Distinct Objects into Identical Containers

Corollary 15 There are \( \frac{(rn)!}{(r!)^n n!} \) ways to distribute \( rn \) distinct objects into \( n \) identical containers so that each container contains exactly \( r \) objects.

- Consider \( (x_1 + x_2 + \cdots + x_n)^{rn} \).
  - Let \( x_i \) denote the containers (distinct, for now).
  - Each distinct object is associated with one \( x_1 + x_2 + \cdots + x_n \).
  - It means that object can be assigned to one of the \( n \) distinct containers.

- To see that objects are treated as being distinct, just observe how the coefficients are added up.
Distinct Objects into Identical Containers (continued)

• What does the coefficient of

\[ x_1^r x_2^r \cdots x_n^r \]

mean?

• It is the number of ways \( rn \) distinct objects can be distributed into \( n \) distinct containers, each of which contains \( r \) objects.

• By Theorem 14 (p. 72), it is

\[
\binom{r n}{r, r, \ldots, r} \equiv \frac{(r n)!}{r! r! \cdots r!}.
\]

• Finally, divide the above count by \( n! \) to remove the identities of the containers.
Distinct Objects into Identical Containers (concluded)

Corollary 16 \( \frac{(rn)!}{(r!)^n n!} \) is an integer.

- Immediate from Corollary 15 (p. 75).
An Alternative Proof of Corollary 16 (p. 77)\textsuperscript{a}

\[
\frac{(rn)!}{(r!)^nn!} = \frac{1}{n!} \frac{(rn)!}{[r(n-1)]!r!} \frac{[r(n-1)]!}{[r(n-2)]!r!} \cdots \frac{[r(1)]!}{[r(n-n)]!r!} = \prod_{k=0}^{n-1} \left( \frac{r(n-k)}{r} \right) \\
= \prod_{k=0}^{n-1} \frac{(r(n-k))}{n-k} = \prod_{k=0}^{n-1} \frac{[r(n-k)]!}{(n-k)r![r(n-k-1)]!} = \prod_{k=0}^{n-1} \frac{r(n-k)[r(n-k)-1]!}{(n-k)r[r-1]![r(n-k-1)]!} = \prod_{k=0}^{n-1} \left( \frac{r(n-k)-1}{r-1} \right).
\]

\textsuperscript{a}Contributed by Mr. Ansel Lin (B93902003) on September 20, 2004.
Distinct Objects into Identical Containers (continued)

- Take \( n = 3 \) and \( r = 2 \).
- So we have

\[
(x_1 + x_2 + x_3)^6 = (x_1^6 + \cdots + x_3^6)
+ 6 (x_1^5 x_2 + \cdots + x_2 x_3^5)
+ 15 (x_1^4 x_2^2 + \cdots + x_2^2 x_3^4)
+ 20 (x_1^3 x_2^3 + \cdots + x_2^3 x_3^3)
+ 30 (x_1^4 x_2 x_3 + \cdots + x_1 x_2 x_3^4)
+ 60 (x_1^3 x_2^2 x_3 + \cdots + x_1 x_2^2 x_3^3)
+ 90 x_1^2 x_2^2 x_3^2.
\]
An Example (concluded)

• Indeed, the 7 coefficients are

\[
\binom{6}{6}, \binom{6}{5,1}, \binom{6}{4,2}, \binom{6}{3,3}, \binom{6}{4,1,1}, \binom{6}{3,2,1}, \binom{6}{2,2,2},
\]

consistent with the multinomial theorem (p. 72).

• In particular, the coefficient of \(x_1^2x_2^2x_3^2\) is

\[
90 = \binom{6}{2,2,2}.
\]

• Thus the desired count is

\[
\frac{90}{3!} = 15.
\]

---

\(^a\)Corrected by Mr. Chun-Kai Hsu (B06702073) on March 27, 2023.
Combinations (Selections) with Repetition

**Theorem 17** Suppose there are \( n \) distinct objects and \( r \geq 0 \) is an integer. The number of selections of \( r \) of these objects, with repetition, is

\[
C(n + r - 1, r) = \binom{n + r - 1}{r}.
\]

- Note that the order of selection is not important.
- Imagine there are \( n \) distinct types of objects.
The Proof (continued)

- Permute

\[
\underbrace{xx \cdots x}_{r} \underbrace{\text{ } \cdots \text{ } x}_{n-1}.
\]

- Think of the \(i\)th interval as containing the \(i\)th type of objects.

- So

\[
\underline{xx \mid xxx \mid x \mid \mid \mid}
\]

means, out of 7 distinct objects, we pick 2 type-1 objects, 3 type-2 objects, and 1 type-3 object.
The Proof (concluded)

• Our goal equals the number of permutations of

\[ \underbrace{xx \cdots x}_{r} \underbrace{\vdots}_{n-1}. \]

• By formula (2) on p. 16, it is

\[
\frac{(r + n - 1)!}{r! (n - 1)!} = \binom{n + r - 1}{r} = C(n + r - 1, r).
\]
Combinatorial Proof of the Hockeystick Identity (P. 38)\textsuperscript{a}

**Corollary 18** For $m, n \geq 0$, $\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}$.

- The number of ways to select $m$ objects out of $n + 2$ types is $\binom{n+m+1}{m}$ by Theorem 17 (p. 81).
- Alternatively, let us focus on how the objects of the first $n + 1$ types are chosen.
- There are $\binom{n+m}{m}$ ways to select $m$ objects out of the first $n + 1$ types.
- There are $\binom{n+m-1}{m-1}$ ways to select $m - 1$ objects out of the first $n + 1$ types and 1 object out of the last type.

