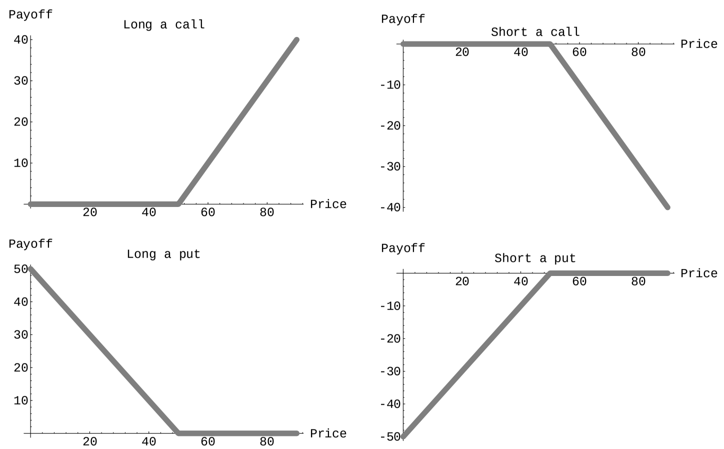
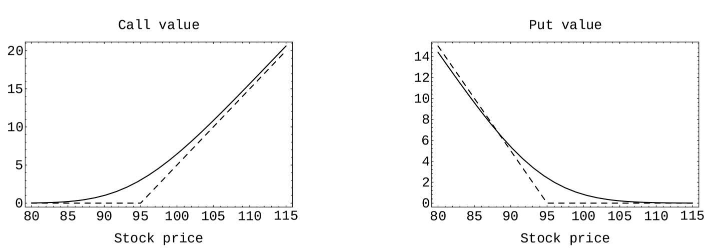




Cash Dividends

- Exchange-traded stock options are not cash dividend-protected (or simply protected).
  - The option contract is not adjusted for cash dividends.
- The stock price falls by an amount roughly equal to the amount of the cash dividend as it goes ex-dividend.
- Cash dividends are detrimental for calls.
- The opposite is true for puts.

Stock Splits and Stock Dividends

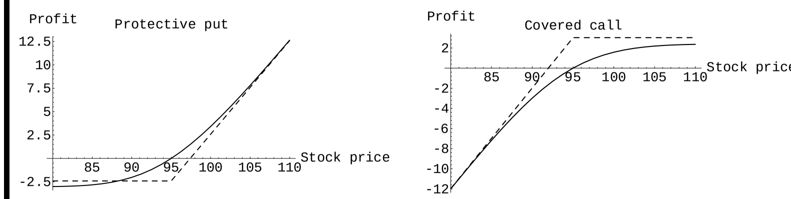
- Options are adjusted for stock splits.
- After an n-for-m stock split, the strike price is only m/n times its previous value, and the number of shares covered by one contract becomes n/m times its previous value.
- Exchange-traded stock options are adjusted for stock dividends.
- Options are assumed to be unprotected.

Example

- Consider an option to buy 100 shares of a company for $50 per share.
- A 2-for-1 split changes the term to a strike price of $25 per share for 200 shares.

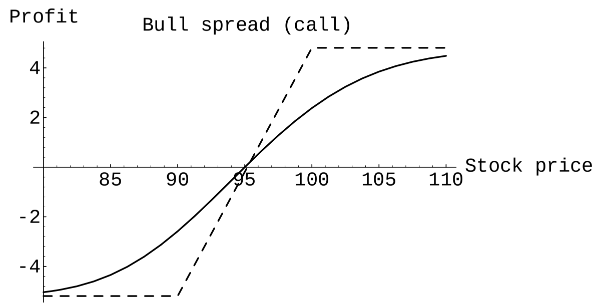
Covered Position: Hedge

- A hedge combines an option with its underlying stock in such a way that one protects the other against loss.
- Protective put: A long position in stock with a long put.
- Covered call: A long position in stock with a short call.
- Both strategies break even only if the stock price rises, so they are bullish.
- Writing a cash-secured put means writing a put while putting in enough money to cover the strike price if the put is exercised.



