Extracting Spot Rates from Yield Curve

- Start with the short rate $S(1)$.
  - Available since short-term Treasuries are zero-coupon bonds.
- Compute $S(2)$ from the two-period coupon bond price P by solving

$$P = \frac{C}{1+S(1)} + \frac{C+100}{(1+S(2))^2}.$$

Extracting Spot Rates from Yield Curve (concluded)

- In general, $S(n)$ can be computed from Eq. (24), repeated below,

$$P = \sum_{i=1}^{n} \frac{C}{(1+S(i))^i} + \frac{F}{(1+S(n))^n},$$

given the market price P of the n-period coupon bond and $S(1), S(2), \ldots, S(n-1)$.

- The running time is $O(n)$.
- The procedure is called bootstrapping.

Some Problems

- Treasuries of the same maturity might be selling at different yields (the multiple cash flow problem).
- Some maturities might be missing from the data points (the incompleteness problem).
- Treasuries might not be of the same quality.
- Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve.
  - Lack economic justifications.

Yield Spread

- Consider a *risky* bond with the cash flow $C_1, C_2, \ldots, C_n$ and selling for P.
- Were this bond riskless, it would fetch

$$P^* = \sum_{t=1}^{n} \frac{C_t}{(1+S(t))^t}.$$

- Since riskiness must be compensated, $P < P^*$.
- Yield spread is the difference between the IRR of the risky bond and that of a riskless bond with comparable maturity.

Static Spread

- The static spread is the amount s by which the spot rate curve has to shift in parallel in order to price the risky bond correctly,

$$P = \sum_{t=1}^{n} \frac{C_t}{(1 + s + S(t))^t}.$$

- Unlike the yield spread, the static spread incorporates information from the term structure.
- Can be computed by the Newton-Raphson method.

Of Spot Rate Curve and Yield Curve

- y_k: yield to maturity for the k-period coupon bond.
- $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (yield curve is normal).
- $S(k) \leq y_k$ if $y_1 > y_2 > \cdots$ (yield curve is inverted).
- $S(k) \geq y_k$ if $S(1) < S(2) < \cdots$ (spot rate curve is normal).
- $S(k) \leq y_k$ if $S(1) > S(2) > \cdots$ (spot rate curve is inverted).
- If the yield curve is flat, the spot rate curve coincides with the yield curve.

Coupon Effect on the Yield to Maturity

- Under a normal spot rate curve, a coupon bond has a lower yield than a zero-coupon bond of equal maturity.
- Picking a zero-coupon bond over a coupon bond based purely on the zero's higher yield to maturity is flawed.

Shapes

- The spot rate curve often has the same shape as the yield curve.
  - If the spot rate curve is inverted (normal, resp.), then the yield curve is inverted (normal, resp.).
- But only a trend not a mathematical truth.
- Consider a 3-period coupon bond that pays $1 per period and repays the principal of $100 at maturity.
- Assume spot rates $S(1) = 0.1$, $S(2) = 0.9$, and $S(3) = 0.901$.

Shapes (concluded)

- Yields to maturity are $y_1 = 0.1$, $y_2 = 0.8873$, and $y_3 = 0.8851$, not strictly increasing!
- When the final principal payment is relatively insignificant, the spot rate curve and the yield curve do share the same shape.
  – Bonds of high coupon rates and long maturities.
- By the agreement in shape, remember the above proviso.

Forward Rates

- The yield curve contains information regarding future interest rates currently "expected" by the market.
- Invest \$1 for j periods to end up with $[\,1 + S(j)\,]^j$ dollars at time j.
  – The maturity strategy.
- Invest \$1 in bonds for i periods and at time i invest the proceeds in bonds for another $j - i$ periods where $j > i$.
- Will have $[\,1 + S(i)\,]^i [\,1 + S(i,j)\,]^{j-i}$ dollars at time j.
  – $S(i,j)$: $(j - i)$-period spot rate i periods from now.
  – The rollover strategy.