\textsuperscript{a}Contributed by Mr. Jerry Lin (B01902113) on March 13, 2014.
The Proof (concluded)

• There are \( \binom{n+m-2}{m-2} \) ways to select \( m-2 \) objects out of the first \( n+1 \) types and \( 2 \) objects of the last type.

• ....

• So,

\[
\binom{n+m}{m} + \binom{n+m-1}{m-1} + \binom{n+m-2}{m-2} + \cdots + \binom{n+0}{0} = \binom{n+m+1}{m}.
\]
Integer Solutions of a Linear Equation

The following three problems are equivalent:

1. The number of nonnegative integer solutions of

\[ x_1 + x_2 + \cdots + x_n = r. \]

2. The number of selections, with repetition, of size \( r \) from a collection of \( n \) distinct objects (Theorem 17 on p. 81).

3. The number of ways \( r \) identical objects can be distributed among \( n \) distinct containers.\(^a\)

They all equal \( \binom{n+r-1}{r} \).\(^b\)

\(^a\)The case of distinct objects and identical containers will be covered on p. 276 (see p. 75 for a special case).

\(^b\)See p. 501 and p. 506 for alternative proofs.
Application: The Multinomial Theorem (P. 72)

• The theorem is about the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$ in the expansion of

$$(x_1 + x_2 + \cdots + x_t)^r.$$ 

• But how many distinct terms\(^a\) are there?

• Each term has the form $x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$ such that
  - $n_1 + n_2 + \cdots + n_t = r$, and
  - $0 \leq n_1, n_2, \ldots, n_t$.

• For example, consider

$$r = 2.$$ 

\(^a\)That is, summands.
Application: The Multinomial Theorem (continued)

- Now,

\[(x_1 + x_2 + x_3)^2 = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3.\]

- E.g., the solution \(n_1 = 1, n_2 = 1, n_3 = 0\) to \(n_1 + n_2 + n_3 = 2\) contributes to the term

\[x_1^1x_2^1x_3^0 = x_1x_2.\]

- So there are 6 nonnegative integer solutions to \(n_1 + n_2 + n_3 = 2\) because there are 6 terms.
Application: The Multinomial Theorem (concluded)

• The desired number of terms is, in general,

\[
\binom{r + t - 1}{r}
\]

from the equivalencies on p. 86.

• Indeed, \( \binom{2+3-1}{2} = 6 \).
Positive Integer Solutions of a Linear Equation

- Consider
  \[ x_1 + x_2 + \cdots + x_n = r, \]
  where \( x_i > 0 \) for \( 1 \leq i \leq n \).

- Define \( x'_i \triangleq x_i - 1 \).

- The original problem becomes
  \[ x'_1 + x'_2 + \cdots + x'_n = r - n, \]
  where \( x'_i \geq 0 \) for \( 1 \leq i \leq n \).

- The number of solutions is therefore (p. 86)
  \[
  \binom{n + (r - n) - 1}{r - n} = \binom{r - 1}{r - n} = \binom{r - 1}{n - 1}.
  \] (15)
Application: Subsets with Restrictions

How many $n$-element subsets of $\{1, 2, \ldots, r\}$ contain no consecutive integers?

- Say $r = 4$ and $n = 2$.

- Then the valid 2-element subsets of $\{1, 2, 3, 4\}$ are

  $\{1, 3\}, \{1, 4\}, \{2, 4\}$.
The Proof (continued)

- For each valid subset \( \{ i_1, i_2, \ldots, i_n \} \), where \( 1 \leq i_1 < i_2 < \ldots < i_n \leq r \), define
  \[
  d_k = i_{k+1} - i_k.
  \]

- As “placeholders,” introduce
  \[
  i_0 = 1, \\
i_{n+1} = r.
  \]

- Then, by telescoping,
  \[
  d_0 + d_1 + \cdots + d_n = i_{n+1} - i_0 = r - 1.
  \]
The Proof (continued)

• Observe that

\[ 0 \leq d_0, d_n \]
\[ 2 \leq d_1, d_2, \ldots, d_{n-1}. \]

• Define

\[ d'_0 \triangleq d_0, \]
\[ d'_k \triangleq d_k - 2, \quad k = 1, 2, \ldots, n - 1, \]
\[ d'_n \triangleq d_n. \]
The Proof (concluded)

• So equivalently,

\[ d'_0 + d'_1 + \cdots + d'_n = r - 1 - 2(n - 1) \]

with \(0 \leq d'_0, d'_1, \ldots, d'_n\).

