Bond Price Volatility

Purposes

- How interest rates affect bond prices.
- Key to the risk management of interest-rate-sensitive securities.
- Assume level-coupon throughout.

The Key Question: Price Volatility

- What is the sensitivity of the percentage price change to changes in interest rates?
- Define price volatility by

$$-(\partial P/P)/\partial y.$$

Price Volatility of Bonds

- The price volatility of a coupon bond is

$$-\frac{(C/y)\,n - \left(C/y^2\right)\left((1+y)^{n+1} - (1+y)\right) - nF}{(C/y)\left((1+y)^{n+1} - (1+y)\right) + F(1+y)}, \quad (14)$$

  where F is the par value, and C is the coupon payment per period.
- For bonds without embedded options,

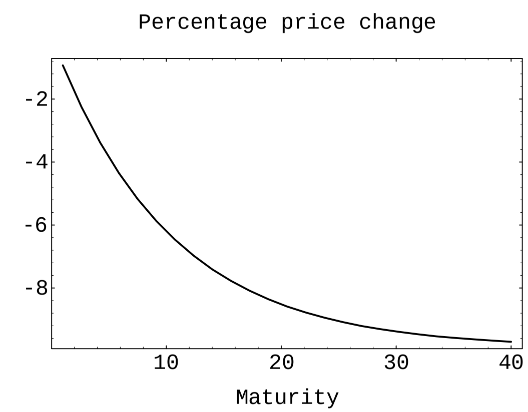
$$-(\partial P/P)/\partial y > 0.$$

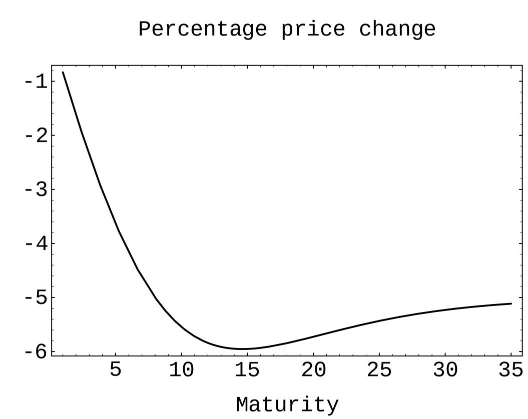
Behavior of Price Volatility (1)

- Price volatility increases as the coupon rate decreases.
  - Zero-coupon bonds are the most volatile.
  - Bonds selling at a deep discount are more volatile than those selling near or above par.
- Price volatility increases as the required yield decreases.
  - So bonds traded with higher yields are less volatile.

Behavior of Price Volatility (2)

- For bonds selling above par or at par, price volatility increases as the term to maturity lengthens (see figure on next page).
  - Bonds with a longer maturity are more volatile.
  - (But the *yields* of long-term bonds are less volatile than those of short-term bonds.)
- For bonds selling below par, price volatility first increases then decreases (see the figure on p. 70).
  - Longer maturity here cannot be equated with higher price volatility.





Macaulay Duration

- The Macaulay duration (MD) is a weighted average of the times to an asset's cash flows.
- The weights are the cash flows' present values divided by the asset's price.
- Formally,

$$\text{MD} \equiv \frac{1}{P} \sum_{i=1}^{n} \frac{iC_i}{(1+y)^i}.$$

- The Macaulay duration, in periods, is equal to

$$\text{MD} = -(1+y)\,\frac{\partial P/P}{\partial y}. \quad (15)$$

MD of Bonds

- The MD of a coupon bond is

$$\text{MD} = \frac{1}{P}\left[\sum_{i=1}^{n} \frac{iC}{(1+y)^i} + \frac{nF}{(1+y)^n}\right]. \quad (16)$$

- Can be simplified to

$$\text{MD} = \frac{c(1+y)\,[\,(1+y)^n - 1\,] + ny(y-c)}{cy\,[\,(1+y)^n - 1\,] + y^2},$$

where c is the period coupon rate.

- The MD of a zero-coupon bond is term to maturity n.
- The MD of a coupon bond is less than its maturity.

Finesse

- Equations (15) and (16) hold only if the coupon C, the par value F, and the maturity n are all independent of the yield y—if the cash flow is independent of yields.
- To see this point, suppose the market yield declines.
- The MD will be lengthened.
- For securities whose maturity actually decreases as a result, the MD may decrease.

How Not To Think of MD

- The MD has its origin in measuring the length of time a bond investment is outstanding.
- But you use it that way at your peril.
- The MD should be seen mainly as measuring price volatility.
- Many, if not most, duration-related terminology cannot be comprehended otherwise.