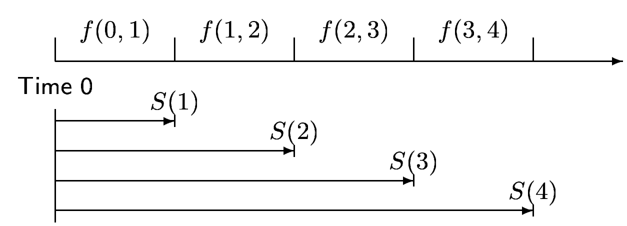
Forward Rates (concluded)

- When $S(i,j)$ equals

$$f(i,j) \equiv \left[\frac{(1 + S(j))^j}{(1 + S(i))^i} \right]^{1/(j-i)} - 1, \qquad (25)$$

  we will end up with $[\,1 + S(j)\,]^j$ dollars again.
- By definition, $f(0,j) = S(j)$.
- $f(i,j)$ is called the (implied) forward rates.
  – More precisely, the $(j - i)$-period forward rate i periods from now.

Time Line



Forward Rates and Future Spot Rates

- We did not assume any a priori relation between $f(i,j)$ and future spot rate $S(i,j)$.
  - This is the subject of the term structure theories.
- We merely looked for the future spot rate that, *if realized*, will equate two investment strategies.
- $f(i,i+1)$ are instantaneous forward rates or one-period forward rates.

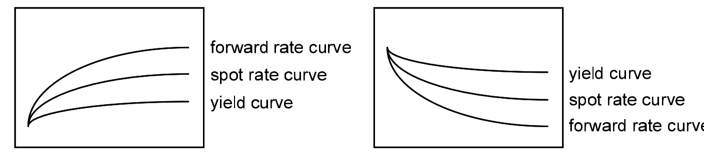
Spot Rates and Forward Rates

- When the spot rate curve is normal, the forward rate dominates the spot rates,

$$f(i,j) > S(j) > \cdots > S(i). \tag{26}$$

- When the spot rate curve is inverted, the forward rate is dominated by the spot rates,

$$f(i,j) < S(j) < \cdots < S(i). \tag{27}$$



(a) (b)

Forward Rates=Spot Rates=Yield Curve

- The future value of \$1 at time n can be derived in two ways.
- Buy n-period zero-coupon bonds and receive $[1+S(n)]^n$.
- Buy one-period zero-coupon bonds today and a series of such bonds at the forward rates as they mature.
- The future value is $[1+S(1)][1+f(1,2)]\cdots[1+f(n-1,n)]$.

Forward Rates=Spot Rates=Yield Curve (concluded)

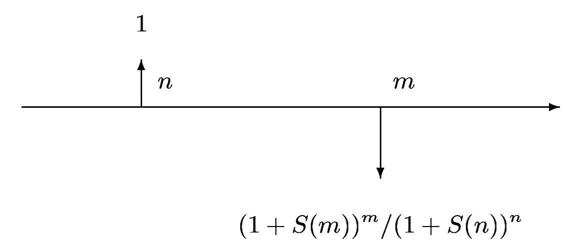
- Since they are identical,

$$S(n) = ((1+S(1))(1+f(1,2)) \cdots (1+f(n-1,n)))^{1/n} - 1. \quad (28)$$

- Hence, the forward rates, specifically the one-period forward rates, determine the spot rate curve.
- Other equivalency can be derived similarly.
- Show that $f(T, T+1) = d(T)/d(T+1) - 1$.

Locking in the Forward Rate $f(n,m)$

- Buy one n-period zero-coupon bond for $1/(1+S(n))^n$ and sell $(1+S(m))^m/(1+S(n))^n$ m-year zero-coupon bonds.
- No net initial investment because the cash inflow equals the cash outflow $1/(1+S(n))^n$.
- At time n there will be a cash inflow of $1.
- At time m there will be a cash outflow of $(1+S(m))^m/(1+S(n))^n$ dollars.
- This implies the rate $f(n,m)$ between times n and m.



Forward Contracts

- We generated the cash flow of a financial instrument called forward contract.
- Agreed upon today, it enables one to borrow money at time n in the future and repay the loan at time $m > n$ with an interest rate equal to the forward rate $f(n,m)$.
- Can the spot rate curve be an arbitrary curve?

Spot and Forward Rates under Continuous Compounding

- The pricing formula:

$$P = \sum_{i=1}^{n} Ce^{-iS(i)} + Fe^{-nS(n)}.$$

- The market discount function:

$$d(n) = e^{-nS(n)}. \tag{29}$$

- The spot rate is an arithmetic average of forward rates,

$$S(n) = \frac{f(0,1) + f(1,2) + \cdots + f(n-1,n)}{n}. \tag{30}$$

Spot and Forward Rates under Continuous Compounding (concluded)

- The formula for the forward rate:

$$f(i,j) = \frac{jS(j) - iS(i)}{j-i}. \tag{31}$$

- The one-period forward rate:

$$f(j,j+1) = -\ln \frac{d(j+1)}{d(j)}. \tag{32}$$

-

$$f(T) \equiv \lim_{\Delta T \to 0} f(T, T+\Delta T) = S(T) + T\,\frac{\partial S}{\partial T}. \tag{33}$$

- $f(T) > S(T)$ if and only if $\partial S/\partial T > 0$.