• The answer to the desired number is (p. 86)

\[
\binom{(n + 1) + (r - 1 - 2(n - 1)) - 1}{r - 1 - 2(n - 1)} = \binom{r - n + 1}{n - 2n + 1} = \binom{r - n + 1}{n}. \tag{16}
\]
Application: Political Majority\textsuperscript{a}

In how many ways can $2n + 1$ seats in a parliament be divided among 3 parties so that the coalition of \textit{any} 2 parties form a majority?

- If $n = 2$, there are 5 seats.
- Clearly, no party should have 3 or more seats.
- The only valid distribution of the 5 seats to 3 parties is: 2, 2, 1.
- The number of ways is therefore 3.

\textsuperscript{a}Recall p. 68.
The Proof (continued)

- This is a problem of distributing identical objects (the seats) among distinct containers (the parties) (p. 86).
- So without the majority condition, the number is
  \[
  \binom{3 + (2n + 1) - 1}{2n + 1} = \binom{2n + 3}{2}.
  \]
- Observe that the majority condition is violated if and only if a party gets \( n + 1 \) or more seats (why?).
The Proof (concluded)

• If a given party gets \( n + 1 \) or more seats, the number of ways of distributing the seats is

\[
\binom{3 + n - 1}{n} = \binom{n + 2}{2}.
\]

  – Allocate \( n + 1 \) seats to that party first.
  – Then allocate the remaining \( 2n + 1 - (n + 1) = n \) seats to the 3 parties.
  – Refer to p. 86 for the formula.

• The desired number of no dominating party is

\[
\binom{2n + 3}{2} - 3 \binom{n + 2}{2} = \frac{n}{2} (n + 1) = \binom{n + 1}{2}. \tag{17}
\]
Political Majority: An Alternative Proof$^a$

- Recall that the majority condition holds if and only if no party gets $n + 1$ or more seats.
- So each party can hold up to $n$ seats.
- Give each party $n$ slots to hold real seats.
- As there are $2n + 1$ seats, there will be

$$3n - (2n + 1) = n - 1$$

empty slots in the end.

$^a$Contributed by Mr. Weicheng Lee (B01902065) on March 14, 2013.
Political Majority: An Alternative Proof (concluded)

• So the answer to the desired number is the number of ways to distribute the $n - 1$ empty slots to 3 parties.

• The count is (p. 86)

$$\binom{3 + (n - 1) - 1}{n - 1} = \binom{n + 1}{n - 1} = \binom{n + 1}{2}.$$
Integer Solutions of a Linear Inequality

• Consider

\[ x_1 + x_2 + \cdots + x_n \leq r, \]

where \( x_i \geq 0 \) for \( 1 \leq i \leq n \).

• It is equivalent to

\[ x_1 + x_2 + \cdots + x_n + x_{n+1} = r, \]

where \( x_i \geq 0 \) for \( 1 \leq i \leq n + 1 \).

• The number of integer solutions of the original inequality is therefore (p. 86)

\[
\binom{(n + 1) + r - 1}{r} = \binom{n + r}{r}.
\]  

(18)
The Hockeystick Identity (P. 38) Reproved

• By Eq. (18) on p. 100, there are \( \binom{n+1+m}{m} \) nonnegative integer solutions to

\[
x_1 + x_2 + \cdots + x_{n+1} \leq m, \quad m \geq 0.
\]

• By p. 86, there are \( \binom{n+k}{k} \) nonnegative integer solutions to

\[
x_1 + x_2 + \cdots + x_{n+1} = k.
\]

• Any solution to \( x_1 + x_2 + \cdots + x_{n+1} \leq m \) is a solution to \( x_1 + x_2 + \cdots + x_{n+1} = k \) for some \( 0 \leq k \leq m \).
The Proof (concluded)

- The opposite is also true.
- It is also clear the correspondence is one-to-one.
- So

\[ \sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}. \]

- This is exactly the hockeystick identity.
Compositions of Positive Integers

• Let \( m \) be a positive integer.

• A composition for \( m \) is a sum of positive integers whose order is relevant and which sum to \( m \).

• For \( m = 3 \), the number of compositions is 4:
  \[ 3, 2 + 1, 1 + 2, 1 + 1 + 1. \]

• For \( m = 4 \), the number of compositions is 8:
  \[ 4, 3 + 1, 2 + 2, 1 + 3, 1 + 1 + 2, 1 + 2 + 1, 2 + 1 + 1, 1 + 1 + 1 + 1. \]

• Is the number of compositions for general \( m \) equal to \( 2^{m-1} \)?
The Number of Compositions

**Theorem 19** The number of compositions for \( m > 0 \) is \( 2^{m-1} \).

- Every composition with \( i \) summands corresponds to a positive integer solution to

\[
x_1 + x_2 + \cdots + x_i = m.
\]

- So the number of solutions is \( \binom{m-1}{m-i} \) by Eq. (15) on p. 90.
- The total number of compositions is therefore

\[
\sum_{i=1}^{m} \binom{m-1}{m-i} = 2^{m-1}
\]

by Eq. (9) on p. 57.
An Alternative Proof for Theorem 19 (p. 104)\textsuperscript{a}

