Recurrence Relations

(Difference Equations)
Pure mathematics is the subject in which we do not know what we are talking about, or whether what we are saying is true.
— Bertrand Russell (1872–1970)
Recurrence Relations Arise Naturally

• When a problem has a recursive nature, recurrence relations often arise.
  – A problem can be solved by solving 2 subproblems of the same nature.

• When an algorithm is of the divide-and-conquer type, a recurrence relation describes its running time.
  – Sorting, the fast Fourier transform, etc.

• Certain combinatorial objects are constructed recursively such as hypercubes.\textsuperscript{a}

\textsuperscript{a}See p. 665.
First-Order Linear Homogeneous Recurrence Relations

• Consider the recurrence relation

\[ a_{n+1} = da_n, \]

where \( n \geq 0 \) and \( d \) is a constant.

• The \textbf{general solution} is given by

\[ a_n = Cd^n \]  \hspace{1cm} (81)

for any constant \( C \).

  – It satisfies the relation: \( Cd^{n+1} = dCd^n \).

• There are infinitely many solutions, one for each choice of \( C \).
First-Order Linear Homogeneous Recurrence Relations (concluded)

- Now suppose we impose the initial condition $a_0 = A$.
- Then the (unique) particular solution is $a_n = Ad^n$.
  - Because $A = a_0 = Cd^0 = C$.
- Note that

\[ a_n = na_{n-1} \]

is not a first-order linear homogeneous recurrence relation.
  - Its solution is $n!$ when $a_0 = 1$. 
First-Order Linear Nonhomogeneous Recurrence Relations

- Consider the recurrence relation
  \[ a_{n+1} + d a_n = f(n). \]
  - \( n \geq 0. \)
  - \( d \) is a constant.
  - \( f(n) : \mathbb{N} \to \mathbb{R}. \)
- A general solution no longer exists.
$k$th-Order Linear Homogeneous Recurrence Relations with Constant Coefficients

- Consider the $k$th-order recurrence relation

\[ C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = 0, \quad (82) \]

where $C_n, C_{n-1}, \ldots, C_{n-k} \in \mathbb{R}$, $C_n \neq 0$, and $C_{n-k} \neq 0$.

- Add $k$ initial conditions for $a_0, a_1, \ldots, a_{k-1}$.

- Clearly,

\[ a_k, a_{k+1}, \ldots \]

are well-defined.

- Indeed, $a_n$ can be calculated with $O(kn)$ operations.
\[a_{n-k} \quad a_{n-k+1} \quad \cdots \quad a_{n-1}\]

\[a_n\]
A solution \( y \) for \( a_n \) is general if for any particular solution \( y^* \), the undetermined coefficients of \( y \) can be found so that \( y \) is identical to \( y^* \).

Any general solution for \( a_n \) that satisfies the \( k \) initial conditions and Eq. (82) is a particular solution.

In fact, it is the unique particular solution because any solution agreeing at \( n = 0, 1, \ldots, k - 1 \) must agree for all \( n \geq 0 \).
Conditions for the General Solution

**Theorem 69** Let \( a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(k)} \) be \( k \) particular solutions of Eq. (82). If

\[
\begin{vmatrix}
  a_0^{(1)} & a_0^{(2)} & \cdots & a_0^{(k)} \\
  a_1^{(1)} & a_1^{(2)} & \cdots & a_1^{(k)} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{k-1}^{(1)} & a_{k-1}^{(2)} & \cdots & a_{k-1}^{(k)}
\end{vmatrix} \neq 0, \quad (83)
\]

then \( a_n = c_1 a_n^{(1)} + c_2 a_n^{(2)} + \cdots + c_k a_n^{(k)} \) is the general solution, where \( c_1, c_2, \ldots, c_k \) are arbitrary constants.

\(^a\)Samuel Goldberg (1986), *Introduction to Difference Equations.*
Fundamental Sets

• The particular solutions of Eq. (82) on p. 548,

\[ a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(k)}, \]

that also satisfy inequality (83) in Theorem 69 (p. 551) are said to form a fundamental set of solutions.