Unbiased Expectations Theory

- Forward rate equals the average future spot rate,

$$f(a,b) = E[\, S(a,b)\,]. \tag{34}$$

- Does not imply that the forward rate is an accurate predictor for the future spot rate.
- Implies that the maturity strategy and the rollover strategy produce the same result at the horizon on the average.

Unbiased Expectations Theory and Spot Rate Curve

- Implies that a normal spot rate curve is due to the fact that the market expects the future spot rate to rise.
  - $f(j,j+1) > S(j+1)$ if and only if $S(j+1) > S(j)$ from Eq. (25) on p. 114.
  - So $E[\, S(j,j+1)\,] > S(j+1) > \cdots > S(1)$ if and only if $S(j+1) > \cdots > S(1)$.
- Conversely, the spot rate is expected to fall if and only if the spot rate curve is inverted.

More Implications

- The theory has been rejected by most empirical studies with the possible exception of the period prior to 1915.
- Since the term structure has been upward sloping about 80% of the time, the theory would imply that investors have expected interest rates to rise 80% of the time.
- Riskless bonds, regardless of their different maturities, are expected to earn the same return on the average.
- That would mean investors are indifferent to risk.

A "Bad" Expectations Theory

- The expected returns on all possible riskless bond strategies are equal for *all* holding periods.
- So

$$(1+S(2))^2 = (1+S(1))\, E[\,1+S(1,2)\,] \qquad (35)$$

because of the equivalency between buying a two-period bond and rolling over one-period bonds.

- After rearrangement, $E[\,1+S(1,2)\,] = (1+S(2))^2/(1+S(1))$.

A "Bad" Expectations Theory (continued)

- Now consider two one-period strategies.
  - Strategy one buys a two-period bond and sells it after one period.
  - The expected return is $E[\,(1+S(1,2))^{-1}\,]\,(1+S(2))^2$.
  - Strategy two buys a one-period bond with a return of $1+S(1)$.
- The theory says the returns are equal:

$$\frac{(1+S(2))^2}{1+S(1)} = \frac{1}{E\,[\,(1+S(1,2))^{-1}\,]}.$$

A "Bad" Expectations Theory (concluded)

- Combining this with Eq. (35) to obtain

$$E\left[\frac{1}{1+S(1,2)}\right] = \frac{1}{E[\,1+S(1,2)\,]}.$$

- But this is impossible save for a certain economy.
  - Jensen's inequality states that $E[\,g(X)\,] > g(E[\,X\,])$ for any nondegenerate random variable X and strictly convex function g (i.e., $g''(x) > 0$).
  - Use $g(x) \equiv (1+x)^{-1}$ to prove our point.

Local Expectations Theory

- The expected rate of return of any bond over a single period equals the prevailing one-period spot rate:

$$\frac{E\left[\,(1+S(1,n))^{-(n-1)}\,\right]}{(1+S(n))^{-n}}=1+S(1) \;\text{ for } \textit{all } n>1. \tag{36}$$

- This theory is the basis of many interest rate models.
- Holding premium:

$$\frac{E\left[\,(1+S(1,n))^{-(n-1)}\,\right]}{(1+S(n))^{-n}}-(1+S(1)).$$

  - Zero under the local expectations theory.

Duration Revisited

- Let $P(y) \equiv \sum_i C_i/(1+S(i)+y)^i$ be the price associated with the cash flow $C_1, C_2, \ldots$.
- Define duration as

$$-\frac{\partial P(y)/P(0)}{\partial y}\bigg|_{y=0}=\frac{\sum_i \frac{iC_i}{(1+S(i))^{i+1}}}{\sum_i \frac{C_i}{(1+S(i))^i}}.$$

  - The curve is shifted in parallel to $S(1)+\Delta y, S(2)+\Delta y, \ldots$ before letting Δy go to zero.
- The percentage price change roughly equals duration times the size of the parallel shift in the spot rate curve.