- Let \( f(m) \) denote the number of compositions for \( m > 0 \).
- A composition for \( m \) is either (1) \( m \) or (2) \( i \) plus a composition for \( m - i \) ("\( i + \cdots \)") for \( i = 1, 2, \ldots, m - 1 \).
- Then
  \[
  f(m) = 1 + \sum_{i=1}^{m-1} f(m - i) = 1 + \sum_{i=1}^{m-1} f(i).
  \]
- The above implies that \( f(m + 1) - f(m) = f(m) \) so
  \[
  f(m + 1) = 2f(m).
  \]

\textsuperscript{a}Contributed by Mr. Chih-Ning Chou (B01902046) on March 7, 2013.
The Proof (concluded)

- As a result,
  \[ f(m) = 2^{m-1} f(1). \]

- Finally, as \( f(1) = 1 = 2^0 \),
  \[ f(m) = 2^{m-1}. \]
A Third Proof for Theorem 19 (p. 104)$^a$

- Start with $m$ $x$’s and $m - 1$ ’s.
- Consider this arrangement:

$$\underbrace{x \mid x \mid x \mid \cdots \mid x}_{2m - 1}$$

- Think of the ’s as dividers.
- Now remove some of the ’s.

---

$^a$Contributed by Mr. Jerry Lin (B01902113) on March 6, 2014.
The Proof (concluded)

- For example,

\[ xx | xxx | x | x \]

means the composition

\[ 2 + 3 + 1 + 1 \]

for 7.

- Each removal of some \(|\)’s leads to a unique composition.

- As there are

\[ 2^{m-1} \]

ways to remove the \(|\)’s, this is the number of compositions for \(m\).
Palindromes of Positive Integers

• Let $m$ be a positive integer.

• A palindrome for $m$ is a composition for $m$ that reads the same left to right as right to left.
  - For $m = 4$, the number of palindromes is 4:
    
    $\boxed{4}, 1 + \boxed{2} + 1, 2 + \boxed{2}, 1 + 1 + 1 + 1.$

    
    - For $m = 5$, the number of palindromes is 4:
      
      $\boxed{5}, 1 + \boxed{3} + 1, 2 + \boxed{1} + 2, 1 + 1 + \boxed{1} + 1 + 1.$

    - The center elements are boxed above.
Palindromes of Positive Integers (concluded)

- The numbers to the left of the center element mirror those to the right, and with the same sum.
- Palindrome is possibly the hardest form of wordplay.\(^a\)
- For example,\(^b\)
  
  A man, a plan, a canal, Panama!

- See https://www.youtube.com/watch?v=PcTVk0zrzQs for Bach’s *The Music Offering* as another example.
- It is used in the Entropy Game of the Dutch National Olympiad in Informatics (2023).

\(^a\)Bryson (2001, p. 228).
\(^b\)Ignore the spaces and punctuation marks.
The Number of Palindromes

**Theorem 20** *The number of palindromes for* \( m > 0 \) *is* \( 2^{\lfloor m/2 \rfloor} \).

- Assume \( m \) is even first.

- The central element of a composition of \( m \) can be \( m, m - 2, \ldots, 2 \) or “+” (think of it as a 0).\(^a\)

- When the central element is \( m \), the number of palindromes is clearly 1.

- Suppose the central element is some other even number \( 0 \leq i < m \).

\(^a\)The central element must be even (why)!
The Proof (concluded)

• Then the numbers to its left sum to \((m - i)/2\).\(^a\)

• They form a composition (p. 103).

• Hence the number of palindromes is \(2^{(m-i)/2-1}\) by Theorem 19 (p. 104).

• The total number of palindromes for \(m\) is thus

\[
1 + \left( 1 + 2 + 2^2 + \cdots + 2^{(m-2)/2-1} + 2^{m/2-1} \right) = 2^{m/2}.
\]

• Follow the same argument when \(m\) is odd to obtain a count of \(2^{(m-1)/2}\).

\(^a\)By symmetry, the numbers to its right sum to \((m - i)/2\), too.
Runs

- Consider a permutation of 10 Os and 5 Es:
  
  $$0 0 E 0 0 0 0 E E E 0 0 0 E 0.$$  

- It has 7 runs:

  $0 0 E 0 0 0 0 E E E 0 0 0 E 0.$  

  run run run run run run run  

- In general, a run is a *maximal* consecutive list of identical objects.
The Number of Runs

Theorem 21  There are

\[
\binom{m - 1}{m - \lceil r/2 \rceil} \binom{n - 1}{n - \lfloor r/2 \rfloor} + \binom{n - 1}{n - \lceil r/2 \rceil} \binom{m - 1}{m - \lfloor r/2 \rfloor}
\]

ways that \( m \) identical objects of type 1 and \( n \) identical objects of type 2 can give rise to \( r \) runs.

- Suppose the run starts with a type-1 object.
- Let \( x_i \) denote the number of type-1 objects in run \( i = 1, 3, \ldots, 2 \lceil r/2 \rceil - 1 \).
The Proof (continued)

• The number of runs with the said counts \( x_1, x_3, \ldots \) equals the number of positive-integer solutions to

\[
x_1 + x_3 + \cdots + x_{2\lceil r/2 \rceil - 1} = m.
\]

  - There are \( \lceil r/2 \rceil \) terms.