• Solving a linear homogeneous recurrence equation thus reduces to finding a fundamental set!
$k$th-Order Linear Homogeneous Recurrence Relations with Constant Coefficients: Distinct Roots

- Let $r_1, r_2, \ldots, r_k$ be the (characteristic) roots of the characteristic equation

$$C_n x^k + C_{n-1} x^{k-1} + \cdots + C_{n-k} = 0. \quad (84)$$

- If $r_1, r_2, \ldots, r_k$ are distinct, then the general solution has the form

$$a_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n,$$

for constants $c_1, c_2, \ldots, c_k$ determined by the initial conditions.
The Proof

- Assume $a_n$ has the form $cr^n$ for nonzero $c$ and $r$.
- After substitution into recurrence equation (82) on p. 548, $r$ satisfies characteristic equation (84).
- Let $r_1, r_2, \ldots, r_k$ be the $k$ distinct (nonzero) roots.
- Hence $a_n = r_i^n$ is a solution for $1 \leq i \leq k$. 
The Proof (continued)

- Solutions $r_i^n$ form a fundamental set because

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
r_1 & r_2 & \cdots & r_k \\
\vdots & \vdots & \ddots & \vdots \\
r_1^{k-1} & r_2^{k-1} & \cdots & r_k^{k-1}
\end{vmatrix} \neq 0.
\]

- The $k \times k$ matrix is called a **Vandermonde matrix**, which is nonsingular whenever $r_1, r_2, \ldots, r_k$ are distinct.\(^a\)

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\(^a\)This is a standard result in linear algebra.
The Proof (concluded)

• Hence

\[ a_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n \]

is the general solution.

• The \( k \) coefficients \( c_1, c_2, \ldots, c_k \) are determined uniquely by the \( k \) initial conditions \( a_0, a_1, \ldots, a_{k-1} \):

\[
\begin{bmatrix}
  a_0 \\
  a_1 \\
  \vdots \\
  a_{k-1}
\end{bmatrix}
= \begin{bmatrix}
  1 & 1 & \cdots & 1 \\
  r_1 & r_2 & \cdots & r_k \\
  \vdots & \vdots & \ddots & \vdots \\
  r_1^{k-1} & r_2^{k-1} & \cdots & r_k^{k-1}
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_k
\end{bmatrix}.
\]

(85)
The Fibonacci Relation

- Consider

\[ a_{n+2} = a_{n+1} + a_n. \]

- The initial conditions are \( a_0 = 0 \) and \( a_1 = 1 \) for Fibonacci numbers (p. 224).\(^a\)

- The characteristic equation is

\[ r^2 - r - 1 = 0. \]

- Its two roots are \( (1 \pm \sqrt{5})/2 \).\(^b\)

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\(^a\)Clearly \( a_n \) can be calculated with \( O(n) \) operations.

\(^b\)The golden ratio \( (1 + \sqrt{5})/2 \) has fascinated mathematicians since Pythagoras (570 B.C.–495 B.C.).
The Fibonacci Relation (continued)

• Because the roots are distinct, the fundamental set is

\[
\left\{ \left( \frac{1 + \sqrt{5}}{2} \right)^n, \left( \frac{1 - \sqrt{5}}{2} \right)^n \right\}.
\]

• The general solution is hence

\[
a_n = c_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n.
\]  \hspace{1cm} (86)
The Fibonacci Relation (continued)

• For example,

\[ a_n = \left( \frac{1 + \sqrt{5}}{2} \right)^n \]

satisfies the Fibonacci relation, as

\[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+2} = \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} + \left( \frac{1 + \sqrt{5}}{2} \right)^n. \]
The Fibonacci Relation (concluded)

• Finally, solve

\[ 0 = a_0 = c_1 + c_2 \]
\[ 1 = a_1 = c_1 \frac{1 + \sqrt{5}}{2} + c_2 \frac{1 - \sqrt{5}}{2} \]

for \( c_1 = 1/\sqrt{5} \) and \( c_2 = -1/\sqrt{5} \).

