tau}N(-x + \sigma\sqrt{\tau}) - SN(-x),$$

where

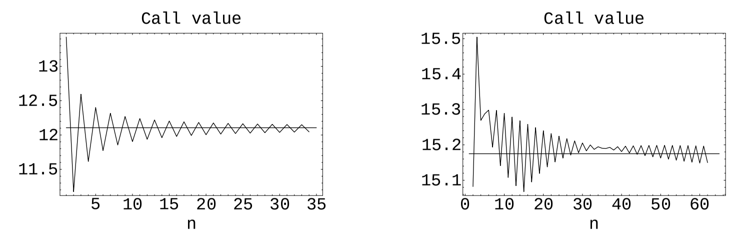
$$x \equiv \frac{\ln(S/X) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

BOPM and Black-Scholes Model

- The Black-Scholes formula needs five parameters: S, X, σ, τ, and r.
- Binomial tree algorithms take six inputs: S, X, u, d, $\hat{r}$, and n.
- The connections are

$$u = e^{\sigma\sqrt{\tau/n}},\ d = e^{-\sigma\sqrt{\tau/n}},\ \hat{r} = r\tau/n.$$

- The binomial tree algorithms converge reasonably fast.
- Oscillations can be eliminated by the judicious choices of u and d (see text).



Implied Volatility

- Volatility is the sole parameter not directly observable.
- The Black-Scholes formula can be used to compute the market's opinion of the volatility.
  - Solve for σ given the option price, S, X, τ, and r with numerical methods.
  - How about American options?
- This volatility is called the implied volatility.
- Implied volatility is often preferred to historical volatility in practice.

Problems; the Smile

- Options written on the same underlying asset usually do not produce the same implied volatility.
- A typical pattern is a "smile" in relation to the strike price.
  - The implied volatility is lowest for at-the-money options and becomes higher the further the option is in- or out-of-the-money.
- This pattern cannot be accounted for by the early exercise feature of American options.
  - Why is this even an issue?

Problems; the Smile (concluded)

- To address this issue, volatilities are often combined to produce a composite implied volatility.
- This practice is not sound theoretically.
- The existence of different implied volatilities for options on the same underlying asset shows the Black-Scholes model cannot be literally true.

Binomial Tree Algorithms for American Puts

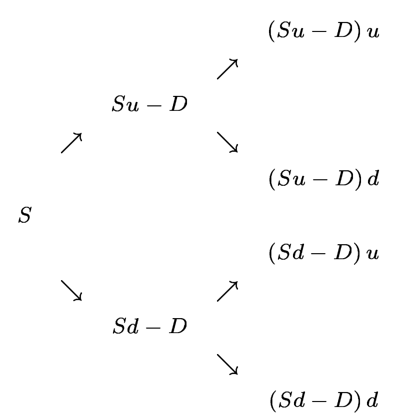
- Early exercise has to be considered.
- The binomial tree algorithm starts with the terminal payoffs $\max(0, X - Su^j d^{n-j})$ and applies backward induction.
- At each intermediate node, it checks for early exercise by comparing the payoff if exercised with continuation.

Options on a Stock That Pays Dividends

- Early exercise must be considered.
- Proportional dividend payout model is tractable (see text).
  - The dividend amount is a constant proportion of the prevailing stock price.
- In general, the corporate dividend policy is a complex issue.

Known Dividends

- Constant dividends introduce complications.
- Use D to denote the amount of the dividend.
- Suppose an ex-dividend date falls in the first period.
- At the end of that period, the possible stock prices are $Su - D$ and $Sd - D$.
- Follow the stock price one more period.
- The number of possible stock prices is not three but four: $(Su - D)\,u$, $(Su - D)\,d$, $(Sd - D)\,u$, $(Sd - D)\,d$.
  - The binomial tree no longer combines.



An Ad-Hoc Approximation

- Use the Black-Scholes formula with the stock price reduced by the PV of the dividends (Roll, 1977).
- Essentially decompose the stock price into a riskless one paying known dividends and a risky one.
- The riskless component at any time is the PV of future dividends during the life of the option.
  - σ equal to the volatility of the process followed by the risky component.
- The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution.

An Ad-Hoc Approximation (concluded)

- Start with the current stock price minus the PV of future dividends before expiration.
- Develop the binomial tree for the new stock price as if there were no dividends.
- Then add to each stock price on the tree the PV of all future dividends before expiration.
- American option prices can be computed as before on this tree of stock prices.

Continuous Dividend Yields

- Dividends are paid continuously.
  - Approximates a broad-based stock market portfolio.
- The payment of a continuous dividend yield at rate q reduces the growth rate of the stock price by q.
  - A stock that grows from S to S_τ with a continuous dividend yield of q would grow from S to $S_\tau e^{q\tau}$ without the dividends.
- A European option has the same value as one on a stock with price $Se^{-q\tau}$ that pays no dividends.

Continuous Dividend Yields (continued)

- The Black-Scholes formulae hold with S replaced by $Se^{-q\tau}$ (Merton, 1973):

$$C = Se^{-q\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \qquad (55)$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau} N(-x), \qquad (55')$$

where

$$x \equiv \frac{\ln(S/X) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

- Formulas (55) and (55') remain valid as long as the dividend yield is predictable.
- Replace q with the average annualized dividend yield.

Continuous Dividend Yields (concluded)

- To run binomial tree algorithms, pick the risk-neutral probability as

$$\frac{e^{(r-q)\,\Delta t} - d}{u - d}, \qquad (56)$$

where $\Delta t \equiv \tau/n$.
  - Because the stock price grows at an expected rate of $r - q$ in a risk-neutral economy.
- The u and d remain unchanged.
- Other than the change in Eq. (56), binomial tree algorithms stay the same.