• There are

\[
\binom{m-1}{\lceil r/2 \rceil - 1} = \binom{m-1}{m - \lceil r/2 \rceil}
\]

solutions by Eq. (15) on p. 90.
The Proof (continued)

• Now let $x_i$ denote the number of type-2 objects in run $i = 2, 4, \ldots, 2\lfloor r/2 \rfloor$.

• The number of runs with the said counts $x_2, x_4, \ldots$ equals that of positive-integer solutions to

$$x_2 + x_4 + \cdots + x_{2\lfloor r/2 \rfloor} = n.$$

  – There are $\lfloor r/2 \rfloor$ terms.

• Similarly, the number of solutions equals

$$\binom{n - 1}{\lfloor r/2 \rfloor - 1} = \binom{n - 1}{n - \lfloor r/2 \rfloor}.$$
The Proof (concluded)

• Therefore the number of runs that start with a type-1 object equals

\[
\binom{m - 1}{m - \lfloor r/2 \rfloor} \binom{n - 1}{n - \lfloor r/2 \rfloor}.
\]

• Repeat the argument for the case where the 1st run starts with a type-2 object.

• The count is

\[
\binom{n - 1}{n - \lfloor r/2 \rfloor} \binom{m - 1}{m - \lfloor r/2 \rfloor}
\]

(by swapping \( m \) and \( n \)).
The Catalan\textsuperscript{a} Numbers (1838)

- A binomial random walk starts at the origin (p. 42).
- What is the number of ways it can end at the origin in $2n$ steps \textit{without} being in the negative territory?
- A left move lowers the position, whereas a right move increases the position.
- So it is equivalent to the number of ways

\[
\underbrace{RR \cdots R}_{n} \underbrace{LL \cdots L}_{n}
\]

can be permuted so that no prefix has more $L$s than $R$s.

\textsuperscript{a}Eugène Charles Catalan (1814–1894). It was known to Euler (1707–1783) and, even earlier, Mongolian mathematician Minggatu (1730).
The Catalan Numbers (concluded)

• For example,

\[ \begin{array}{c}
0 \\
\quad \downarrow 1 \\
\quad \downarrow 2 \\
\quad \downarrow 1 \\
\quad \downarrow 0 \\
\quad \downarrow 1 \\
\quad \downarrow 1 \\
\quad \downarrow 0 \\
\quad \downarrow 1 \\
\quad \downarrow R \\
\quad \downarrow LRLRRLLL. 
\end{array} \]
Formula for the Catalan Number\(^a\)

The number is\(^b\)

\[ b_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n}, \quad n \geq 1. \quad (19) \]

with \(b_0 = 1\).

- \(\underbrace{RR \cdots R}_{n} \underbrace{LL \cdots L}_{n}\) can be permuted in \(\binom{2n}{n}\) ways by formula (2) on p. 16.\(^c\)

- Some of the permutations are illegal, such as \(RLLRR\).

\(^a\)Attributed to Jacques Touchard (1885–1968).

\(^b\)The subscript in \(b_n\) is \(n\) not \(2n!\).

\(^c\)Alternatively, recall formula (5) on p. 43.
The Proof (continued)

- We now prove that \( \binom{2n}{n-1} \) of the permutations are illegal.
- For every illegal permutation, we consider the first \( L \) move that makes the particle land at \(-1\).
  - Such as \( RL[L]LRR \).
- Swap \( L \) and \( R \) for this offending \( L \) and all earlier moves.
  - Such as \( L[R][R]LRR \).
- The result is a permutation of
  
  \[
  \underbrace{RR \cdots R}_{n+1} \underbrace{LL \cdots L}_{n-1}.
  \]
The Proof (concluded)

• There are \( \binom{2n}{n-1} \) ways to permute

\[
\underbrace{RR\cdots R}_{n+1} \underbrace{LL\cdots L}_{n-1}
\]

by Eq. (2) on 16.

• But the correspondence is one-to-one between the permutations of

\[
\underbrace{RR\cdots R}_{n+1} \underbrace{LL\cdots L}_{n-1}
\]

and illegal permutations (see next page).

• So there are \( \binom{2n}{n-1} \) illegal walks.
The Reflection Principle

\textsuperscript{a}André (1887).
A Simple Corollary

Corollary 22  *For* $n \geq 1$,

$$b_n = \frac{\sum_{i=0}^{n} \binom{n}{i}^2}{n + 1}.$$

- See Eq. (14) on p. 62.
Application: No Return to Origin until End

What is the number of ways a binomial random walk that is never in the negative territory \textit{and} returns to the origin the \textit{first} time after $2n$ steps?

- Let $n \geq 1$.
- The answer is $b_{n-1}$. 
Application: No Return to Origin until End (concluded)

What is the number of ways a binomial random walk returns to the origin the first time after $2n$ steps?