• The particular solution is thus

\[ a_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n, \quad (87) \]

known as the **Binet formula.**\(^a\)

\(^a\)So \( a_n \) can now be calculated with \( O(\log n) \) operations. There is no need to calculate \( \sqrt{5} \)!
Don’t Believe It?

\[ a_2 = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^2 - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^2 \]

\[ = \frac{1}{\sqrt{5}} \frac{1 + 2\sqrt{5} + 5}{4} - \frac{1}{\sqrt{5}} \frac{1 - 2\sqrt{5} + 5}{4} = 1. \]

\[ a_3 = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^3 - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^3 \]

\[ = \frac{1}{\sqrt{5}} \frac{1 + 3\sqrt{5} + 15 + 5\sqrt{5}}{8} - \frac{1}{\sqrt{5}} \frac{1 - 3\sqrt{5} + 15 - 5\sqrt{5}}{8} = 2. \]
Back to Inequality (83) (p. 551)

- Let us confirm the criterion for the general solution in inequality (83):

\[
\begin{vmatrix}
\frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^0 & -\frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^0 \\
\frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right) & -\frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)
\end{vmatrix}
\]

\[
= -\frac{1}{5} \left( \frac{1-\sqrt{5}}{2} \right) + \frac{1}{5} \left( \frac{1+\sqrt{5}}{2} \right)
\]

\[
= \frac{1}{\sqrt{5}} 
\]

\[
\neq 0.
\]
Impacts of the Initial Conditions

- Different initial conditions give rise to different solutions.
- Suppose $a_0 = 1$ and $a_1 = 2$.
- Then solve

\[
\begin{align*}
1 &= a_0 = c_1 + c_2, \\
2 &= a_1 = c_1 \frac{1 + \sqrt{5}}{2} + c_2 \frac{1 - \sqrt{5}}{2},
\end{align*}
\]

for

\[
\begin{align*}
c_1 &= \frac{[(1 + \sqrt{5})/2]^2}{\sqrt{5}},  \\
c_2 &= -\frac{[(1 - \sqrt{5})/2]^2}{\sqrt{5}}.
\end{align*}
\]
Impacts of the Initial Conditions (continued)

• Finally,

\[
a_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+2} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+2}.
\]

• This formula differs from the Binet formula (87) on p. 560.
Impacts of the Initial Conditions (continued)

• Suppose \( a_0 = a_1 = 1 \) instead.

• Then solve

\[
1 = a_0 = c_1 + c_2, \\
1 = a_1 = c_1 \frac{1 + \sqrt{5}}{2} + c_2 \frac{1 - \sqrt{5}}{2},
\]

for

\[
c_1 = \left[ (1 + \sqrt{5})/2 \right]/\sqrt{5}, \quad c_2 = -\left[ (1 - \sqrt{5})/2 \right]/\sqrt{5}.
\]
Impacts of the Initial Conditions (concluded)

- Finally,

\[ a_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \]  \hspace{1cm} (89)

- This formula differs from Eq. (88) on p. 564.
Generating Function for the Fibonacci Relation

• From \( a_{n+2} = a_{n+1} + a_n \), we obtain

\[
\sum_{n=0}^{\infty} a_{n+2}x^{n+2} = \sum_{n=0}^{\infty} \left( a_{n+1}x^{n+2} + a_nx^{n+2} \right).
\]

• Let \( f(x) \) be the generating function for \( \{ a_n \}_{n=0,1,2,...} \).

• Then

\[
f(x) - a_0 - a_1x = x[f(x) - a_0] + x^2f(x).
\]

• Hence

\[
f(x) = \frac{-a_0x + a_0 + a_1x}{1 - x - x^2}.
\] (90)
A Formula for the Fibonacci Numbers $a_n$

$$a_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

$$= \left\lfloor \frac{n}{2} \right\rfloor - 1 \sum_{m=0}^{[n/2]-1} \binom{n - m - 1}{m}.$$

- The left-hand side is the Binet formula (87) on p. 560.
A Formula for the Fibonacci Numbers $a_n$ (concluded)

- The generating function (90) on p. 567 gives

$$\frac{-a_0 x + a_0 + a_1 x}{1 - x - x^2}$$

$$= \frac{x}{1 - x(1 + x)}$$

$$= x + x^2(1 + x) + x^3(1 + x)^2 + \cdots + x^{n-1}(1 + x)^{n-2} + x^n(1 + x)^{n-1} + \cdots$$

$$= \cdots + \left[ \binom{n - \left\lfloor n/2 \right\rfloor}{\left\lfloor n/2 \right\rfloor - 1} + \cdots + \binom{n - 2}{1} + \binom{n - 1}{0} \right] x^n + \cdots .$$

$a$Recall that $a_0 = 0$ and $a_1 = 1.$
Binary Sequences without Consecutive 0s

• Let \( a_n \) denote the number of binary sequences of length \( n \) without consecutive 0s.

• There are \( a_{n-1} \) valid sequences with the \( n \)th symbol being 1.

