The Multinomial Theorem

Theorem 14

\[(x_1 + x_2 + \cdots + x_t)^n = \sum_{\substack{0 \leq n_1, n_2, \ldots, n_t \leq n \\ n_1 + n_2 + \cdots + n_t = n}} \frac{n!}{n_1! n_2! \cdots n_t!} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}.\]

- Expand \((x_1 + x_2 + \cdots + x_t)^n\).
- Each term in the expansion must have the form
  \[(\text{coefficient}) \times x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t},\]
  where \(0 \leq n_1, n_2, \ldots, n_t \leq n\) and \(n_1 + n_2 + \cdots + n_t = n\).
The Proof (concluded)

- The coefficient of

\[ x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t} \]

equals the number of ways to pick \( n_1 \) \( x_1 \)'s, \( n_2 \) \( x_2 \)'s, and so on.

- By formula (2) on p. 16, there are

\[
\binom{n}{n_1, n_2, \ldots, n_t} \triangleq \frac{n!}{n_1! n_2! \cdots n_t!}
\]

ways.
Coefficient of $a^2b^3c^2d^5$ in $(a + 2b - 3c + 2d + 5)^{16}$

- Make $x_1 = a$, $x_2 = 2b$, $x_3 = -3c$, $x_4 = 2d$, and $x_5 = 5$ symbolically.

- The coefficient of $a^2(2b)^3(-3c)^2(2d)^55^4$ is

\[
\binom{16}{2, 3, 2, 5, 4} = \frac{16!}{2!3!2!5!4!} = 302,702,400
\]

by the multinomial theorem with $n = 16$.

- The desired coefficient is then

\[
302,702,400 \times 2^3 \times (-3)^2 \times 2^5 \times 5^4
= 435,891,456,000,000.
\]
Distinct Objects into Identical Containers

**Corollary 15** There are \( \frac{(rn)!}{(r!)^n n!} \) ways to distribute \( rn \) distinct objects into \( n \) identical containers so that each container contains exactly \( r \) objects.

- Consider \( (x_1 + x_2 + \cdots + x_n)^{rn} \).
  - Let \( x_i \) denote the containers (distinct, for now).
  - Each object is associated with one \( x_1 + x_2 + \cdots + x_n \).
  - It means an object can be assigned to one of the \( n \) containers.

- What does the coefficient of
  \[ x_1^r x_2^r \cdots x_n^r \]
  mean?
Distinct Objects into Identical Containers (continued)

- It is the number of ways $rn$ distinct objects can be distributed into $n$ distinct containers, each of which contains $r$ objects.

- By Theorem 14 (p. 72), it is

$$\binom{rn}{r, r, \ldots, r} \triangleq \frac{(rn)!}{r! \cdot r! \cdots r!}.$$  

- Finally, divide the above count by $n!$ to remove the identities of the containers.
Distinct Objects into Identical Containers (concluded)

Corollary 16 \[ \frac{(rn)!}{(r!)^n n!} \] is an integer.

- Immediate from Corollary 15 (p. 75).
An Alternative Proof of Corollary 16 (p. 77)\textsuperscript{a}

\[
\frac{(rn)!}{(r!)^nn!} = \frac{1}{n!} \frac{(rn)!}{[r(n-1)]!r!} \frac{[r(n-2)]!r!}{[r(n-2)]!} \ldots \frac{[r(1)]!}{[r(n-n)]!r!} \\
= \prod_{k=0}^{n-1} \left( \frac{r(n-k)}{n-k} \right) \\
= \prod_{k=0}^{n-1} \frac{r(n-k)}{n-k} = \prod_{k=0}^{n-1} \frac{[r(n-k)]!}{(n-k)r![r(n-k-1)]!} \\
= \prod_{k=0}^{n-1} \frac{r(n-k)[r(n-k)-1]!}{(n-k)r[r-1]![r(n-k-1)]!} = \prod_{k=0}^{n-1} \left( \frac{r(n-k)-1}{r-1} \right).
\]

\textsuperscript{a}Contributed by Mr. Ansel Lin (B93902003) on September 20, 2004.
Distinct Objects into Identical Containers (continued)

- Take \( n = 3 \) and \( r = 2 \).
- So we have

\[
(x_1 + x_2 + x_3)^6 = (x_1^6 + \cdots + x_3^6) \\
+ 6 (x_1^5 x_2 + \cdots + x_2 x_3^5) \\
+ 15 (x_1^4 x_2^2 + \cdots + x_2^2 x_3^4) \\
+ 20 (x_1^3 x_2^3 + \cdots + x_2^3 x_3^3) \\
+ 30 (x_1^4 x_2 x_3 + \cdots + x_1 x_2 x_3^4) \\
+ 60 (x_1^3 x_2^2 x_3 + \cdots + x_1 x_2^2 x_3^3) \\
+ 90 x_1^2 x_2^2 x_3^2.
\]
An Example (concluded)