Conversion

- To convert the MD to be year-based, modify Eq. (16) to

$$\frac{1}{P}\left[\sum_{i=1}^{n} \frac{i}{k} \frac{C}{\left(1+\frac{y}{k}\right)^{i}} + \frac{n}{k} \frac{F}{\left(1+\frac{y}{k}\right)^{n}}\right],$$

  where y is the *annual* yield and k is the compounding frequency per annum.
- Equation (15) also becomes

$$\text{MD} = -\left(1+\frac{y}{k}\right) \frac{\partial P/P}{\partial y}.$$

- By definition,

$$\text{MD (in years)} = \frac{\text{MD (in periods)}}{k}.$$

Modified Duration

- Modified duration is defined as

$$\text{modified duration} \equiv -\frac{\partial P/P}{\partial y} = \frac{\text{MD}}{(1+y)}. \qquad (17)$$

- By Taylor expansion,

$$\text{percentage price change} \approx -\text{modified duration} \times \text{yield change}$$

- The modified duration of a portfolio equals $\sum_i \omega_i D_i$, where D_i is the modified duration of the ith asset and ω_i is the market value of that asset expressed as a percentage of the market value of the portfolio.

Example

- Consider a bond whose modified duration is 11.54 with a yield of 10%.
- If the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change will be

$$-11.54 \times 0.001 = -0.01154 = -1.154\%.$$

Effective Duration

- The effective duration is defined as

$$\frac{P_- - P_+}{P_0(y_+ - y_-)}. \qquad (18)$$

  – P_- is the price if the yield is decreased by Δy, P_+ is the price if the yield is increased by Δy, P_0 is the initial price, y is the initial yield, and Δy is small.
- A general numerical formula for volatility.
- One can compute the effective duration of just about any financial instrument.

Effective Duration (concluded)

- Most useful where yield changes alter the cash flow or securities whose cash flow is so complex that simple formulae are unavailable.
- Duration of a security can be longer than its maturity or negative!
- Neither makes sense under the maturity interpretation.
- An alternative is to use

$$\frac{P_0 - P_+}{P_0 \, \Delta y}. \tag{19}$$

  - More economical but less accurate.

The Practices

- Duration is usually expressed in percentage terms for quick mental calculation.
- Given duration $D_{\%}$, the percentage price change expressed in percentage terms is approximated by $-D_{\%} \times \Delta r$ when the yield increases instantaneously by $\Delta r\%$.
  - Price will drop by 20% if $D_{\%} = 10$ and $\Delta r = 2$ because $10 \times 2 = 20$.
- In fact, $D_{\%}$ equals modified duration as originally defined.

Meeting Liabilities

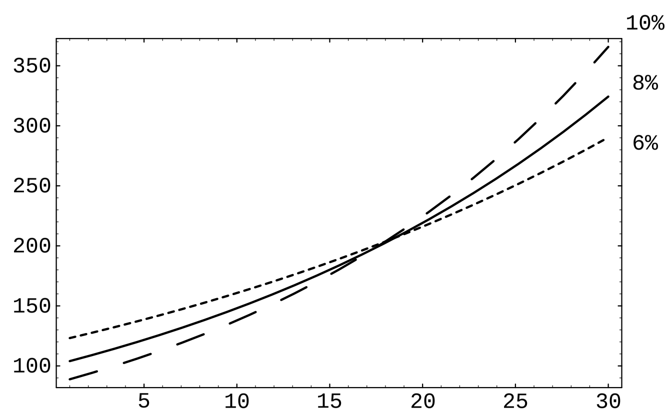
- Buy coupon bonds to meet a future liability.
- What happens at the horizon date when the liability is due?
- Say interest rates rise subsequent to the purchase:
  - The interest on interest from the reinvestment of the coupon payments will increase.
  - But a capital loss will occur for the sale of the bonds.
- The reverse is true if interest rates fall.
- Uncertainties in meeting the liability.

Immunization

- A portfolio immunizes a liability if its value at horizon covers the liability for small rate changes now.
- A bond portfolio whose MD equals the horizon and whose PV equals the PV of the single future liability.
  - At horizon, losses from the interest on interest will be compensated by gains in the sale price when interest rates fall.
  - Losses from the sale price will be compensated by the gains in the interest on interest when interest rates rise (see figure on p. 84).

Immunization (concluded)

- A \$100,000 liability 12 years from now should be matched by a portfolio with a MD of 12 years and a future value of \$100,000.



