Financial Application: Compound Interest\textsuperscript{a}

- Consider $a_{n+1} = (1 + r) a_n$.
  - Deposit grows at a period interest rate of $r > 0$.
  - The initial deposit is $a_0$ dollars.
- The solution is obviously
  
  $a_n = (1 + r)^n a_0$.

- The deposit thus grows exponentially with time.

\textsuperscript{a}“In the fifteenth century mathematics was mainly concerned with questions of commercial arithmetic and the problems of the architect,” wrote Joseph Alois Schumpeter (1883–1950) in \textit{Capitalism, Socialism and Democracy} (1942).
Financial Application: Amortization

• Consider $a_{n+1} = (1 + r) a_n - M$.
  – The initial loan amount is $a_0$ dollars.
  – The monthly payment is $M$ dollars.
  – The outstanding loan principal after the $n$th payment is $a_n$.

• By Eq. (94) on p. 599, the solution is

$$a_n = (1 + r)^n a_0 - M \frac{(1 + r)^n - 1}{r}.$$
The Proof (concluded)

- What is the unique monthly payment $M$ for the loan to be paid off after $k$ monthly payments?

- Set $a_k = 0$ to obtain

$$a_k = (1 + r)^k a_0 - M \left( \frac{(1 + r)^k - 1}{r} \right) = 0.$$ 

- Hence

$$M = \frac{(1 + r)^k a_0 r}{(1 + r)^k - 1}.$$ 

- This is a standard formula for home mortgages and annuities.\(^a\)

\(^a\)Lyuu (2002).
Trial and Error a Third Time

- Consider the more general \( a_{n+1} - Aa_n = BC^n \).

- Calculations show that

  \[
  a_1 = Aa_0 + B, \\
  a_2 = Aa_1 + BC = A^2a_0 + B(A + C), \\
  a_3 = Aa_2 + BC^2 = A^3a_0 + B(A^2 + AC + C^2).
  \]

- Conjecture that is easily verified by substitution:

  \[
  a_n = \begin{cases} 
  A^n a_0 + B \frac{A^n - C^n}{A - C}, & \text{if } A \neq C \\
  A^n a_0 + BA^{n-1}, & \text{if } A = C 
  \end{cases}
  \quad (95)
  \]
Application: Runs of Binary Strings

- A run is a maximal consecutive list of identical objects (p. 113).
  - Binary string “0 0 1 1 1 0” has 3 runs.

- Let $r_n$ denote the total number of runs determined by the $2^n$ binary strings of length $n$.

- First, $r_1 = 2$.
  - Each of “0” and “1” has 1 run.

- Next, $r_2 = 6$.
  - “00” and “11” each has 1 run, while “01” and “10” each has 2 runs.
The Proof (continued)

- In general, suppose we append a bit to every \((n-1)\)-bit string \(b_1b_2\cdots b_{n-1}\) to make \(b_1b_2\cdots b_{n-1}b_n\).

- First, suppose \(b_{n-1} = b_n\) (i.e., the last 2 bits are identical).

- Then the total number of runs does not change.
  - The total number of runs remains \(r_{n-1}\).
The Proof (continued)

• Next, suppose $b_{n-1} \neq b_n$ (i.e., the last 2 bits are distinct).

• Then the total number of runs increases by 1 for each $(n - 1)$-bit string.
  – There are $2^{n-1}$ of them.
  – So the total number of runs becomes $r_{n-1} + 2^{n-1}$. 
The Proof (continued)

• Hence

\[ r_n = 2r_{n-1} + 2^{n-1}, \quad n \geq 2. \]  \hfill (96)

• By Eq. (95) on p. 604,

\[ r_n = 2^n r_0 + 2^{n-1}n. \]

• To make sure that \( r_1 = 2 \), it is easy to see that \( r_0 = 1/2 \).

• Hence

\[ r_n = 2^{n-1} + 2^{n-1}n = 2^{n-1}(n + 1). \]
The Proof (concluded)

• The recurrence (96) is identical to that for the number of edges of a Hasse diagram (p. 597).

• But the initial condition was different: \(a_1 = 1\).