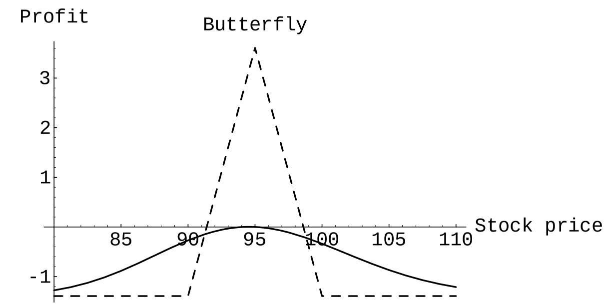
Covered Position: Spread

- A spread consists of options of the same type and on the same underlying asset but with different strike prices or expiration dates.
- We use X_L, X_M, and X_H to denote the strike prices with $X_L < X_M < X_H$.
- A bull call spread consists of a long X_L call and a short X_H call with the same expiration date.
  - The initial investment is $C_L - C_H$.
  - The maximum profit is $(X_H - X_L) - (C_L - C_H)$.
  - The maximum loss is $C_H - C_L$.



Covered Position: Spread (continued)

- Writing an X_H put and buying an X_L put with identical expiration date creates the bull put spread.
- A bear spread amounts to selling a bull spread.
- It profits from declining stock prices.
- Three calls or three puts with different strike prices and the same expiration date create a butterfly spread.
  - The spread is long one X_L call, long one X_H call, and short two X_M calls.

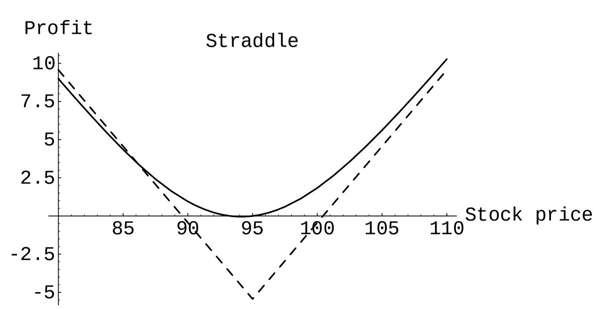


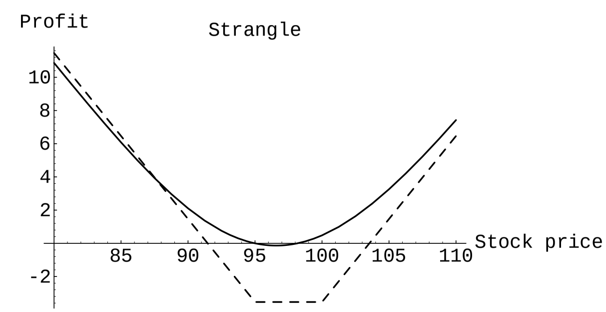
Covered Position: Spread (concluded)

- A butterfly spread pays off a positive amount at expiration only if the asset price falls between X_L and X_H.
- A butterfly spread with a small $X_H - X_L$ approximates a state contingent claim, which pays $1 only when a particular price results.
- The price of a state contingent claim is called a state price.

Covered Position: Combination

- A combination consists of options of different types on the same underlying asset, and they are either all bought or all written.
- Straddle: A long call and a long put with the same strike price and expiration date.
- Since it profits from high volatility, a person who buys a straddle is said to be long volatility.
- Selling a straddle benefits from low volatility.
- Strangle: Identical to a straddle except that the call's strike price is higher than the put's.





Arbitrage in Option Pricing

Arbitrage

- The no-arbitrage principle says there should be no free lunch.
- It supplies the argument for option pricing.
- A riskless arbitrage opportunity is one that, without any initial investment, generates nonnegative returns under all circumstances and positive returns under some.
- In an efficient market, such opportunities do not exist.
- The portfolio dominance principle says portfolio A should be more valuable than B if A's payoff is at least as good under all circumstances and better under some.

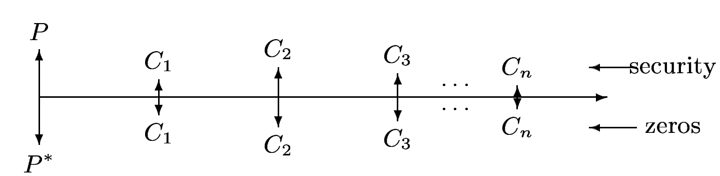
A Corollary

- A portfolio yielding a zero return in every possible scenario must have a zero PV.
  - Short the portfolio if its PV is positive.
  - Buy it if its PV is negative.
  - In both cases, a free lunch is created.