The Proof

- Assume the liability is L at time m and the current interest rate is y.
- Want a portfolio such that

  (1) its FV is L at the horizon m;

  (2) $\partial \text{FV}/\partial y = 0$;

  (3) FV is convex around y.
- Condition (1) says the obligation is met.
- Conditions (2) and (3) mean L is the portfolio's minimum FV at horizon for small rate changes.

The Proof (continued)

- Let $\text{FV} \equiv (1+y)^m P$, where P is the PV of the portfolio.
- Now,

$$\frac{\partial \text{FV}}{\partial y} = m(1+y)^{m-1} P + (1+y)^m \frac{\partial P}{\partial y}. \qquad (20)$$

- Imposing Condition (2) leads to

$$m = -(1+y)\, \frac{\partial P/P}{\partial y}. \qquad (21)$$

- The MD is equal to the horizon m.

The Proof (concluded)

- Employ a coupon bond for immunization.
- Since

$$\text{FV} = \sum_{i=1}^{n} \frac{C}{(1+y)^{i-m}} + \frac{F}{(1+y)^{n-m}},$$

it follows that

$$\frac{\partial^2 \text{FV}}{\partial y^2} > 0 \tag{22}$$

for $y > -1$.

- Since FV is convex for $y > -1$, the minimum value of FV is indeed L.

Rebalancing

- Immunization has to be rebalanced constantly to ensure that the MD remains matched to the horizon.
- The MD decreases as time passes.
- But, except for zero-coupon bonds, the decrement is not identical to that in the time to maturity.
  - Consider a coupon bond whose MD matches horizon.
  - Since the bond's maturity date lies beyond the horizon date, its MD will remain positive at horizon.
  - So immunization needs to be reestablished even if interest rates never change.

Hedging

- Hedging offsets the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged.
- Define dollar duration as

$$\text{modified duration} \times \text{price (\% of par)} = -\frac{\partial P}{\partial y}.$$

- The approximate dollar price change per \$100 of par value is

$$\text{price change} \approx -\text{dollar duration} \times \text{yield change}.$$

Convexity

- Convexity is defined as

$$\text{convexity (in periods)} \equiv \frac{\partial^2 P}{\partial y^2} \frac{1}{P}. \tag{23}$$

- The convexity of a coupon bond is positive.
- For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude.
- Hence, between two bonds with the same duration, the one with a higher convexity is more valuable.

Convexity (concluded)

- Convexity measured in periods and convexity measured in years are related by

$$\text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}$$

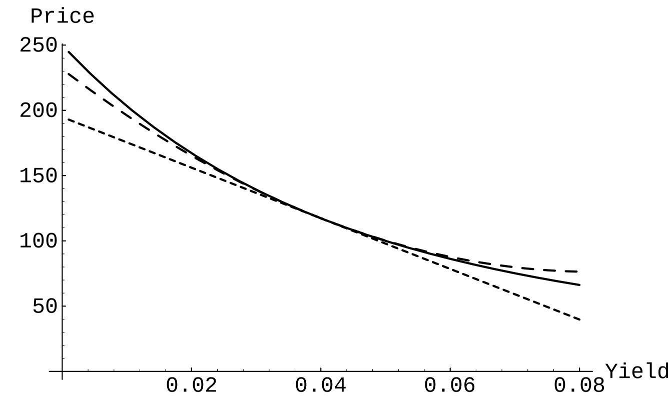
when there are k periods per annum.

- The convexity of a coupon bond increases as its coupon rate decreases.
- For a given yield and duration, the convexity decreases as the coupon decreases.

Use of Convexity

- The approximation $\Delta P/P \approx -$ duration $\times$ yield change works for small yield changes.
- To improve upon it for larger yield changes, use

$$\begin{aligned}\frac{\Delta P}{P} &\approx \frac{\partial P}{\partial y}\frac{1}{P}\Delta y + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}\frac{1}{P}(\Delta y)^2 \\ &= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.\end{aligned}$$



The Practices

- Convexity is usually expressed in percentage terms for quick mental calculation.
- Given convexity $C_{\%}$, the percentage price change expressed in percentage terms is approximated by $-D_{\%} \times \Delta r + C_{\%} \times (\Delta r)^2/2$ when the yield increases instantaneously by $\Delta r\%$.
  - Price will drop by 17% if $D_{\%} = 10$, $C_{\%} = 1.5$, and $\Delta r = 2$ because

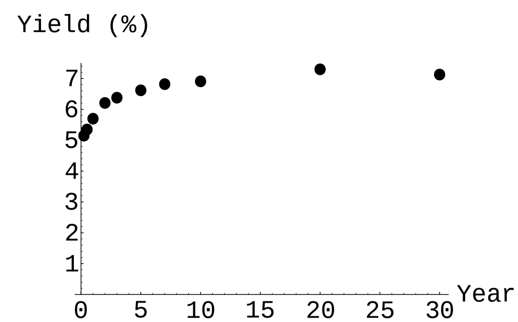
$$-10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.$$

- In fact, $C_{\%}$ equals convexity divided by 100.