• Its slightly different solution appeared in Eq. (93) on p. 596: \(a_n = n2^{n-1}\).
Method of Undetermined Coefficients

• Recall Eq. (92) on p. 593, repeated below:

\[ C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = f(n). \]  

(97)

• Let \( a_n^{(h)} \) denote the general solution of the associated homogeneous relation (with \( f(n) = 0 \)).

• Let \( a_n^{(p)} \) denote a particular solution of the nonhomogeneous relation.

• Then

\[ a_n = a_n^{(h)} + a_n^{(p)}. \]

• All the entries in the table on p. 595 fit the claim.
Conditions for the General Solution

Similar to Theorem 69 (p. 551), we have the following.

**Theorem 70** Let \( a_n^{(p)} \) be any particular solution of the nonhomogeneous recurrence relation Eq. (97) on p. 610. Let

\[
a_n^{(h)} = C_1 a_n^{(1)} + C_2 a_n^{(2)} + \cdots + C_k a_n^{(k)}
\]

be the general solution of its homogeneous version as specified in Theorem 69. Then \( a_n^{(h)} + a_n^{(p)} \) is the general solution of Eq. (97) on p. 610.
Solution Techniques

- Typically, one finds the general solution of its homogeneous version $a_n^{(h)}$ first.
- Then one finds a particular solution $a_n^{(p)}$ of the nonhomogeneous recurrence relation Eq. (97) on p. 610.
- Make sure $a_n^{(p)}$ is “independent” of $a_n^{(h)}$.
- Finally, use the initial conditions to nail the coefficients of $a_n^{(h)}$.
- Output $a_n^{(h)} + a_n^{(p)}$. 
\[ a_{n+1} - Aa_n = B \]

Revisited

- Recall that the general solution is \( a_n^{(h)} = cA^n \) by Eq. (81) on p. 545.

- A particular solution is (verify it)

\[
a_n^{(p)} = \begin{cases} 
  B/(1 - A), & \text{if } A \neq 1, \\
  Bn, & \text{if } A = 1. 
\end{cases} \tag{98}
\]

- So \( a_n = cA^n + a_n^{(p)} \).

- In particular,

\[
c = a_0 - a_0^{(p)} = \begin{cases} 
  a_0 - B/(1 - A), & \text{if } A \neq 1, \\
  a_0, & \text{if } A = 1. 
\end{cases}
\]
\[ a_{n+1} - Aa_n = B \] Revisited (concluded)

- The solution matches Eq. (94) on p. 599.
- We can also write the solution as

\[
a_n = \begin{cases} 
    A^n [a_0 - a_0^{(p)}] + a_n^{(p)}, & \text{if } A \neq 1, \\
    a_0 + a_n^{(p)}, & \text{if } A = 1.
\end{cases}
\] (99)
Nonhomogeneous $a_n - 3a_{n-1} = 5 \times 7^n$ with $a_0 = 2$

- $a_n^{(h)} = c \times 3^n$, because the characteristic equation has the nonzero root 3.
- We propose $a_n^{(p)} = a \times 7^n$.
- Place $a \times 7^n$ into the relation to obtain $a \times 7^n - 3a \times 7^{n-1} = 5 \times 7^n$.
- Hence $a = 35/4$ and $a_n^{(p)} = (35/4) \times 7^n = (5/4) \times 7^{n+1}$.
- The general solution is $a_n = c \times 3^n + (5/4) \times 7^{n+1}$.
- Now, $c = -27/4$ because $a_0 = 2 = c + (5/4) \times 7$.
- So the solution is $a_n = -(27/4) \times 3^n + (5/4) \times 7^{n+1}$.
Nonhomogeneous \( a_n - 3a_{n-1} = 5 \times 3^n \) with \( a_0 = 2 \)

- As before, \( a_n^{(h)} = c \times 3^n \).
- But \( a_n^{(h)} \) and \( f(n) = 5 \times 3^n \) are not “independent” this time.
- So propose \( a_n^{(p)} = an \times 3^n \).
- Plug \( an \times 3^n \) into the relation to obtain
  \[ an \times 3^n - 3a(n - 1) \times 3^{n-1} = 5 \times 3^n. \]
- Hence \( a = 5 \) and \( a_n^{(p)} = 5n \times 3^n \).
- The general solution is \( a_n = c \times 3^n + 5n \times 3^n \).
- Finally, \( c = 2 \) with use of \( a_0 = 2 \).
Nonhomogeneous $a_{n+1} - 2a_n = n + 1$ with $a_0 = 4$