The PV Formula Justified

$P = \sum_{i=1}^{n} C_i d(i)$ for a certain cash flow $C_1, C_2, \ldots, C_n$.

- If the price $P^* < P$, short the zeros that match the security's n cash flows and use P^* of the proceeds P to buy the security.
- Since the cash inflows of the security will offset exactly the obligations of the zeros, a riskless profit of $P - P^*$ dollars has been realized now.
- If the price $P^* > P$, a riskless profit can be realized by reversing the trades.



Two More Examples

- An American option cannot be worth less than the intrinsic value.
  - Otherwise, one can buy the option, promptly exercise it and sell the stock with a profit.
- A put or a call must have a nonnegative value.
  - Otherwise, one can buy it for a positive cash flow now and end up with a nonnegative amount at expiration.

Relative Option Prices

- These relations hold regardless of the probabilistic model for stock prices.
- Assume, among other things, that there are no transactions costs or margin requirements, borrowing and lending are available at the riskless interest rate, interest rates are nonnegative, and there are no arbitrage opportunities.
- Let the current time be time zero.
- $\mathrm{PV}(x)$ stands for the PV of x dollars at expiration; hence $\mathrm{PV}(x) = xd(\tau)$ where τ is the time to expiration.

Maturity and Option Value

Lemma 1 *An American call (put) with a longer time to expiration cannot be worth less than an otherwise identical call (put) with a shorter time to expiration.*[a]

- Suppose instead that $C_{t_1} > C_{t_2}$ where $t_1 < t_2$.
- Buy C_{t_2} and sell C_{t_1} to generate a net cash flow of $C_{t_1} - C_{t_2}$ at time zero.
- When the time to t_2 is τ and the short call either expires or is exercised, the position is worth $C_\tau - \max(S_\tau - X, 0)$.

[a] May not hold for European options.

The Proof (concluded)

- If this value is positive, close out the position with a profit by selling the remaining call.
- If $\max(S_\tau - X, 0) > C_\tau \geq 0$, the short call is exercised.
- In this case, exercise the remaining call and have a net cash flow of zero.
- In both cases, the total payoff is positive without any initial investment.

Strike Price and Option Value

Lemma 2 *A call (put) option with a higher (lower) strike price cannot be worth more than an otherwise identical call (put) with a lower (higher) strike price.*

- This proposition certainly holds at expiration.
- Let the two strike prices be $X_1 < X_2$.
- Suppose $C_{X_1} < C_{X_2}$ instead.
- Buy the low-priced C_{X_1} and write the high-priced C_{X_2}, generating a positive return.
- If the holder of C_{X_2} exercises it before expiration, exercise C_{X_1} to generate a cash flow of $X_2 - X_1 > 0$.

Difference in Strike Price and Option Value

Lemma 3 *The difference in the values of two otherwise identical options cannot be greater than the difference in their strike prices.*

- Let the two strike prices be $X_1 < X_2$.
- Assume $C_{X_1} - C_{X_2} > X_2 - X_1$ instead.
- Buy the lower-priced C_{X_2}, write the higher-priced C_{X_1}, generating a positive return, and deposit $X_2 - X_1$ in a riskless bank account.

The Proof (continued)

- Suppose the holder of C_{X_1} exercises the option before expiration.
  - If $C_{X_2} > S - X_1$, then sell C_{X_2} to realize a cash flow of $C_{X_2} - (S - X_1) > 0$.
  - Otherwise, exercise C_{X_2} and realize a cash flow of $X_1 - X_2 < 0$.
  - In both scenarios, close out the position with the money in the bank to realize a nonnegative net cash flow.

The Proof (concluded)

- Suppose the holder of C_{X_1} does not exercise the option early.
  - At expiration, our cash flow is 0, $X_1 - S < 0$, and $X_1 - X_2 < 0$, respectively, if $S \leq X_1$, $X_1 < S < X_2$, and $X_2 \leq S$.
  - The net cash flow remains nonnegative after adding the money in the bank account, which is at least $X_2 - X_1$.