- Let $n \geq 1$.
- The answer is

$$2b_{n-1} = \frac{1}{2n-1} \binom{2n}{n}.$$  \hspace{1cm} (20)

- It may return to the origin by way of the negative territory.
- It may return to the origin by way of the positive territory.
Application: Nonnegative Partial Sums

What is the number of ways we can arrange $n$ “+1” and $n$ “−1” such that all $2n$ partial sums are nonnegative?

- For example, the six partial sums of $(1, 1, -1, 1, -1, -1)$ are $(1, 2, 1, 2, 1, 0)$.
- Let $n \geq 1$.
- The answer is $b_n$ by definition (p. 118).
- The number remains $b_n$ if we have only $n - 1$ “−1”.
  - In the original problem, the last number must be −1.
  - So it is “redundant.”
Application: Nonpositive Partial Sums

What is the number of ways we can arrange $n$ “+1” and $n$ “−1” such that all $2n$ partial sums are nonpositive?

- For example, the six partial sums of $(-1, -1, 1, -1, 1, 1)$ are $(-1, -2, -1, -2, -1, 0)$.

- Let $n \geq 1$.

- The answer is $b_n$.

- The number remains $b_n$ if we have only $n - 1$ “+1”.
  - In the original problem, the last number must be 1.
  - So it is “redundant.”
Combinatorics and “Higher” Mathematics
For relaxation,
General Bradley did algebra problems,
and he worked at integral calculus
when he was flying an airplane
— or flying in his airplane.
He said it relaxed him, made him think.
— Chet Hansen, Major,
aide to 5-star General Omar Bradley (1893–1981)
Growth of Factorials

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A Logplot (Base Two)

Logplot of $n!$
A Useful Lower Bound for $n!$

**Lemma 23** $n! > \left( \frac{n}{e} \right)^n$.

Proof:

\[
\ln(n!) = \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n
\]

\[
= \sum_{k=1}^{n} \ln k
\]

\[
> \sum_{k=1}^{n} \int_{k-1}^{k} \ln x \, dx \quad \text{as } \ln x \text{ is increasing}
\]

\[
= \int_{0}^{n} \ln x \, dx
\]

\[
= [x \ln x - x]_{x=0}^{n}
\]

\[
= n \ln n - n.
\]
How Good Is the Bound?

$n!$ over lower bound

Good, but probably not of the same order as $n!$. 
Lemma 24  \( n! > e^{(n/e)^n} \).

Proof:

\[
\ln(n!) = \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n \\
= \sum_{k=2}^{n} \ln k \\
> \sum_{k=2}^{n} \int_{k-1}^{k} \ln x \, dx \\
\geq \int_{1}^{n} \ln x \, dx \\
= [x \ln x - x]_{x=1}^{n} \\
= n \ln n - n + 1.
\]
A Useful Upper Bound for $C(n, m)$

**Lemma 25** $C(n, m) < (ne/m)^m$ for any $0 < m \leq n$.\(^a\)

**Proof:**

$$C(n, m) = \frac{n!}{(n-m)!m!} = \frac{n(n-1) \cdots (n-m+1)}{m!} \leq \frac{n^m}{m!} < \frac{n^m}{(m/e)^m} \text{ by Lemma 23 (p. 133)}$$

$$= \left(\frac{ne}{m}\right)^m.$$\(^a\) The tighter bound $(ne/m)^m/e$ follows Lemma 24 (p. 136).
Stirling’s Formula\textsuperscript{a} (1730)

- The notation $f(x) \sim g(x)$ means
  \[
  \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1,
  \]
  i.e.,
  \[
  f(x) = g(x) + o(g(x))
  \]
as $x \to \infty$.\textsuperscript{b}

- Stirling’s formula says:

**Theorem 26** \(n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n\).

**Corollary 27** \(e = \lim_{n \to \infty} \frac{n}{(n!)^{1/n}}\).

\textsuperscript{a}James Stirling (1692–1770); but due to Abraham DeMoivre (1667–1754)!

\textsuperscript{b}It does not imply $f(x) - g(x) \to 0$. 
Goodness of Approximation to $n!$

$n!$ over approximation
Approximation of $C(n, m)$

- Stirling’s formula can be used to approximate $C(n, m)$ better than Lemma 25 (p. 137) under some conditions.

- For that purpose, a more refined Stirling’s formula is stated below without proof:\(^{a}\)

\[
\sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{\frac{1}{12n}}. \tag{21}
\]

\(^{a}\)Robbins (1955).
The Proof (concluded)