- From Eq. (94) on p. 599, $a_n^{(h)} = c \times 2^n$.
- Guess $a_n^{(p)} = an + b$.
- Substitute this particular solution into the relation to yield

$$a(n + 1) + b - 2(an + b) = n + 1.$$  

- Rearrange the above to obtain

$$(-a - 1)n + (a - b - 1) = 0.$$  

- This holds for all $n$ if $a = -1$ and $b = -2$. 

The Proof (concluded)

• Hence \( a_n^{(p)} = -n - 2 \).

• The general solution is

\[
a_n = c \times 2^n - n - 2.
\]

• Use the initial condition

\[
4 = a_0 = c - 2
\]

to obtain \( c = 6 \).

• The solution to the complete relation is

\[
a_n = 6 \times 2^n - n - 2.
\]
Nonhomogeneous $a_{n+1} - a_n = 2n + 3$ with $a_0 = 1$

- This equation is very similar to the previous one:

$$a_{n+1} - 2a_n = n + 1.$$ 

- First, $a_n^{(h)} = d \times 1^n = d$.

- If one guesses $a_n^{(p)} = an + b$ as before, then

$$a_{n+1} - a_n = a(n + 1) + b - an - b = a,$$

which cannot be right.\(^a\)

- So we guess $a_n^{(p)} = an^2 + bn + c$.

\(^a\)Contributed by Mr. Yen-Chieh Sung (B01902011) on June 17, 2013.
The Proof (continued)

- Substitute this particular solution into the relation to yield

\[ a(n + 1)^2 + b(n + 1) + c - (an^2 + bn + c) = 2n + 3. \]

- Simplify the above to obtain

\[ 2an + (a + b) = 2n + 3. \]

- The solutions are \( a = 1 \) and \( b = 2 \).

- Hence \( a_n^{(p)} = n^2 + 2n + c \).

- The general solution is \( a_n = n^2 + 2n + c. \)

---

\[ ^a \text{We merge} \ d \text{into} \ c. \]
The Proof (concluded)

- Use the initial condition
  
  \[ 1 = a_0 = c \]

  to obtain \( c = 1 \).

- The solution to the complete relation is
  
  \[ a_n = n^2 + 2n + 1 = (n + 1)^2. \]

- It is very different from the solution to the previous example:
  
  \[ a_n = 6 \times 2^n - n - 2. \]
Nonhomogeneous $a_{n+2} - 3a_{n+1} + 2a_n = 2$ with $a_0 = 0$ and $a_1 = 2$

- The characteristic equation $r^2 - 3r + 2 = 0$ has roots 2 and 1.
- So $a_n^{(h)} = c_11^n + c_22^n = c_1 + c_22^n$.
- Guess $a_n^{(p)} = an + b$.
- Substitute $a_n^{(p)}$ into the relation to yield
  
  $$a(n + 2) + b - 3[a(n + 1) + b] + 2(an + b) = 2.$$  

- Rearrange the above to obtain $a = -2$.
- Hence $a_n^{(p)} = -2n + b$. 

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The Proof (concluded)

- The general solution is now \( a_n = c_1 + c_22^n - 2n \).\(^a\)

- Use the initial conditions

\[
\begin{align*}
0 &= a_0 &= c_1 + c_2, \\
2 &= a_1 &= c_1 + 2c_2 - 2.
\end{align*}
\]

To obtain \( c_1 = -4 \) and \( c_2 = 4 \).