- Indeed, the coefficients are
  \[
  \binom{6}{6}, \binom{6}{5,1}, \binom{6}{4,2}, \binom{6}{3,3}, \binom{6}{4,1,1}, \binom{6}{3,2,1}, \binom{6}{2,2,2},
  \]
  consistent with the multinomial theorem (p. 72).

- The coefficient of $x_1^2x_2^2x_3^3$ is 90.

- Thus the desired count is
  \[
  \frac{90}{3!} = 15.
  \]
Combinations (Selections) with Repetition

**Theorem 17** Suppose there are $n$ distinct objects and $r \geq 0$ is an integer. The number of selections of $r$ of these objects, with repetition, is

$$C(n + r - 1, r) = \binom{n + r - 1}{r}.$$  

- Note that the order of selection is not important.
- Imagine there are $n$ distinct types of objects.
The Proof (continued)

• Permute

\[
\begin{array}{c|c|c|c}
  \text{xx} & \ldots & x & n-1 \\
  r & & & \\
\end{array}
\]

• Think of the \(i\)th interval as containing the \(i\)th type of objects.

• So

\[
\begin{array}{|c|c|c|c|}
  xx & xxx & x & \vdots \\
\end{array}
\]

means, out of 7 distinct objects, we pick 2 type-1 objects, 3 type-2 objects, and 1 type-3 object.
The Proof (concluded)

• Our goal equals the number of permutations of

\[ \underbrace{xx \cdots x}_{r} \bigg| \underbrace{\cdots}_{n-1} \bigg|. \]

• By formula (2) on p. 16, it is

\[
\frac{(r + n - 1)!}{r!(n - 1)!} = \binom{n + r - 1}{r} = C(n + r - 1, r).
\]
Combinatorial Proof of the Hockeystick Identity (P. 39)\textsuperscript{a}

**Corollary 18** For \( m, n \geq 0 \), \( \sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m} \).

- The number of ways to select \( m \) objects out of \( n + 2 \) types is \( \binom{n+m+1}{m} \) by Theorem 17 (p. 81).
- Alternatively, let us focus on how the objects of the first \( n + 1 \) types are chosen.
- There are \( \binom{n+m}{m} \) ways to select \( m \) objects out of the first \( n + 1 \) types.
- There are \( \binom{n+m-1}{m-1} \) ways to select \( m - 1 \) objects out of the first \( n + 1 \) types and 1 object out of the last type.

\textsuperscript{a}Contributed by Mr. Jerry Lin (B01902113) on March 13, 2014.
The Proof (concluded)

• There are \( \binom{n+m-2}{m-2} \) ways to select \( m - 2 \) objects out of the first \( n + 1 \) types and 2 objects of the last type.

• ....

• So,

\[
\binom{n+m}{m} + \binom{n+m-1}{m-1} + \binom{n+m-2}{m-2} + \cdots + \binom{n+0}{0} = \binom{n+m+1}{m}.
\]
Integer Solutions of a Linear Equation

The following three problems are equivalent:

1. The number of nonnegative integer solutions of

\[ x_1 + x_2 + \cdots + x_n = r. \]

2. The number of selections, with repetition, of size \( r \) from a collection of \( n \) distinct objects (Theorem 17 on p. 81).

3. The number of ways \( r \) identical objects can be distributed among \( n \) distinct containers.\(^a\)

They all equal \( \binom{n+r-1}{r} \).\(^b\)

\(^a\)The case of distinct objects and identical containers will be covered on p. 277 (see p. 75 for a special case).

\(^b\)See p. 504 and p. 509 for alternative proofs.
Application: The Multinomial Theorem (P. 72)

• It concerned the coefficient of $x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}$ in the expansion of

$$\left(x_1 + x_2 + \cdots + x_t\right)^r.$$ 

• But let us ask how many distinct forms of summands are there?

• Each term has the form $x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}$ such that

  - $n_1 + n_2 + \cdots + n_t = r$, and
  - $0 \leq n_1, n_2, \ldots, n_t$.