Duration Revisited (continued)

- The simple linear relation between duration and MD in Eq. (17) on p. 76 breaks down.
- One way to regain it is to resort to a different kind of shift, the proportional shift:

$$\frac{\Delta(1+S(i))}{1+S(i)}=\frac{\Delta(1+S(1))}{1+S(1)}$$

  for all i.

  - $\Delta(x)$ denotes the change in x when the short-term rate is shifted by Δy.

Duration Revisited (concluded)

- Duration now becomes

$$\frac{1}{1+S(1)}\left[\frac{\sum_i \frac{iC_i}{(1+S(i))^i}}{\sum_i \frac{C_i}{(1+S(i))^i}}\right]. \tag{37}$$

- Define Macaulay's second duration to be the number within the brackets in Eq. (37).
- Then

$$\text{duration}=\frac{\text{Macaulay's second duration}}{(1+S(1))}.$$

Immunization Revisited

- Recall that a future liability can be immunized by matching PV and MD under flat spot rate curves.
- If only parallel shifts are allowed, this conclusion continues to hold under general spot rate curves.
- Assume liability L is T periods from now.
- Assume $L = 1$ for simplicity.
- Assume the matching portfolio consists only of zero-coupon bonds maturing at t_1 and t_2 with $t_1 < T < t_2$.

Immunization Revisited (continued)

- Let there be n_i bonds maturing at time t_i, $i = 1, 2$.
- The portfolio's PV is

$$V \equiv n_1 e^{-S(t_1)\,t_1} + n_2 e^{-S(t_2)\,t_2} = e^{-S(T)\,T}.$$

- Its MD is

$$\frac{n_1 t_1 e^{-S(t_1)\,t_1} + n_2 t_2 e^{-S(t_2)\,t_2}}{V} = T.$$

- These two equations imply

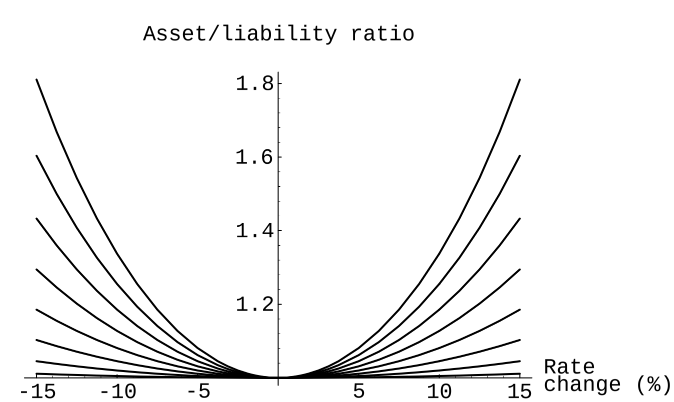
$$n_1 e^{-S(t_1)\,t_1} = \frac{V(t_2 - T)}{t_2 - t_1} \quad \text{and} \quad n_2 e^{-S(t_2)\,t_2} = \frac{V(t_1 - T)}{t_1 - t_2}.$$

Immunization Revisited (concluded)

- Now shift the spot rate curve uniformly by $\delta \neq 0$.
- The portfolio's PV becomes

$$\begin{aligned} & n_1 e^{-(S(t_1)+\delta)\,t_1} + n_2 e^{-(S(t_2)+\delta)\,t_2} \\ = \; & e^{-\delta t_1} \frac{V(t_2 - T)}{t_2 - t_1} + e^{-\delta t_2} \frac{V(t_1 - T)}{t_1 - t_2} \\ = \; & \frac{V}{t_2 - t_1} \left(e^{-\delta t_1}(t_2 - T) + e^{-\delta t_2}(T - t_1) \right). \end{aligned}$$

- The liability's PV after shift is $e^{-(S(T)+\delta)\,T} = e^{-\delta T} V$.
- And $\frac{V}{t_2 - t_1} \left(e^{-\delta t_1}(t_2 - T) + e^{-\delta t_2}(T - t_1) \right) > e^{-\delta T} V$.



Two Intriguing Implications

- A duration-matched position under parallel shifts implies free lunch as any interest rate change generates profits.
- No investors would hold the T-period bond because a portfolio of t_1- and t_2-period bonds has a higher return for any interest rate shock.
  - They would own only bonds of the shortest and longest maturities.
- The logic seems impeccable.
- What gives?