- The solution to the complete relation is

\[
a_n = -4 + 2^{n+2} - 2n.
\]

\(^a\)We merge \( b \) into \( c_1 \).
The Method of Generating Functions\textsuperscript{a}

- Consider the relation $a_n - 3a_{n-1} = n$ with $a_0 = 1$.
- Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for $a_0, a_1, \ldots$.
- From the recurrence equation,
  \[
  \sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} 3a_{n-1} x^n = \sum_{n=1}^{\infty} n x^n.
  \]
- $f(x) - a_0 - 3xf(x) = \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$ from p. 479.
- Hence
  \[
  f(x) = \frac{x}{(1-x)^2} + 1 \frac{1}{1 - 3x}.
  \]

\textsuperscript{a} Recall p. 567.
The Method of Generating Functions (continued)

• Now,

\[
f(x) = \frac{1}{1 - 3x} + \frac{x}{(1 - x)^2(1 - 3x)}
\]

\[
= \frac{7/4}{1 - 3x} + \frac{-1/4}{1 - x} + \frac{-1/2}{(1 - x)^2}
\]

by a partial fraction decomposition.

– The following equivalent form is not a partial fraction decomposition:

\[
\frac{7/4}{-3x + 1} + \frac{x - 3}{(1 - x)^2}.
\]
The Method of Generating Functions (continued)

Now,

\[
\frac{7/4}{1 - 3x} = (\frac{7}{4}) \frac{1}{1 - 3x} = (\frac{7}{4}) \sum_{n=0}^{\infty} (3x)^n,
\]

\[
\frac{-1/4}{1 - x} = -(\frac{1}{4}) \frac{1}{1 - x} = -(\frac{1}{4}) \sum_{n=0}^{\infty} x^n,
\]

\[
\frac{-1/2}{(1 - x)^2} = -(\frac{1}{2}) \frac{1}{(1 - x)^2} = -(\frac{1}{2}) \sum_{n=0}^{\infty} (n + 1) x^n, \quad \text{from p. 478.}
\]
The Method of Generating Functions (concluded)

- Now,

\[
f(x) = \left(\frac{7}{4}\right) \sum_{n=0}^{\infty} 3^n x^n - \left(\frac{1}{4}\right) \sum_{n=0}^{\infty} x^n - \left(\frac{1}{2}\right) \sum_{n=0}^{\infty} (n + 1) x^n.
\]

- So

\[
a_n = \left(\frac{7}{4}\right) 3^n - \left(\frac{1}{4}\right) - \left(\frac{1}{2}\right)(n + 1).
\]

- The methodology should be clear.
The Method of Generating Functions for
\[ a_{n+1} - a_n = 3^n \text{ with } a_0 = 1 \]

- Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) be the generating function for \( a_0, a_1, \ldots \).

- From the recurrence equation,
  \[ \sum_{n=0}^{\infty} a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} 3^n x^{n+1}. \]

- \( f(x) - a_0 - xf(x) = x \sum_{n=0}^{\infty} (3x)^n = \frac{x}{1-3x}. \)

- This implies that
  \[ f(x) = \frac{x}{1-3x} + \frac{1}{1-x} = \frac{1/2}{1-3x} + \frac{1/2}{1-x} = (1/2) \sum_{n=0}^{\infty} (3^n + 1) x^n. \]

- Hence \( a_n = (3^n + 1)/2. \)
The Method of Generating Functions for
\[ a_{n+1} - Aa_n = B \] Again

- Assume \( A \neq 1 \).

- We next obtain Eq. (99) on p. 614,
\[
a_n = A^n [a_0 - a_0^{(p)}] + a_n^{(p)},
\]
by the method of generating functions.

- Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) be the generating function for \( a_0, a_1, \ldots \).
The Proof (continued)

• Then

\[
\sum_{n=0}^{\infty} a_{n+1}x^{n+1} - \sum_{n=0}^{\infty} Aa_n x^{n+1} = \sum_{n=0}^{\infty} Bx^{n+1}.
\]

• So

\[
f(x) - a_0 - Ax f(x) = Bx \frac{1}{1-x}
\]

from p. 475.
The Proof (continued)

- Simplify the identity to yield

\[
f(x) = \frac{a_0}{1 - Ax} + \frac{Bx}{(1 - x)(1 - Ax)}
\]

\[
= \frac{a_0}{1 - Ax} + \frac{B}{1 - A} \left( \frac{1}{1 - x} - \frac{1}{1 - Ax} \right)
\]

\[
= \frac{a_0}{1 - Ax} + a_n^{(p)} \left( \frac{1}{1 - x} - \frac{1}{1 - Ax} \right)
\]

\[
= \left[ a_0 - a_n^{(p)} \right] \frac{1}{1 - Ax} + a_n^{(p)} \frac{1}{1 - x},
\]

where \(a_n^{(p)} = B/(1 - A)\), matching Eq. (98) on p. 613.
The Proof (concluded)

• From p. 475,

\[
f(x) = \left[ a_0 - a_n^{(p)} \right] \sum_{n=0}^{\infty} A^n x^n + a_n^{(p)} \sum_{n=0}^{\infty} x^n.
\]

– Note that \( a_n^{(p)} \) is independent of \( n \).

