Conversion between Compounding Methods

- Suppose r_1 is the annual rate with continuous compounding.
- Suppose r_2 is the equivalent rate compounded m times per annum.
- Then $(1+\frac{r_2}{m})^m = e^{r_1}$.
- Therefore,

$$r_1 = m \ln\left(1+\frac{r_2}{m}\right), \tag{2}$$

$$r_2 = m\left(e^{r_1/m}-1\right). \tag{3}$$

But Are They Really "Equivalent"?

- Recall r_1 and r_2 on the previous page.
- They are based on different cash flows.
- In what sense are they equivalent?

Annuities

- An annuity pays out the same C dollars at the end of each year for n years.
- With a rate of r, the FV at the end of the nth year is

$$\sum_{i=0}^{n-1} C(1+r)^i = C\,\frac{(1+r)^n-1}{r}. \tag{4}$$

General Annuities

- If m payments of C dollars each are received per year (the general annuity), then Eq. (4) becomes

$$C\,\frac{\left(1+\frac{r}{m}\right)^{nm}-1}{\frac{r}{m}}.$$

- The PV of a general annuity is

$$\sum_{i=1}^{nm} C\left(1+\frac{r}{m}\right)^{-i} = C\,\frac{1-\left(1+\frac{r}{m}\right)^{-nm}}{\frac{r}{m}}. \tag{5}$$

Amortization

- It is a method of repaying a loan through regular payments of interest *and* principal.
- The size of the loan (the original balance) is reduced by the principal part of each payment.
- The interest part of each payment pays the interest incurred on the remaining principal balance.
- As the principal gets paid down over the term of the loan, the interest part of the payment diminishes.

Example: Home Mortgage

- By paying down the principal consistently, the risk to the lender is lowered.
- When the borrower sells the house, the remaining principal is due the lender.
- Consider the equal-payment case, i.e., fixed-rate, level-payment, fully amortized mortgages (traditional mortgages).

A Numerical Example

- Consider a 15-year, $250,000 loan at 8.0% interest rate.
- Solving Eq. (5) on p. 30 with $\text{PV} = 250000$, $n = 15$, $m = 12$, and $r = 0.08$ gives a monthly payment of $C = 2389.13$.
- The amortization schedule is shown on p. 34.
- In every month (1) the principal and interest parts add up to $2,389.13, (2) the remaining principal is reduced by the amount indicated under the Principal heading, and (3) the interest is computed by multiplying the remaining balance of the previous month by 0.08/12.

Month	Payment	Interest	Principal	Remaining principal
				250,000.000
1	2,389.13	1,666.667	722.464	249,277.536
2	2,389.13	1,661.850	727.280	248,550.256
3	2,389.13	1,657.002	732.129	247,818.128
		...		
178	2,389.13	47.153	2,341.980	4,730.899
179	2,389.13	31.539	2,357.591	2,373.308
180	2,389.13	15.822	2,373.308	0.000
Total	430,043.438	180,043.438	250,000.000	

Two Methods of Calculating the Remaining Principal

- Go down the amortization schedule.
- Right after the kth payment, the remaining principal is the PV of the future $nm - k$ cash flows,

$$\sum_{i=1}^{nm-k} C\left(1+\frac{r}{m}\right)^{-i} = C\,\frac{1-\left(1+\frac{r}{m}\right)^{-nm+k}}{\frac{r}{m}}. \qquad (6)$$

Yields

- The term yield denotes the return of investment.
- Two widely used yields are the bond equivalent yield (BEY) and the mortgage equivalent yield (MEY).
- BEY corresponds to the r in Eq. (1) that equates PV with FV when $m = 2$.
- MEY corresponds to the r in Eq. (1) that equates PV with FV when $m = 12$.

Internal Rate of Return

- It is the interest rate which equates an investment's PV with its price P,

$$P = \frac{C_1}{(1+y)} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \cdots + \frac{C_n}{(1+y)^n}. \qquad (7)$$

- It assumes all cash flows are reinvested at the *same* rate as the internal rate of return,
- The foundation upon which pricing methodologies are built.

Holding Period Return (HPR)

- Calculate the FV by whatever means and then find the yield y that satisfies $\text{PV} = \text{FV} \times e^{-yn}$.
- Explicit assumptions about the reinvestment rates must be made.
- If the reinvestment assumptions turn out to be wrong, the yield will not be realized.
  - This is the reinvestment risk.
- Financial instruments without intermediate cash flows do not have reinvestment risks.

Numerical Methods for Yields

- Solve $f(y) = 0$ for $y \geq -1$, where

$$f(y) \equiv \sum_{t=1}^{n} \frac{C_t}{(1+y)^t} - P. \tag{8}$$

  - P is the market price.
- The function $f(y)$ is monotonic in y if $C_t > 0$.
- A unique solution exists.