• There are \( a_{n-2} \) valid sequences with the \( n \)th symbol being 0 because any such sequence must end with 10.

• Hence \( a_n = a_{n-1} + a_{n-2} \), a Fibonacci sequence.

• Because \( a_1 = 2 \) and \( a_2 = 3 \), we must have \( a_0 = 1 \) to retrofit the Fibonacci sequence.

• The formula appeared as Eq. (88) on p. 564.
Number of Subsets without Consecutive Numbers

• How many subsets of \( \{1, 2, \ldots, n\} \) contain no 2 consecutive integers?

• A binary sequences \( b_1 b_2 \cdots b_n \) of length \( n \) can be interpreted as the set \( \{ i : b_i = 0 \} \subseteq \{ 1, 2, \ldots, n \} \).

• So a subset of \( \{1, 2, \ldots, n\} \) without consecutive integers implies a binary sequence without consecutive 0s, and vice versa.

• Hence there are \( a_n \) subsets of \( \{1, 2, \ldots, n\} \) that contain no 2 consecutive integers, where \( a_n \) is the Fibonacci number with \( a_0 = 1 \) and \( a_1 = 2 \).\(^a\)

\(^a\)Recall p. 570.
Number of Subsets without Consecutive Numbers (continued)

• By formula (88) on p. 564,

\[ a_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+2} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+2}. \]

• We knew there are \( \binom{n-m+1}{m} \) \( m \)-element subsets of \( \{1, 2, \ldots, n\} \) that contain no consecutive integers.\(^a\)

\(^a\)Recall Eq. (15) on p. 94.
Number of Subsets without Consecutive Numbers (concluded)

• Hence $a_n$ also equals

$$\sum_{m=0}^{\lceil n/2 \rceil} \left( \begin{array}{c} n - m + 1 \\ m \end{array} \right).$$

• In summary,

$$\frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+2} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+2} = \sum_{m=0}^{\lceil n/2 \rceil} \left( \begin{array}{c} n - m + 1 \\ m \end{array} \right).$$

• This was the identity on p. 568.
Number of Subsets without Cyclically Consecutive Numbers

• How many subsets of \( \{1, 2, \ldots, n\} \) contain no 2 consecutive integers when 1 and \( n \) are considered consecutive?

• Let \( a_n \) be the solution for the problem on p. 571.

• So \( a_n \) is the Fibonacci number with \( a_0 = 1 \) and \( a_1 = 2 \) (formula appeared in Eq. (88) on p. 564).

• Now assume \( n \geq 3 \) first.

• There are \( a_{n-1} \) acceptable subsets that do not contain \( n \).
Number of Subsets without Cyclically Consecutive Numbers (continued)

- If \( n \) is included, an acceptable subset cannot contain 1 or \( n - 1 \).
- Hence there are \( a_{n-3} \) such subsets.
- The total is therefore \( L_n \triangleq a_{n-1} + a_{n-3} \), the **Lucas number**.\(^a\)
- It can be easily checked that

\[
L_n = a_{n-1} + a_{n-3} \\
= a_{n-2} + a_{n-3} + a_{n-4} + a_{n-5} \\
= L_{n-1} + L_{n-2}.
\]

\(^a\)Corrected by Mr. Gong-Ching Lin (B00703082) on May 19, 2012.
Number of Subsets without Cyclically Consecutive Numbers (continued)

- Furthermore, $L_0 = 2$ and $L_1 = 1$.
  - $L_3 = a_2 + a_0 = 3 + 1 = 4$ and $L_4 = a_3 + a_1 = 5 + 2 = 7$.
  - So

\[
\begin{align*}
L_2 & = L_4 - L_3 = 3, \\
L_1 & = L_3 - L_2 = 1, \\
L_0 & = L_2 - L_1 = 2.
\end{align*}
\]
Number of Subsets without Cyclically Consecutive Numbers (continued)

• The general solution is

\[ L_n = c_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n \]

by Eq. (86) on p. 558.

• Solve

\[ \begin{align*}
2 &= L_0 = c_1 + c_2, \\
1 &= L_1 = c_1 \frac{1 + \sqrt{5}}{2} + c_2 \frac{1 - \sqrt{5}}{2},
\end{align*} \]

for \( c_1 = 1 \) and \( c_2 = 1 \).
Number of Subsets without Cyclically Consecutive Numbers (concluded)

- The solution is

\[ L_n = \left(\frac{1 + \sqrt{5}}{2}\right)^n + \left(\frac{1 - \sqrt{5}}{2}\right)^n. \]
Number of Palindromes Revisited

• A palindrome is a composition for \( n \in \mathbb{Z}^+ \) that reads the same left to right as right to left (p. 109).