Term Structure of Interest Rates

Term Structure of Interest Rates

- Concerned with how interest rates change with maturity.
- The set of yields to maturity for bonds of equal quality and differing solely in their terms to maturity forms the term structure.
- Fundamental to the valuation of fixed-income securities.



Term Structure of Interest Rates (concluded)

- Often refers exclusively to the yields of zero-coupon bonds.
- A yield curve plots yields to maturity against maturity.
- A par yield curve is constructed from bonds trading near par.

Four Shapes

- A normal yield curve is upward sloping.
- An inverted yield curve is downward sloping.
- A flat yield curve is flat.
- A humped yield curve is upward sloping at first but then turns downward sloping.

Spot Rates

- The i-period spot rate $S(i)$ is the yield to maturity of an i-period zero-coupon bond.
- The PV of one dollar i periods from now is $(1+S(i))^{-i}$.
- The one-period spot rate is called the short rate.
- A spot rate curve is a plot of spot rates against maturity.

Problems with the PV Formula

- In the bond price formula,
$$\sum_{i=1}^{n} \frac{C}{(1+y)^i} + \frac{F}{(1+y)^n},$$
every cash flow is discounted at the same yield y.
- Consider two riskless bonds with different yields to maturity because of their different cash flow streams.
- The yield-to-maturity methodology discounts their contemporaneous cash flows with different rates.
- But shouldn't they be discounted at the same rate?
- Enter the spot rate methodology.

Spot Rate Discount Methodology

- A cash flow $C_1, C_2, \ldots, C_n$ is equivalent to a package of zero-coupon bonds with the ith bond paying C_i dollars at time i.
- So a level-coupon bond has the price
$$P = \sum_{i=1}^{n} \frac{C}{[1+S(i)]^i} + \frac{F}{[1+S(n)]^n}. \tag{24}$$
- This pricing method incorporates information from the term structure.
- Discount each cash flow at the corresponding spot rate.

Discount Factors

- Any riskless security having a cash flow $C_1, C_2, \ldots, C_n$ should have a market price of

$$P = \sum_{i=1}^{n} C_i d(i),$$

where $d(i) \equiv [\,1 + S(i)\,]^{-i}$, $i = 1, 2, \ldots, n$, are called discount factors.

- $d(i)$ is the PV of one dollar i periods from now.
- The discount factors are often interpolated to form a continuous function called the discount function.

Fundamental Statistical Concepts

Moments

- The variance of a random variable X is defined as

$$\text{Var}[\,X\,] \equiv E\left[\,(X - E[\,X\,])^2\,\right].$$

- The covariance between random variables X and Y is

$$\text{Cov}[\,X, Y\,] \equiv E\,[\,(X - \mu_X)(Y - \mu_Y)\,],$$

where μ_X and μ_Y are the means of X and Y, respectively.

- Random variables X and Y are uncorrelated if $\text{Cov}[\,X, Y\,] = 0$.

Variance of Sum

- Variance of a weighted sum of random variables equals

$$\text{Var}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j\, \text{Cov}[\,X_i, X_j\,]. \qquad (25)$$

- It becomes $\sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2\, \text{Var}[\,X_i\,]$ when X_i are uncorrelated.

Conditional Expectation

- "$X \mid I$" denotes X conditional on the information set I.
- The information set can be another random variable's value or the past values of X, say.
- The conditional expectation $E[\, X \mid I \,]$ is the expected value of X conditional on I; it is a random variable.
- The law of iterated conditional expectations:

$$E[\, X \,] = E[\, E[\, X \mid I \,]\,].$$

- If I_2 contains at least as much information as I_1, then

$$E[\, X \mid I_1 \,] = E[\, E[\, X \mid I_2 \,] \mid I_1 \,]. \tag{26}$$

The Normal Distribution

- A random variable X has the normal distribution with mean μ and variance σ^2 if its probability density function is $e^{-(x-\mu)^2/(2\sigma^2)}/(\sigma\sqrt{2\pi})$.
- Expressed by $X \sim N(\mu, \sigma^2)$.
- The standard normal distribution has zero mean, unit variance, and the distribution function

$$N(z) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\, dx.$$

Moment Generating Function

- The moment generating function of random variable X is

$$\theta_X(t) \equiv E[\, e^{tX} \,].$$

- The moment generating function of $X \sim N(\mu, \sigma^2)$ is

$$\theta_X(t) = \exp\left[\, \mu t + \frac{\sigma^2 t^2}{2} \,\right]. \tag{27}$$