The Bisection Method

- Start with a and b where $a < b$ and $f(a)\, f(b) < 0$.
- Then $f(\xi)$ must be zero for some $\xi \in [\,a, b\,]$.
- If we evaluate f at the midpoint $c \equiv (a+b)/2$, either (1) $f(c) = 0$, (2) $f(a)\, f(c) < 0$, or (3) $f(c)\, f(b) < 0$.
- In the first case we are done, in the second case we continue the process with the new bracket $[\,a, c\,]$, and in the third case we continue with $[\,c, b\,]$.
- The bracket is halved in the latter two cases.
- After n steps, we will have confined ξ within a bracket of length $(b-a)/2^n$.

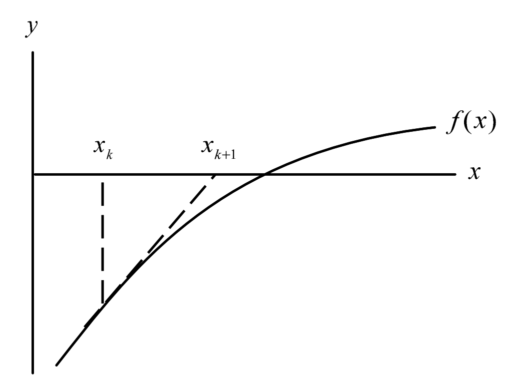
The Newton-Raphson Method

- Converges faster than the bisection method.
- Start with a first approximation x_0 to a root of $f(x) = 0$.
- Then

$$x_{k+1} \equiv x_k - \frac{f(x_k)}{f'(x_k)}. \tag{9}$$

- When computing yields,

$$f'(x) = -\sum_{t=1}^{n} \frac{tC_t}{(1+x)^{t+1}}.$$



The Secant Method

- A variant of the Newton-Raphson method.
- Replace differentiation with difference.
- Start with two approximations x_0 and x_1.
- Then compute the $(k+1)$st approximation with

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}.$$

- Its convergence rate, 1.618, is slightly worse than the Newton-Raphson method's 2 but better than the bisection method's 1.

Solving Systems of Nonlinear Equations

- It is not easy to extend the bisection method to higher dimensions.
- The Newton-Raphson method can be extended to higher dimensions.
- Let (x_k, y_k) be the kth approximation to the solution of the two simultaneous equations,

$$\begin{aligned} f(x,y) &= 0, \\ g(x,y) &= 0. \end{aligned}$$

Solving Systems of Nonlinear Equations (concluded)

- The $(k+1)$st approximation (x_{k+1}, y_{k+1}) satisfies the following linear equations,

$$\begin{bmatrix} \partial f(x_k,y_k)/\partial x & \partial f(x_k,y_k)/\partial y \\ \partial g(x_k,y_k)/\partial x & \partial g(x_k,y_k)/\partial y \end{bmatrix} \begin{bmatrix} \Delta x_{k+1} \\ \Delta y_{k+1} \end{bmatrix} = - \begin{bmatrix} f(x_k,y_k) \\ g(x_k,y_k) \end{bmatrix}, \tag{10}$$

  where $\Delta x_{k+1} \equiv x_{k+1} - x_k$ and $\Delta y_{k+1} \equiv y_{k+1} - y_k$.
- The above has a unique solution for $(\Delta x_{k+1}, \Delta y_{k+1})$ when the 2×2 matrix is invertible.
- The $(k+1)$st approximation is $(x_k + \Delta x_{k+1}, y_k + \Delta y_{k+1})$.

Zero-Coupon Bonds (Pure Discount Bonds)

- The price of a zero-coupon bond that pays F dollars in n periods is

$$F/(1+r)^n,$$

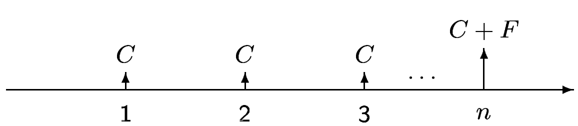
  where r is the interest rate per period.
- Can meet future obligations without reinvestment risk.
- Coupon bonds can be thought of as a matching package of zero-coupon bonds, at least theoretically.

Example

- The interest rate is 8% compounded semiannually.
- A zero-coupon bond that pays the par value 20 years from now will be priced at $1/(1.04)^{40}$, or 20.83%, of its par value.
- It will be quoted as 20.83.
- If the bond matures in 10 years instead of 20, its price would be 45.64.

Level-Coupon Bonds

- Coupon rate.
- Par value, paid at maturity.
- F denotes the par value and C denotes the coupon.
- Cash flow:



Pricing Formula

- $$P = \sum_{i=1}^{n} \frac{C}{\left(1+\frac{r}{m}\right)^i} + \frac{F}{\left(1+\frac{r}{m}\right)^n} = C\,\frac{1-\left(1+\frac{r}{m}\right)^{-n}}{\frac{r}{m}} + \frac{F}{\left(1+\frac{r}{m}\right)^n}. \quad (11)$$
  - n: number of cash flows.
  - m: number of payments per year.
  - r: annual interest rate compounded m times per annum.
- $C = Fc/m$ when c is the annual coupon rate.