• So

\[
a_n = A^n \left[ a_0 - a_n^{(p)} \right] + a_n^{(p)},
\]

matching the earlier solution (99) on p. 614 as desired.
Convolution

• Consider the following recurrence equation,

\[ b_{n+1} = b_0 b_n + b_1 b_{n-1} + \cdots + b_{n-1} b_1 + b_n b_0. \]

• Let \( f(x) = \sum_{n=0}^{\infty} b_n x^n \).

• Then

\[ \sum_{n=0}^{\infty} b_{n+1} x^{n+1} = \sum_{n=0}^{\infty} (b_0 b_n + b_1 b_{n-1} + \cdots + b_n b_0) x^{n+1}. \]

• So \( f(x) - b_0 = x f^2(x) \) from p. 485.
The Proof (continued)

• When $b_0 = 1$,

$$f(x) = \left(1 \pm \sqrt{1 - 4x}\right)/(2x).$$

• Pick

$$f(x) = \left(1 - \sqrt{1 - 4x}\right)/(2x)$$

to match $b_0$.\(^a\)

• By Eq. (69) on p. 496,

$$\sqrt{1 - 4x} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n (-4x)^n = \sum_{n=0}^{\infty} \binom{1/2}{n} (-4)^n x^n.$$

\(^a f(0) = \infty \) if one picked $f(x) = (1 + \sqrt{1 - 4x})/(2x)$ instead (Graham, Knuth, & Patashnik, 1989).
The Proof (concluded)

• Now, by Eq. (65) on p. 494,

\[
\binom{1/2}{n}(-4)^n = \frac{1}{n!} \left( \frac{1}{2} - 1 \right) \cdots \left( \frac{1}{2} - n + 1 \right) (-4)^n = -\frac{1}{2n-1} \binom{2n}{n}.
\]

• So

\[
f(x) = \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2(2n-1)} x^{n-1} = \sum_{n=1}^{\infty} \frac{\binom{2n-2}{n-1}}{n} x^{n-1} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{n+1} x^n,
\]

the Catalan numbers (recall Eq. (18) on p. 120)!
An Example

• It is easy to verify that

\[ f(x) = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + \cdots. \]

• The coefficients indeed match

\[ \frac{0}{1}, \frac{2}{2}, \frac{4}{3}, \frac{6}{4}, \frac{8}{5}, \frac{10}{6}, \cdots. \]
A Binary Tree

\[\text{Gustav Kirchhoff (1824–1887).}\]
Number of Rooted Binary Trees

- There is a distinct node called the **root**.
- Every node has at most two descendants.
- A rooted binary tree is **ordered** if the left and right branches are considered distinct.
- What is the number $b_n$ of rooted ordered binary trees on $n$ nodes?
Illustration: $b_3 = 5$
Number of Rooted Binary Trees: The Formula

• $b_0 = 1$, as it is the empty tree.

• Recursively,

$$b_{n+1} = b_0 b_n + b_1 b_{n-1} + \cdots + b_{n-1} b_1 + b_n b_0.$$ 

- $b_i b_{n-i}$: $i$ nodes on the left and $n - i$ nodes on the right, $0 \leq i \leq n$.

• So $b_n$ is the $n$th Catalan number by Eq. (100) on p. 635:

$$b_n = \frac{\binom{2n}{n}}{n+1}.$$
An Introduction to Graph Theory
If 50 million people believe a foolish thing, it’s still a foolish thing.
— George Bernard Shaw (1856–1950)
Graphs\(^a\)

- Let \( V \) be a finite nonempty set of nodes.
- Let \( E \subseteq V \times V \) be a set of edges.
- \( G = (V, E) \) is the directed graph (or digraph) made up of the node set \( V \) and the edge set \( E \).
- When \( E \) is considered to consist of *unordered* pairs, \((V, E)\) is called an **undirected graph**.\(^b\)

---

\(^a\)Founded by Leonhard Euler in 1736.