• Let \( a_n \) denote the number of palindromes for \( n \).

• Clearly, \( a_1 = 1 \) and \( a_2 = 2 \).

• Given each palindrome for \( n \), we can do two things to obtain a palindrome for \( n + 2 \).
  – Add 1 to the first and last summands.
    * So \( 1 + 3 + 1 \) becomes \( 2 + 3 + 2 \).
  – Insert summand 1 to the start and end.
    * So \( 1 + 3 + 1 \) becomes \( 1 + 1 + 3 + 1 + 1 \).
The Proof (continued)

- This mapping is a one-to-two correspondence (why?).
- Hence
  \[ a_{n+2} = 2a_n, \quad n \geq 1. \]
- The characteristic equation
  \[ r^2 - 2 = 0 \]
  has two roots \( \pm \sqrt{2} \).
The Proof (continued)

- The general solution is hence

\[ a_n = c_1 \left( \sqrt{2} \right)^n + c_2 \left( -\sqrt{2} \right)^n. \]

- Solve\(^a\)

\[
\begin{align*}
1 & = a_1 = \sqrt{2} (c_1 - c_2), \\
2 & = a_2 = 2(c_1 + c_2),
\end{align*}
\]

for \( c_1 = \left( 1 + \frac{1}{\sqrt{2}} \right)/2 \) and \( c_2 = \left( 1 - \frac{1}{\sqrt{2}} \right)/2. \)

\(^a\)This time, we are not retrofitting.
The Proof (concluded)

• The number of palindromes for $n$ therefore equals

$$a_n = \frac{1 + \frac{1}{\sqrt{2}}}{2} (\sqrt{2})^n + \frac{1 - \frac{1}{\sqrt{2}}}{2} (-\sqrt{2})^n$$

$$= \begin{cases} 
\frac{1 + \frac{1}{\sqrt{2}}}{2} 2^{n/2} + \frac{1 - \frac{1}{\sqrt{2}}}{2} 2^{n/2}, & \text{if } n \text{ is even,} \\
\frac{1 + \frac{1}{\sqrt{2}}}{2} \sqrt{2} 2^{(n-1)/2} - \frac{1 - \frac{1}{\sqrt{2}}}{2} \sqrt{2} 2^{(n-1)/2}, & \text{if } n \text{ is odd,} 
\end{cases}$$

$$= \begin{cases} 
2^{n/2}, & \text{if } n \text{ is even,} \\
2^{(n-1)/2}, & \text{if } n \text{ is odd,} 
\end{cases}$$

$$= 2^\lfloor n/2 \rfloor.$$ 

• It matches Theorem 20 (p. 111).
An Example: A Third-Order Relation

- Consider

\[ 2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n \]

with \( a_0 = 0 \), \( a_1 = 1 \), and \( a_2 = 2 \).

- The characteristic equation

\[ 2r^3 - r^2 - 2r + 1 = 0 \]

has three distinct real roots: 1, -1, and 0.5.

- The general solution is

\[ a_n = c_1 1^n + c_2 (-1)^n + c_3 (1/2)^n \]

\[ = c_1 + c_2 (-1)^n + c_3 (1/2)^n. \]
An Example: A Third-Order Relation (concluded)

- Solving the three initial conditions, we have\(^a\)

\[
\begin{bmatrix}
0 \\
1 \\
2
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 \\
1 & -1 & 0.5 \\
1^2 & (-1)^2 & 0.5^2
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}.
\]

- The solutions are

\[
\begin{align*}
c_1 &= 2.5, \\
c_2 &= 1/6, \\
c_3 &= -8/3.
\end{align*}
\]

\(^a\text{This is Eq. (85) on p. 556.}\)
The Case of Complex Roots

• Consider

\[ a_n = 2(a_{n-1} - a_{n-2}) \]

with \( a_0 = 1 \) and \( a_1 = 2 \).

• The characteristic equation

\[ r^2 - 2r + 2 = 0 \]

has two distinct complex roots \( 1 \pm i \).