Upper Bounds

Lemma 4 *A call is never worth more than the stock price, an American put is never worth more than the strike price, and a European put is never worth more than the PV of the strike price.*

- If the call value exceeded the stock price, a covered call position earns arbitrage profits (long stock, short call).
- If the put value exceeded the strike price, writing a cash-secured put earns arbitrage profits.
- The tighter bound holds for European puts because the cash can earn riskless interest until expiration.

Put-Call Parity (Castelli, 1877)

$$C = P + S - \mathrm{PV}(X). \qquad (42)$$

- Consider the portfolio of one short European call, one long European put, one share of stock, and a loan of $\mathrm{PV}(X)$.
- All options are assumed to carry the same strike price and time to expiration, τ.
- The initial cash flow is therefore $C - P - S + \mathrm{PV}(X)$.
- At expiration, if the stock price $S_\tau \leq X$, the put will be worth $X - S_\tau$ and the call will expire worthless.

The Proof (concluded)

- On the other hand, if $S_\tau > X$, the call will be worth $S_\tau - X$ and the put will expire worthless.
- After the loan, now X, is repaid, the net future cash flow is zero in either case.
- The no-arbitrage principle implies that the initial investment to set up the portfolio must be nil as well.

Consequences of Put-Call Parity

- There is only one kind of European option because the other can be replicated from it in combination with the underlying stock and riskless lending or borrowing.
  - Combinations such as this create synthetic securities.
- $S = C - P + \mathrm{PV}(X)$ says a stock is equivalent to a portfolio containing a long call, a short put, and lending $\mathrm{PV}(X)$.
- $C - P = S - \mathrm{PV}(X)$ implies a long call and a short put amount to a long position in stock and borrowing the PV of the strike price (buying stock on margin).

Intrinsic Value

Lemma 5 *An American call or a European call on a non-dividend-paying stock is never worth less than its intrinsic value.*

- The put-call parity implies $C = (S - X) + (X - \text{PV}(X)) + P \geq S - X$.
- Since $C \geq 0$, it follows that $C \geq \max(S - X, 0)$, the intrinsic value.
- An American call also cannot be worth less than its intrinsic value.

Intrinsic Value (concluded)

A European put on a non-dividend-paying stock may be worth less than its intrinsic value, but:

Lemma 6 *For European puts,* $P \geq \max(\text{PV}(X) - S, 0)$.

- Use the put-call parity.

Early Exercise of American Calls

European calls and American calls are identical when the underlying stock pays no dividends.

Theorem 7 (Merton, 1973) *An American call on a non-dividend-paying stock should not be exercised before expiration.*

- By an exercise in text, $C \geq \max(S - \text{PV}(X), 0)$.
- If the call is exercised, the value is the smaller $S - X$.

Comments

- The above theorem does not mean American calls should be kept until maturity.
  - A call that is deep in the money might turn out to be out of the money at expiration, hence worthless.
  - In this case, keeping the option might be less profitable than exercising it earlier.
- What it does imply is that when early exercise is being considered, a *better* alternative is to sell it.
- Early exercise may become optimal for American calls on a dividend-paying stock.
  - Stock price declines as the stock goes ex-dividend.

Early Exercise of American Calls: Dividend Case

Surprisingly, an American call should be exercised only at a few dates.

Theorem 8 *An American call will only be exercised at expiration or just before an ex-dividend date.*

- It suffices to show $C > S - X$ at any time other than the expiration date or just before an ex-dividend date.
- Assume otherwise: $C \leq S - X$.
- Now, buy the call, short the stock, and lend $Xd(\tau)$, where τ is time to the next dividend date.
- The initial cash flow is positive because $X > Xd(\tau)$.

The Proof (concluded)

- Close out the position just before the next ex-dividend date by calling the loan, worth X, and selling the call, worth at least $\max(S_\tau - X, 0)$ by Lemma 5 (p. 191).
- The proceeds are sufficient to buy the stock at S_τ.
- The initial cash flow thus represents an arbitrage profit.
- Between ex-dividend dates, selling is better than exercising.