\(^b\)Assumed unless stated otherwise.
Graphs (continued)

- A graph is **loop-free** if it contains no (self-)loops.
- A **multigraph** allows parallel edges between nodes.
Graphs (concluded)

- A loop-free undirected graph without parallel edges between nodes is said to be **simple**.
- A node is **isolated** if it has no incident edges.
- For an undirected graph, we typically use \{x, y\} to represent an edge.
- For a digraph, we always use (x, y) to represent an edge.
Illustration of Graphs

- In the following graph $G$,

$$V = \{ a, b, c, d, e, f, g, h \}$$

$$E = \{ \{ a, b \}, \{ a, e \}, \{ a, f \}, \{ b, c \}, \{ b, g \}, \{ b, f \}, \{ f, g \}, \{ f, h \}, \{ c, d \}, \{ c, h \}, \{ c, g \}, \{ d, e \}, \{ d, h \}, \{ g, h \}, \{ h, e \} \}.$$
Applications of Graph Theory

- Representation of networks, both structured ones like interconnection networks and unstructured ones like telephone networks or social networks.
- Natural representation of relations (p. 364).
- A computation can be described as a digraph.
- Optimization problems such as circuit layout.
- Physical systems such as ferromagnetism.
  
- ...
Additional Notions

• Let $G = (V, E)$ be a graph (directed or otherwise).

• $G_1 = (V_1, E_1)$ is called a **subgraph** of $G$ if
  - $\emptyset \neq V_1 \subseteq V$.
  - $E_1 \subseteq V_1 \times V_1$.
  - $E_1 \subseteq E$.

• $G_1$ is an **induced subgraph** of $G$ if it is a subgraph of $G$ and $E_1 = E \cap (V_1 \times V_1)$.

• An undirected graph $G$ is **connected** if there is a path between any two distinct nodes of $G$.

• A **component** is a maximal subgraph that is connected.
Illustration of Subgraphs
All Kinds of Walks on Undirected Graphs

- A walk from $x$ to $y$ is a finite sequence of non-loop edges connecting $x$ and $y$.
- The length of a walk is the number of edges in it.
- A walk from $x$ to $y$ where $x \neq y$ is called an open walk.
- A walk from $x$ to itself is called a closed walk.
- A walk without repeated edges is called a trail.
- A closed trail is called a circuit.
All Kinds of Walks on Undirected Graphs (concluded)

- A walk without repeated nodes is a (simple) path.
- A closed path is called a cycle.
  - A cycle must be a circuit, but not vice versa.
- By convention, a cycle has at least 3 distinct edges.
- A cycle of even length is called an even cycle.
- A cycle of odd length is called an odd cycle.
- These definitions apply to digraphs with minimum changes.
- A digraph that has no cycles is acyclic.
Illustration of Walks

- \((b, c, g, b, f)\) is a trail of length 4.
- \((a, b, c)\) is a path of length 2.
- \((a, b, c, d, e, a)\) is a cycle of length 5.
- \((g, b, c, g, h, e, a, f, g)\) is a circuit but not a cycle (as \(g\) is repeated).
Partial Order and Its Digraph Representation

• The digraph representation of a partial order\textsuperscript{a} must be acyclic.\textsuperscript{b}

• Any acyclic digraph entails a partial order.
  – Take the transitive closure of the digraph.
  – The resulting digraph clearly remains acyclic.
  – Add a loop to every node.
  – It is not hard to check that the digraph’s associated relation satisfies the definition of partial order.

\textsuperscript{a}Recall p. 371.