• The general solution is

\[ a_n = c_1(1 + i)^n + c_2(1 - i)^n. \]
The Case of Complex Roots (concluded)

• Solve the two initial conditions for $c_1 = (1 - i)/2$ and $c_2 = (1 + i)/2$.

• The particular solution becomes\(^a\)

$$a_n = (1 + i)^{n-1} + (1 - i)^{n-1}$$
$$= (\sqrt{2})^n \left[ \cos(n\pi/4) + \sin(n\pi/4) \right].$$

\(^a\)An equivalent one is $a_n = (\sqrt{2})^{n+1} \cos((n - 1)\pi/4)$ by Mr. Tunglin Wu (B00902040) on May 17, 2012.
\textit{k}th-Order Linear Homogeneous Recurrence Relations with Constant Coefficients: Repeated Real Roots

- Consider the recurrence relation
  \[ C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = 0, \]
  where \( C_n, C_{n-1}, \ldots \) are real constants, \( C_n \neq 0, C_{n-k} \neq 0 \).

- Let \( r \) be a characteristic root of \textit{multiplicity} \( m \), where \( 2 \leq m \leq k \), of the characteristic equation
  \[ f(x) = C_n x^k + C_{n-1} x^{k-1} + \cdots + C_{n-k} = 0. \]

- The general solution that involves \( r \) has the form
  \[ (A_0 + A_1 n + A_2 n^2 + \cdots + A_{m-1} n^{m-1}) r^n \]
  with \( A_0, A_1, \ldots, A_{m-1} \) are constants to be determined.
The Proof

- If \( f(x) \) has a root \( r \) of multiplicity \( m \), then
  \[
  f(r) = f'(r) = \cdots = f^{(m-1)}(r) = 0.
  \]

- Because \( r \neq 0 \) is a root of multiplicity \( m \), it is easy to check that
  \[
  0 = r^{n-k} f(r),
  \]
  \[
  0 = r(r^{n-k} f(r))',
  \]
  \[
  0 = r(r(r^{n-k} f(r))')',
  \]
  \[
  \vdots
  \]
  \[
  0 = r(\cdots r(r^{n-k} f(r))' \cdots)''.
  \]
The Proof (continued)

- Note that we differentiate and then multiply by $r$ before iterating.
- These give

\[
0 = C_n r^n + C_{n-1} r^{n-1} + \cdots + C_{n-k} r^{n-k}, \\
0 = C_n n r^n + C_{n-1} (n - 1) r^{n-1} + \cdots + C_{n-k} (n - k) r^{n-k}, \\
0 = C_n n^2 r^n + C_{n-1} (n - 1)^2 r^{n-1} + \cdots + C_{n-k} (n - k)^2 r^{n-k}, \\
\vdots 
\]
The Proof (continued)

• Now, \( a_n = r^n \), \( 0 \leq k \leq m - 1 \), is indeed a solution because the \( k \)th row on the previous page says

\[
0 = \binom{n}{k} r^n + \binom{n-1}{k} r^{n-1} + \cdots + \binom{n-k}{k} r^{n-k} = C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k}.
\]
The Proof (continued)

- From Eq. (83) on p. 551, $r^n, nr^n, n^2 r^n, \ldots, n^{m-1} r^n$ form a fundamental set if\(^a\)

\[
\begin{vmatrix}
1 & 0 & \cdots & 0 \\
r & r & \cdots & r \\
r^2 & 2r^2 & \cdots & 2^{m-1}r^2 \\
\vdots & \vdots & \ddots & \vdots \\
r^{m-1} & (m-1)r^{m-1} & \cdots & (m-1)^{m-1}r^{m-1}
\end{vmatrix} \neq 0.
\]

\(^a\)The \(i\)th row sets \(n = i - 1, \ i = 1, 2, \ldots, m.\)
The Proof (concluded)

- The above is a Vandermonde matrix in disguise.

- In fact, after deleting the first row and column, the determinant equals

\[
(m - 1)! \cdot r^{1+2+\cdots+(m-1)}
\]

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 2^{m-2} \\
\vdots & \vdots & \ddots & \vdots \\
1 & (m-1) & \cdots & (m-1)^{m-2}
\end{vmatrix} \neq 0.
\]
Nonhomogeneous Recurrence Relations

• Consider
\[ C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = f(n). \]  \hspace{1cm} (92)

• Suppose \( a_n = a_{n-1} + f(n) \).