- Now from bounds (21) on p. 140,

\[
C(n, m) = \frac{n!}{(n - m)! \cdot m!} < \frac{\sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{\frac{1}{12n}}}{\sqrt{2\pi(n - m)} \left( \frac{n - m}{e} \right)^{n - m} e^{\frac{1}{12(n - m) + 1}} \sqrt{2\pi m} \left( \frac{m}{e} \right)^m e^{\frac{1}{12m + 1}}}
\]

\[
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n - m} \right)^{n - m} \sqrt{\frac{n}{m(n - m)}} \\
&\quad \times e^{\frac{1 - 12n - 144(m - n)^2 - 144mn}{(\cdots)(\cdots)(\cdots)}}
\end{aligned}
\]

\[
< \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n - m} \right)^{n - m} \sqrt{\frac{n}{m(n - m)}}. \quad (22)
\]
Approximation of $C(n, m)$, $1 \leq m \leq n/2$

$$C(n, m)$$

$$> \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n-m} \right)^{n-m} \sqrt{\frac{n}{m(n-m)}} e^{\frac{1}{12(n+1)} - \frac{1}{12(n-m)} - \frac{1}{12m}}$$

$$= \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n-m} \right)^{n-m} \sqrt{\frac{n}{m(n-m)}} e^{\frac{-12m-1}{2(n-m)(2n+1)} - \frac{1}{12m}}$$

$$\geq \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n-m} \right)^{n-m} \sqrt{\frac{n}{m(n-m)}} e^{-\frac{1}{6m} + \frac{1}{(24m+1)}}$$

$$= \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n-m} \right)^{n-m} \sqrt{\frac{n}{m(n-m)}} e^{-\frac{1}{6m}}. \quad (23)$$
The Proof (continued)

• Combine inequalities (22) on p. 141 and (23) on p. 142 under $1 \leq m \leq n/2$ to obtain

\[
\frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n-m} \right)^{n-m} \sqrt{\frac{n}{m(n-m)}} e^{-\frac{1}{6m}}
\]

\[< C(n, m) \]

\[< \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n-m} \right)^{n-m} \sqrt{\frac{n}{m(n-m)}}.\]
The Proof (concluded)

- So

\[ C(n, m) \sim \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n-m} \right)^{n-m} \sqrt{\frac{n}{m(n-m)}} \]  

as \( m \to \infty \) and \( n - m \to \infty \).

- An alternative formulation is

\[ C(n, m) \sim \frac{1}{\sqrt{2\pi p q n}} (p^p q^q)^{-n}, \]

where \( p \triangleq m/n \) and \( q \triangleq 1 - p \).
Application: Probability of Return to Origin

- Suppose the binomial random walk has a probability of \(2^{-1} = 0.5\) of going in either direction (p. 46).
  - This is called a **symmetric random walk**.

- The number of ways it is at the origin after \(2n\) steps is \(\binom{2n}{n}\) by formula (5) on p. 43.\(^a\)

- The probability for this to happen is

\[
\frac{\binom{2n}{n}}{2^{2n}} \approx \frac{1}{\sqrt{2\pi}} \frac{2^n}{2^{2n}} \sqrt{\frac{2}{n}} \approx \sqrt{\frac{1}{\pi n}} = O\left(\frac{1}{\sqrt{n}}\right)
\]  

(25)  

by Eq. (24) on p. 144.

\(^a\)We have seen \(\binom{2n}{n}\) many times before (e.g., p. 58, p. 62, p. 120, and p. 124). We will continue to encounter it.
Application: Probability of Return to Origin (concluded)

- Suppose 100 U.S. Senators vote on a bill randomly.\(^a\)
- What is the probability of a tie?\(^b\)
- By Eq. (25) on p. 145, it equals
  \[
  \frac{\binom{100}{50}}{2^{100}} = 0.0795892 \approx \frac{1}{12}.
  \]
- The probability is surprisingly high.
- It rises to 0.176197 with 20 Senators in late 18th century.

\(^a\)Dixit & Nalebuff (1993).
\(^b\)Which is broken by the Vice President.
Application: Deviation

- Consider the symmetric random walk again.
- Its average position at the end is 0.
- Assume $n$ is even.
- Given $c > 0$, after $n$ steps what is the probability for the walk to end at a position $\geq c\sqrt{n}$ for $n$ sufficiently large?
Application: Deviation (continued)

- The probability that the walk ends at position \( k \) after \( n \) steps is
  \[
  \left( \frac{n}{\frac{n+k}{2}} \right) 2^{-n}
  \]
  by formula (5) on p. 43, where \( k \) is even.

- The probability that the position is at least \( c\sqrt{n} \) is
  \[
  \sum_{k=\lceil c\sqrt{n} \rceil}^{n} \left( \frac{n}{\frac{n+k}{2}} \right) 2^{-n} \approx \frac{1}{2} - \sum_{k=2}^{\lfloor c\sqrt{n} \rfloor} \left( \frac{n}{\frac{n+k}{2}} \right) 2^{-n}
  \]
  by Eq. (10) on p. 59.
  - The integer \( k \) must be even.
Application: Deviation (concluded)

- But

\[
\frac{1}{2} - \sum_{k=2}^{\lfloor c\sqrt{n} \rfloor} \left( \frac{n}{2} + k \right)^{-n} \geq \frac{1}{2} - \frac{c\sqrt{n}}{2} \left( \frac{n}{2} \right)^{-n}
\]

according to the unimodal property (p. 28).\(^a\)

- That \( k \) is even accounts for the 2 in the denominator.

- Finally, the desired probability is

\[
\frac{1}{2} - 2^{-n} \frac{c\sqrt{n}}{2} \left( \frac{n}{2} \right) \geq \frac{1}{2} - c\sqrt{\frac{1}{2\pi}}
\]

by Eq. (25) on p. 145 for \( n \) sufficiently large.