Early Exercise of American Puts

- It might be optimal to exercise an American put even if the underlying stock does not pay dividends.
- Part of the reason: Exercising a put generates an immediate cash income X.
- Early exercise becomes more profitable as the interest rate increases, other things being equal.

Convexity of Option Prices

Lemma 9 *For three otherwise identical calls with strike prices* $X_1 < X_2 < X_3$,

$$\begin{aligned} C_{X_2} &\leq \omega C_{X_1} + (1-\omega)\, C_{X_3} \\ P_{X_2} &\leq \omega P_{X_1} + (1-\omega)\, P_{X_3} \end{aligned}$$

Here $\omega \equiv (X_3 - X_2)/(X_3 - X_1)$. (*Equivalently,* $X_2 = \omega X_1 + (1-\omega)\, X_3$.)

- Suppose the lemma were wrong.
- Write C_{X_2}, buy ωC_{X_1}, and buy $(1-\omega)\, C_{X_3}$ to generate a positive cash flow now.

The Proof (continued)

- If the short call is not exercised before expiration, hold the calls until expiration.
- The cash flow is described below.
- Since the net cash flows are either nonnegative or positive, there is an arbitrage profit.

	$S \leq X_1$	$X_1 < S \leq X_2$	$X_2 < S < X_3$	$X_3 \leq S$
$-C_{X_2}$	0	0	$X_2 - S$	$X_2 - S$
C_{X_1}	0	$\omega(S - X_1)$	$\omega(S - X_1)$	$\omega(S - X_1)$
C_{X_3}	0	0	0	$(1-\omega)(S - X_3)$
Net	0	$\omega(S - X_1)$	$\omega(S - X_1) + (X_2 - S)$	0

The Proof (concluded)

- Suppose the short call is exercised early when the stock price is S.
- If $\omega C_{X_1} + (1-\omega)\, C_{X_3} > S - X_2$, sell the long calls to generate a net cash flow of $\omega C_{X_1} + (1-\omega)\, C_{X_3} - (S - X_2) > 0$.
- Otherwise, exercise the long calls and deliver the stock.
- The net cash flow is $-\omega X_1 - (1-\omega)\, X_3 + X_2 = 0$.
- Again, there is an arbitrage profit.

Shape of Option Values against Strike Prices

- Lemma 3 says the slope of the call value, when plotted against the strike price, is at most one.
- Lemma 3 says the slope of the put value, when plotted against the strike price, is at least minus one.
- Lemma 9 adds that the shape is convex.

Option on Portfolio vs. Portfolio of Options

An option on a portfolio of stocks is cheaper than a portfolio of options.

Theorem 10 *Consider a portfolio of non-dividend-paying assets with weights* ω_i. *Let* C_i *denote the price of a European call on asset* i *with strike price* X_i. *Then the call on the portfolio with a strike price* $X \equiv \sum_i \omega_i X_i$ *has a value at most* $\sum_i \omega_i C_i$. *All options expire on the same date.*

The same result holds for European puts.

Option Pricing Models

The Setting

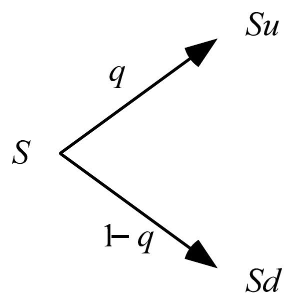
- The no-arbitrage principle is insufficient to pin down the exact option value without further assumptions on the probabilistic behavior of stock prices.
- One major obstacle is that it seems a risk-adjusted interest rate is needed to discount the option's payoff.
- Breakthrough came in 1973 when Black (1938–1995) and Scholes with help from Merton published their celebrated option pricing model.
- Known as the Black-Scholes option pricing model.

Terms and Approach

- C: call value.
- P: put value.
- X: strike price
- S: stock price
- $\hat{r} > 0$: the continuously compounded riskless rate per period.
- $R \equiv e^{\hat{r}}$: gross return.
- Start from the discrete-time binomial model.