• Then the solution is
\[ a_n = a_0 + \sum_{i=1}^{n} f(i). \]

• A closed-form formula exists if one for \( \sum_{i=1}^{n} f(i) \) does.
Nonhomogeneous Recurrence Relations (concluded)

- In general, no failure-free methods exist except for special $f(n)$s.
  - See pp. 441–2 of the textbook (4th ed.).
  - See p. 532 of Rosen (2012) when $f(n)$ is the product of a polynomial in $n$ and the $n$th power of a constant.
### Examples ($c, c_1, c_2, \ldots$ Are Arbitrary Constants)

<table>
<thead>
<tr>
<th>Equation</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{n+1} - a_n = 0$</td>
<td>$a_n = c$</td>
</tr>
<tr>
<td>$a_{n+1} - a_n = 1$</td>
<td>$a_n = n + c$</td>
</tr>
<tr>
<td>$a_{n+1} - a_n = n$</td>
<td>$a_n = n(n - 1)/2 + c$</td>
</tr>
<tr>
<td>$a_{n+2} - 3a_{n+1} + 2a_n = 0$</td>
<td>$a_n = c_1 + c_2 2^n$</td>
</tr>
<tr>
<td>$a_{n+2} - 3a_{n+1} + 2a_n = 1$</td>
<td>$a_n = c_1 + c_2 2^n - n$</td>
</tr>
<tr>
<td>$a_{n+2} - a_n = 0$</td>
<td>$a_n = c_1 + c_2 (-1)^n$</td>
</tr>
<tr>
<td>$a_{n+1} = a_n/(1 + a_n)$</td>
<td>$a_n = c/(1 + cn)$</td>
</tr>
</tbody>
</table>
Trial and Error

• Consider \( a_{n+1} = 2a_n + 2^n \) with \( a_1 = 1 \).

• Calculations show that \( a_2 = 4 \) and \( a_3 = 12 \).

• Conjecture:

\[
a_n = n2^{n-1}.
\]  

(93)

• Verify that, indeed,

\[
(n + 1)2^n = 2(n2^{n-1}) + 2^n,
\]

and \( a_1 = 1 \).
Application: Number of Edges of a Hasse Diagram

- Let $a_n$ be the number of edges of the Hasse diagram for the partial order $(2\{1,2,\ldots,n\}, \subseteq)$.

- Consider the Hasse diagrams $H_1$ for $(2\{1,2,\ldots,n\}, \subseteq)$ and $H_2$ for $(\{T \cup \{n + 1\} : T \subseteq \{1, 2, \ldots, n\}\}, \subseteq)$.
  - $H_1$ and $H_2$ are “isomorphic.”

- The Hasse diagram for $(2\{1,2,\ldots,n+1\}, \subseteq)$ is constructed by adding an edge from each node $T$ of $H_1$ to node $T \cup \{n + 1\}$ of $H_2$.

- Hence $a_{n+1} = 2a_n + 2^n$ with $a_1 = 1$.

- The desired number was solved in Eq. (93) on p. 596.
Illustration with \((2^{\{1,2,3\}}, \subseteq)\)
Trial and Error Again

- Consider $a_{n+1} - Aa_n = B$.

- Calculations show that

$$
\begin{align*}
    a_1 &= Aa_0 + B, \\
    a_2 &= Aa_1 + B = A^2a_0 + B(A + 1), \\
    a_3 &= Aa_2 + B = A^3a_0 + B(A^2 + A + 1).
\end{align*}
$$

- Conjecture that is easily verified by substitution:

$$
    a_n = \begin{cases} 
      A^n a_0 + B \frac{A^{n-1}}{A-1}, & \text{if } A \neq 1, \\
      a_0 + Bn, & \text{if } A = 1.
    \end{cases} \tag{94}
$$
Will the Number of Students Explode?

- A professor teaches a required course that $N$ new students take every year.
- He flunks $0 < f < 1$ of the students taking his course.
- Failed students retake the course annually until passage.
- Does the class size $a_n$ for year $n$ explode or have a limit?
- It satisfies
  \[ a_{n+1} = fa_n + N. \]
- With $a_0 = 0$, $B = N$, and $A = f$ in Eq. (94) on p. 599,
  \[ a_n = N \frac{f^n - 1}{f - 1} \to \frac{N}{1 - f}. \]