Distribution of Sum

- If $X_i \sim N(\mu_i, \sigma_i^2)$ are independent, then $\sum_i X_i \sim N(\sum_i \mu_i, \sum_i \sigma_i^2)$.
- Let $X_i \sim N(\mu_i, \sigma_i^2)$ which may not be independent.
- Then $\sum_{i=1}^n t_i X_i \sim N(\sum_{i=1}^n t_i\, \mu_i, \sum_{i=1}^n \sum_{j=1}^n t_i t_j \,\mathrm{Cov}[\, X_i, X_j \,])$.
- These X_i are said to have a multivariate normal distribution.

Generation of Univariate Normal Distributions

- Let X be uniformly distributed over $(0, 1]$ so that $\text{Prob}[X \leq x] = x$ for $0 < x \leq 1$.
- Repeatedly draw two samples x_1 and x_2 from X until $\omega \equiv (2x_1 - 1)^2 + (2x_2 - 1)^2 < 1$.
- Then $c(2x_1 - 1)$ and $c(2x_2 - 1)$ are independent standard normal variables where $c \equiv \sqrt{-2(\ln \omega)/\omega}$.

Generation of Bivariate Normal Distributions

- Pairs of normally distributed variables with correlation ρ can be generated.
- Let X_1 and X_2 be independent standard normal variables.
- Then

$$\begin{aligned} U &\equiv aX_1 \\ V &\equiv \rho U + \sqrt{1 - \rho^2}\, aX_2 \end{aligned}$$

are the desired random variables with $\text{Var}[U] = \text{Var}[V] = a^2$ and $\text{Cov}[U, V] = \rho a^2$.

Dirty Trick

- Let ξ_i are independent and uniformly distributed over $(0, 1)$.
- A simple method to generate the standard normal variable is to calculate

$$\sum_{i=1}^{12} \xi_i - 6,$$

- Always blame your random number generator last; instead, check your programs first.

The Lognormal Distribution

- A random variable Y is said to have a lognormal distribution if $\ln Y$ has a normal distribution.
- Let $X \sim N(\mu, \sigma^2)$ and $Y \equiv e^X$.
- The mean and variance of Y are

$$\mu_Y = e^{\mu + \sigma^2/2} \quad \text{and} \quad \sigma_Y^2 = e^{2\mu + \sigma^2} \left(e^{\sigma^2} - 1 \right), \qquad (28)$$

respectively.

- They follow from $E[Y^n] = e^{n\mu + n^2\sigma^2/2}$.

Option Basics

Calls and Puts

- A call gives its holder the right to buy a number of the underlying asset by paying a strike price.
- A put gives its holder the right to sell a number of the underlying asset by paying a strike price.
- An embedded option has to be traded along with the underlying asset.
- How to price options?

Exercise

- When a call is exercised, the holder pays the strike price in exchange for the stock.
- When a put is exercised, the holder receives from the writer the strike price in exchange for the stock.
- An option can be exercised prior to the expiration date: early exercise.

American and European

- American options can be exercised at any time up to the expiration date.
- European options can only be exercised at expiration.
- The terms “American” and “European” have nothing to do with geography.
- An American option is worth at least as much as an otherwise identical European option because of the early exercise feature.

Convenient Conventions

- C: call value.
- P: put value.
- X: strike price.
- S: stock price.
- D: dividend.

Payoff

- A call will be exercised only if the stock price is higher than the strike price.
- A put will be exercised only if the stock price is less than the strike price.
- The payoff of a call at expiration is $C = \max(0, S - X)$, and that of a put is $P = \max(0, X - S)$.
- At any time t before the expiration date, we call $\max(0, S_t - X)$ the intrinsic value of a call and $\max(0, X - S_t)$ the intrinsic value of a put.

Payoff (concluded)

- A call is in the money if $S > X$, at the money if $S = X$, and out of the money if $S < X$.
- A put is in the money if $S < X$, at the money if $S = X$, and out of the money if $S > X$.
- Options that are in the money at expiration should be exercised.
  - 11% of option holders let in-the-money options expire worthless.
- Finding an option's value at any time before expiration is a major intellectual breakthrough.