Binomial Option Pricing Model (BOPM)

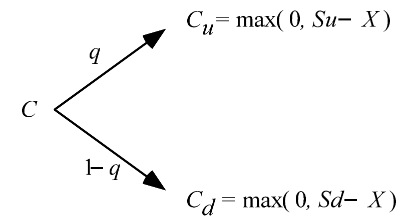
- Time is discrete and measured in periods.
- If the current stock price is S, it can go to Su with probability q and Sd with probability $1 - q$, where $0 < q < 1$ and $d < u$.
  - In fact, $d < R < u$ must hold to rule out arbitrage.
- Six pieces of information suffice to determine the option value based on arbitrage considerations: S, u, d, X, $\hat{r}$, and the number of periods to expiration.



Call on a Non-Dividend-Paying Stock: Single Period

- The expiration date is only one period from now.
- C_u is the price at time one if the stock price moves to Su.
- C_d is the price at time one if the stock price moves to Sd.
- Clearly,

$$\begin{aligned} C_u &= \max(0, Su - X), \\ C_d &= \max(0, Sd - X). \end{aligned}$$



Call Pricing in One Period

- Set up a portfolio of h shares of stock and B dollars in riskless bonds.
  - This costs $hS + B$.
  - We call h the hedge ratio or delta.
- The value of this portfolio at time one is either $hSu + RB$ or $hSd + RB$.
- Choose h and B such that the portfolio replicates the payoff of the call,

$$\begin{aligned} hSu + RB &= C_u, \\ hSd + RB &= C_d. \end{aligned}$$

Call Pricing in One Period (concluded)

- Solve the above equations to obtain

$$h = \frac{C_u - C_d}{Su - Sd} \geq 0, \quad (43)$$

$$B = \frac{uC_d - dC_u}{(u-d)\,R}. \quad (44)$$

- By the no-arbitrage principle, the European call should cost the same as the equivalent portfolio, $C = hS + B$.
- As $uC_d - dC_u < 0$, the equivalent portfolio is a levered long position in stocks.

American Call Pricing in One Period

- Have to consider immediate exercise.
- $C = \max(hS + B, S - X)$.
  - When $hS + B \geq S - X$, the call should not be exercised immediately.
  - When $hS + B < S - X$, the option should be exercised immediately.
- For non-dividend-paying stocks, early exercise is not optimal by Theorem 7 (p. 193), so $C = hS + B$.

Put Pricing in One Period

- Puts can be similarly priced.
- The delta for the put is $(P_u - P_d)/(Su - Sd) \leq 0$, where

$$P_u = \max(0, X - Su),$$

$$P_d = \max(0, X - Sd).$$

- Let $B = \frac{uP_d - dP_u}{(u-d)\,R}$.
- The European put is worth $hS + B$.
- The American put is worth $\max(hS + B, X - S)$.

Risk

- Surprisingly, the option value is independent of q.
- Hence it is independent of the expected gross return of the stock, $qSu + (1-q)\,Sd$.
- It therefore does not directly depend on investors' risk preferences.
- The option value does depend on the sizes of price changes, u and d, the magnitudes of which the investors must agree upon.

Pseudo Probability

- After substitution and rearrangement,

$$hS + B = \frac{\left(\frac{R-d}{u-d}\right) C_u + \left(\frac{u-R}{u-d}\right) C_d}{R}. \tag{45}$$

- Rewrite Eq. (45) as

$$hS + B = \frac{pC_u + (1-p)\, C_d}{R}, \tag{46}$$

where

$$p \equiv \frac{R-d}{u-d}. \tag{47}$$

- As $0 < p < 1$, it may be interpreted as a probability.

Risk-Neutral Probability

- The expected rate of return for the stock is equal to the riskless rate $\hat{r}$ under $q = p$ as $pSu + (1-p)\, Sd = RS$.
- Risk-neutral investors care only about expected returns.
- The expected rates of return of all securities must be the riskless rate when investors are risk-neutral.
- For this reason, p is called the risk-neutral probability.
- The value of an option is the expectation of its discounted future payoff in a risk-neutral economy.
- So the rate used for discounting the future value is the riskless rate in a risk-neutral economy.