\textsuperscript{b}Recall p. 376.
Transitive Closure of a Digraph
Diameter

- Let $G(V, E)$ be an undirected graph.
- The **distance** between nodes $x, y \in V$ (or $d(x, y)$) is the minimum length of all the paths between $x$ and $y$.
- The **diameter** $d(G)$ of $G$ is the maximum distance over all pairs of nodes of $G$.
  - So the distance between any two nodes is at most $d(G)$.
- Diameter can be computed by an efficient all-pair-shortest-paths algorithm.\(^a\)

\(^a\)Roy (1959); Floyd (1962); Warshall (1962).
Complete Graphs

- Let $V$ be a set of $n$ nodes.
- The **complete graph** on $V$, denoted $K_n$, is a loop-free\(^a\) undirected graph.
  - There is an edge between any pair of distinct nodes.
  - $K_n$ has $\binom{n}{2}$ edges.
- The diameter of $K_n$ is clearly one.

\(^a\)Depending on applications, sometimes (self-)loops are allowed.
$K_{17}$
Complete Graphs (concluded)

- There are \( \binom{n}{i} \) ways to pick \( i \) nodes from \( K_n \).\(^a\)

- As there are \( \binom{i}{2} \) pairs of nodes, there are \( 2^{\binom{i}{2}} \) ways to pick the edges.

- Hence \( K_n \) has

\[
\sum_{i=1}^{n} \binom{n}{i} 2^{\binom{i}{2}}
\]

subgraphs.

- Can you simplify it?

\(^aK_n\) is a labeled graph.
An Inequality Relating $|V|$ and $|E|$

Lemma 71 Let $G = (V, E)$ be a simple undirected graph. Then $|V| \geq \frac{1+\sqrt{1+8|E|}}{2}$.

- $G$ has at most $\left(\frac{|V|}{2}\right)$ edges (the complete graph).
- So $V$ must be big enough such that $\left(\frac{|V|}{2}\right) \geq |E|$.
- This results in $|V|^2 - |V| \geq 2 \times |E|$, or
  \[
  \left(|V| - \frac{1}{2}\right)^2 \geq \frac{1}{4} + 2 \times |E| \geq \frac{1 + 8 \times |E|}{4}.
  \]
Complements

- The **complement** of graph $G$, denoted $\overline{G}$, is the subgraph of $K_n$ consisting of the nodes in $G$ and all edges that are *not* in $G$.

- $\overline{K}_n$, consisting of $n$ nodes and no edges, is called a **null graph**.
Degrees

- Let $G = (V, E)$ be an undirected graph.

- For each node $v \in G$, the degree of $v$, or $\text{deg}(v)$, is the number of edges in $G$ that are incident with $v$.

- A self-loop contributes two incident edges.
A Useful Identity

Lemma 72 (The handshaking theorem)

\[ \sum_{v \in V} \deg(v) = 2 \times |E|. \] (101)

- An edge is counted twice, once at each end.

Corollary 73  For finite graphs, the number of nodes of odd degree must be even.
Existence of Nodes with Identical Degree

- Let $G = (V, E)$ be a simple undirected graph without isolated nodes.
- Let $n \triangleq |V| \geq 2$.
- Observe that $1 \leq \deg(v) \leq n - 1$.
- But there are $n$ nodes.
- By the pigeonhole principle (p. 308), there must be 2 nodes with the same degree.
Regular Graphs

- A **d-regular graph** is an undirected graph such that every node has degree $d$.

- An $d$-regular graph $G = (V, E)$ must have an even number of nodes if $d$ is odd.
  - By Eq. (101) on p. 662,
    \[ 2 \times |E| = d \times |V|. \]
  - As $d$ is odd, $|V|$ must be even.
The Hypercube

• The nodes of the $n$-dimensional hypercube $Q_n$ are represented as $n$-bit numbers.$^a$
  
  – There are $2^n$ nodes.

• Two nodes are connected if they differ in one dimension.
  
  – For example, there is an edge between 00100 and 00110.

• The diameter is $n$.

• It is $n$-regular.

$^a$Recall p. 597.
The Hypercube (concluded)

- There are
  \[
  \frac{n2^n}{2} = n2^{n-1}
  \]
  undirected edges.

- The hypercube was once a popular topology for massively parallel processors (MPPs).

- The record is \( n = 16 \) set by Thinking Machine Corp.’s Connection Machine CM-2.\(^a\)

\(^a\)Hillis (1985).
Illustration with $Q_3$