\(^a\)Corrected by Mr. Gong-Ching Lin (B00703082) on March 8, 2012 and Mr. Rajon Geng (B03902010) on March 5, 2016.
An Upper Bound for $C(2n, n)$

Lemma 28 \( \binom{2n}{n} < \frac{4^n}{\sqrt{n \pi}} \).

Proof: From inequality (22) on p. 141,

\[
\binom{2n}{n} < \frac{1}{\sqrt{2\pi}} \left( \frac{2n}{n} \right)^n \left( \frac{2n}{2n - n} \right)^{2n-n} \sqrt{\frac{2n}{n(2n-n)}}
\]

\[
= \frac{1}{\sqrt{n \pi}} 4^n.
\]

Note that Lemma 25 (p. 137) gives a much looser upper bound of $(2e)^n \sim 5.43656^n$. 
A Tight Bound for $C(2n, n)$

Lemma 29 \( \binom{2n}{n} \sim \frac{4^n}{\sqrt{n\pi}} \).\(^a\)

- From inequality (23) on p. 142,

\[
\binom{2n}{n} \cdot \frac{1}{\sqrt{2\pi}} \left( \frac{2n}{n} \right)^n \left( \frac{2n}{2n-n} \right)^{2n-n} \sqrt{\frac{2n}{n(2n-n)}} e^{-\frac{1}{6n}}
\]

\[
= \frac{1}{\sqrt{n\pi}} 4^n e^{-\frac{1}{6n}}.
\]

- Finally, recall Lemma 28 (p. 150).

\(^a\)In fact, $e^{-1/(8n)} < \sqrt{n\pi} \binom{2n}{n}/4^n < 1$ (Hipp & Mattner, 2008).
A Tight Bound for $C'(2n, n)$ (concluded)

\[ \binom{2n}{n} / \left(4^n / \sqrt{n\pi} \right) \]
First Return to Origin\textsuperscript{a}

What is the probability a symmetric binomial random walk returns to the origin the \textit{first} time at step $2n$?

- Formula (20) on p. 126 says the probability is
  \[
  \frac{1}{2n - 1} \binom{2n}{n} 2^{-2n}.
  \]

- The above probability is asymptotically
  \[
  \sim \frac{1}{2\sqrt{n^3 \pi}}
  \]
  by Lemma 29 (p. 151).

\textsuperscript{a}Recall p. 125.
Analytic Number Theory
A proof is that which convinces a reasonable man; a rigorous proof is that which convinces an unreasonable man.
— Mark Kac (1914–1984)
There Are an Infinite Number of Primes

Theorem 30 (Euclid, 300 B.C.) There are infinitely many primes.

• A prime is a positive integer larger than 1 whose only divisors are itself and 1.

• Suppose $p_1, p_2, \ldots, p_k$ are all the primes.

• Let $B = p_1 p_2 \cdots p_k + 1$.

• Because $B > p_i$ for all $i$, $B$ cannot be a prime.

Euclid (325 B.C.–265 B.C.). Some, such as Calude (1994), claim this is the most important result in all mathematics.
The Proof (concluded)

• So there must be a prime $p_j$ such that $p_j$ divides $B = p_1 p_2 \cdots p_k + 1$.

• But that implies $p_j$ must divide 1, a contradiction.
There Are an Infinite Number of Primes: An Alternative Proof\textsuperscript{a}

- Every number $n$ can be uniquely factorized into prime factors $p_1^{k_1} p_2^{k_2} \ldots$.

- So

$$
\left( \sum_{k=0}^{\infty} \frac{1}{2^k} \right) \left( \sum_{k=0}^{\infty} \frac{1}{3^k} \right) \left( \sum_{k=0}^{\infty} \frac{1}{5^k} \right) \ldots
$$

$$
= \sum \frac{1}{2^{k_1} 3^{k_2} 5^{k_3} \ldots}
$$

$$
= \sum_{n \geq 1} \frac{1}{n}.
$$

\textsuperscript{a}Leonhard Euler (1707–1783) in 1737.
The Proof (concluded)

- The right-hand side is an infinite number (why?).
- The left-hand side equals

\[
\frac{1}{1 - \frac{1}{2}} \frac{1}{1 - \frac{1}{3}} \frac{1}{1 - \frac{1}{5}} \cdots
\]

- It is an infinite number only if the number of primes is infinite.
Leonhard Euler (1707–1783)
The Prime Number Theorem\textsuperscript{a}

Let $\pi(n)$ stand for the number of primes up to $n$.

**Theorem 31** $\pi(n) \sim n/\ln n$.

**Corollary 32** The average density of primes from 1 to $n$ is $1/\ln n$.

**Corollary 33** The $n$th prime number is about $n \ln n$.

\textsuperscript{a}Jacques Salomon Hadamard (1865–1963) and Charles De la Vallée Poussin (1866–1962) in 1896. “[Hadamard’s] daughter claimed he could not count beyond four, ‘After that came $n$.’” (Derbyshire, 2003).
\( \pi(n) \ vs. \ n/\ln n \)