Yields to Maturity

- The r that satisfies Eq. (11) with P being the bond price.
- For a 15% BEY, a 10-year bond with a coupon rate of 10% paid semiannually sells for

$$5 \times \frac{1-[\,1+(0.15/2)\,]^{-2\times 10}}{0.15/2} + \frac{100}{[\,1+(0.15/2)\,]^{2\times 10}} = 74.5138$$

percent of par.

Price Behavior (1)

- Bond prices fall when interest rates rise, and vice versa.
- "Only 24 percent answered the question correctly."

Price Behavior (2)

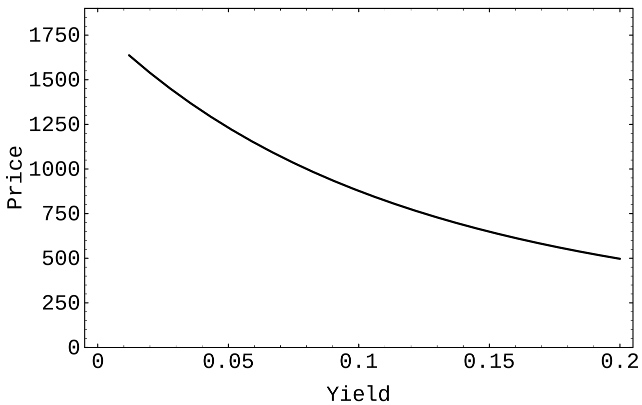
- A level-coupon bond sells
  - at a premium (above its par value) when its coupon rate is above the market interest rate;
  - at par (at its par value) when its coupon rate is equal to the market interest rate;
  - at a discount (below its par value) when its coupon rate is below the market interest rate.

Yield (%)	Price (% of par)
7.5	113.37
8.0	108.65
8.5	104.19
9.0	100.00
9.5	96.04
10.0	92.31
10.5	88.79

Terminology

- Bonds selling at par are called par bonds.
- Bonds selling at a premium are called premium bonds.
- Bonds selling at a discount are called discount bonds.

Price Behavior (3): Convexity



Day Count Conventions: Actual/Actual

- The first "actual" refers to the actual number of days in a month.
- The second refers to the actual number of days in a coupon period.
- The number of days between June 17, 1992, and October 1, 1992, is 106.
  - 13 days in June, 31 days in July, 31 days in August, 30 days in September, and 1 day in October.

Day Count Conventions: 30/360

- Each month has 30 days and each year 360 days.
- The number of days between June 17, 1992, and October 1, 1992, is 104.
  - 13 days in June, 30 days in July, 30 days in August, 30 days in September, and 1 day in October.
- In general, the number of days from date $D_1 \equiv (y_1, m_1, d_1)$ to date $D_2 \equiv (y_2, m_2, d_2)$ is

$$360 \times (y_2 - y_1) + 30 \times (m_2 - m_1) + (d_2 - d_1).$$

- Complications: 31, Feb 28, and Feb 29.

Full Price (Dirty Price, Invoice Price)

- In reality, the settlement date may fall on any day between two coupon payment dates.
- Let

$$\omega \equiv \frac{\text{number of days between the settlement and the next coupon payment date}}{\text{number of days in the coupon period}}. \tag{12}$$

- The price is now calculated by

$$\text{PV} = \sum_{i=0}^{n-1} \frac{C}{\left(1+\frac{r}{m}\right)^{\omega+i}} + \frac{F}{\left(1+\frac{r}{m}\right)^{\omega+n-1}}. \tag{13}$$

Accrued Interest

- The buyer pays the quoted price plus the accrued interest

$$C \times \frac{\text{number of days from the last coupon payment to the settlement date}}{\text{number of days in the coupon period}} = C \times (1-\omega).$$

- The yield to maturity is the r satisfying (13) when P is the invoice price, the sum of the quoted price and the accrued interest.
- The quoted price in the U.S./U.K. does not include the accrued interest; it is called the clean price or flat price.

Example ("30/360")

- A bond with a 10% coupon rate and paying interest semiannually, with clean price 111.2891.
- The maturity date is March 1, 1995, and the settlement date is July 1, 1993.
- There are 60 days between July 1, 1993, and the next coupon date, September 1, 1993.

Example ("30/360") (concluded)

- The accrued interest is $(10/2) \times \frac{180-60}{180} = 3.3333$ per \$100 of par value.
- The yield to maturity is 3%.
- This can be verified by Eq. (13) with $\omega = 60/180$, $m = 2$, $C = 5$, PV$= 111.2891 + 3.3333$, and $r = 0.03$.

Price Behavior (2) Revisited

- Before: A bond selling at par if the yield to maturity equals the coupon rate.
- But it assumed that the settlement date is on a coupon payment date.
- Now suppose the settlement date for a bond selling at par (i.e., the *quoted price* is equal to the par value) falls between two coupon payment dates.
- Then its yield to maturity is less than the coupon rate.
  - The short reason is that exponential growth is replaced by linear growth, hence "overpaying" the coupon.