Binomial Distribution

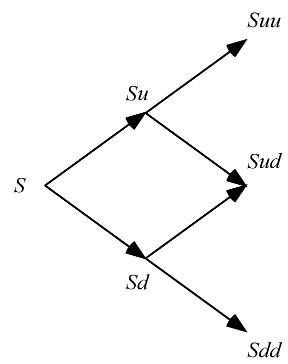
- Denote the binomial distribution with parameters n and p by

$$b(j; n, p) \equiv \binom{n}{j} p^j (1-p)^{n-j} = \frac{n!}{j!\,(n-j)!}\, p^j (1-p)^{n-j}.$$

- Recall that $n! = n \times (n-1) \cdots 2 \times 1$ with the convention $0! = 1$.
- $b(j; n, p)$ is the probability of getting j heads when tossing a coin n times with p being the probability of getting heads.

Option on a Non-Dividend-Paying Stock: Multi-Period

- Consider a call with two periods remaining before expiration.
- Under the binomial model, the stock can take on three possible prices at time two: Suu, Sud, and Sdd.
  - Note that the tree recombines.
- At any node, the next two stock prices only depend on the current price, not the prices of earlier times.
- This memoryless property is a key feature of an efficient market.



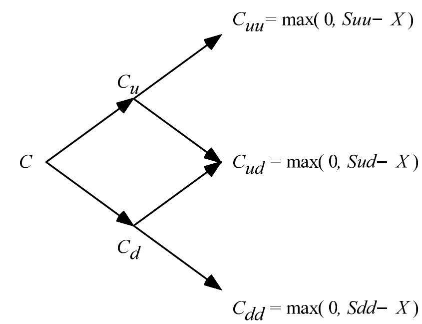
Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- Let C_{uu} be the call's value at time two if the stock price is Suu.
- Thus,

$$C_{uu} = \max(0, Suu - X).$$

- C_{ud} and C_{dd} can be calculated analogously,

$$\begin{aligned} C_{ud} &= \max(0, Sud - X), \\ C_{dd} &= \max(0, Sdd - X). \end{aligned}$$



Option on a Non-Dividend-Paying Stock: Multi-Period (continued)

- The call values at time one can be obtained by applying the same logic:

$$\begin{aligned} C_u &= \frac{pC_{uu} + (1-p)\, C_{ud}}{R}, \qquad (48) \\ C_d &= \frac{pC_{ud} + (1-p)\, C_{dd}}{R}. \end{aligned}$$

- Deltas can be derived from Eq. (43).
- For example, the delta at C_u is

$$(C_{uu} - C_{ud})/(Suu - Sud).$$

Option on a Non-Dividend-Paying Stock: Multi-Period (concluded)

- We now reach the current period.
- An equivalent portfolio of h shares of stock and $\$B$ riskless bonds can be set up for the call that costs C_u (C_d, respectively) if the stock price goes to Su (Sd, respectively).
- The values of h and B can be derived from Eq. (43)–(44).

Early Exercise

- Since the call will not be exercised at time one even if it is American, $C_u \geq Su - X$ and $C_d \geq Sd - X$.
- Therefore,

$$\begin{aligned} hS + B &= \frac{pC_u + (1-p)\,C_d}{R} \geq \frac{[\,pu + (1-p)\,d\,]\,S - X}{R} \\ &= S - \frac{X}{R} > S - X. \end{aligned}$$

- So the call again will not be exercised at present, and

$$C = hS + B = \frac{pC_u + (1-p)\,C_d}{R}.$$

Backward Induction (Zermelo)

- The above expression calculates C from the two successor nodes C_u and C_d and none beyond.
- The same computation happens at C_u and C_d, too, as demonstrated in Eq. (48).
- This recursive procedure is called backward induction.
- Now, C equals

$$\begin{aligned} & [p^2 C_{uu} + 2p(1-p)\,C_{ud} + (1-p)^2 C_{dd}](1/R^2) \\ = & \;[\,p^2 \cdot \max\left(0, Su^2 - X\right) + 2p(1-p) \cdot \max\left(0, Sud - X\right) \\ & +(1-p)^2 \cdot \max\left(0, Sd^2 - X\right)\,](1/R^2). \end{aligned}$$