• For example, consider

$$r = n_1 + n_2 + n_3 = 2.$$
Application: The Multinomial Theorem (continued)

• Now,

\[(x_1 + x_2 + x_3)^2 = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3.\]

– E.g., the solution “\(n_1 = 1, n_2 = 1, n_3 = 0\)” to \(n_1 + n_2 + n_3 = 2\) contributes 1 to the term \(x_1^1x_2^1x_3^0 = x_1x_2\).

– So there are 6 nonnegative integer solutions to \(n_1 + n_2 + n_3 = 2\) because there are 6 terms.
Application: The Multinomial Theorem (concluded)

- The desired number of terms is therefore
  \[
  \binom{r + t - 1}{r}.
  \]
  from the equivalencies on p. 86.

- Indeed, \( \binom{2+3-1}{2} = 6 \).
Positive Integer Solutions of a Linear Equation

- Consider

\[ x_1 + x_2 + \cdots + x_n = r, \]

where \( x_i > 0 \) for \( 1 \leq i \leq n \).

- Define \( x'_i \triangleq x_i - 1 \).

- The original problem becomes

\[ x'_1 + x'_2 + \cdots + x'_n = r - n, \]

where \( x'_i \geq 0 \) for \( 1 \leq i \leq n \)

- The number of solutions is therefore (p. 86)

\[
\binom{n + (r - n) - 1}{r - n} = \binom{r - 1}{r - n} = \binom{r - 1}{n - 1}.
\] (14)
Application: Subsets with Restrictions

How many $n$-element subsets of $\{1, 2, \ldots, r\}$ contain no consecutive integers?

- Say $r = 4$ and $n = 2$.
- Then the valid 2-element subsets of $\{1, 2, 3, 4\}$ are
  \[
  \{1, 3\}, \{1, 4\}, \{2, 4\}.
  \]
The Proof (continued)

• For each valid subset \( \{i_1, i_2, \ldots, i_n\} \), where \( 1 \leq i_1 < i_2 < \cdots < i_n \leq r \), define

\[
d_k = i_{k+1} - i_k.
\]

• As “placeholders,” introduce

\[
i_0 = 1, \\
i_{n+1} = r.
\]

• Then, by telescoping,

\[
d_0 + d_1 + \cdots + d_n = i_{n+1} - i_0 = r - 1.
\]
The Proof (continued)

• Observe that

\[ 0 \leq d_0, d_n \]

\[ 2 \leq d_1, d_2, \ldots, d_{n-1}. \]

• Define

\[ d'_0 \triangleq d_0, \]

\[ d'_k \triangleq d_k - 2, \quad k = 1, 2, \ldots, n - 1, \]

\[ d'_n \triangleq d_n. \]
The Proof (concluded)

• So equivalently,

\[d'_0 + d'_1 + \cdots + d'_n = r - 1 - 2(n - 1)\]

with \(0 \leq d'_0, d'_1, \ldots, d'_n\).

• The answer to the desired number is (p. 86)

\[
\begin{pmatrix}
(n + 1) + (r - 1 - 2(n - 1)) - 1 \\
r - 1 - 2(n - 1)
\end{pmatrix}

= \begin{pmatrix}
(r - n + 1) \\
r - 2n + 1
\end{pmatrix}

= \begin{pmatrix}
(r - n + 1) \\
n
\end{pmatrix}.
\]

(15)
Application: Political Majority\footnote{Recall p. 68.}

In how many ways can $2n + 1$ seats in a parliament be divided among 3 parties so that the coalition of \textit{any} 2 parties form a majority?

- If $n = 2$, there are 5 seats.
- Clearly, no party should have 3 or more seats.
- The only valid distribution of the 5 seats to 3 parties is: 2, 2, 1.
- The number of ways is therefore 3.
The Proof (continued)

- This is a problem of distributing identical objects (the seats) among distinct containers (the parties) (p. 86).

- So without the majority condition, the number is

\[
\binom{3 + (2n + 1) - 1}{2n + 1} = \binom{2n + 3}{2}.
\]

- Observe that the majority condition is violated if and only if a party gets \(n + 1\) or more seats (why?).
The Proof (concluded)

- If a given party gets \( n + 1 \) or more seats, the number of ways of distributing the seats is
  \[
  \binom{3+n-1}{n} = \binom{n+2}{2}.
  \]
  - Allocate \( n + 1 \) seats to that party before allocating the remaining \( n \) seats to the 3 parties.
  - Then refer to p. 86 for the formula.

- The desired number of no dominating party is
  \[
  \binom{2n+3}{2} - 3 \binom{n+2}{2} = \frac{n}{2} (n+1) = \binom{n+1}{2}.
  \] (16)
Political Majority: An Alternative Proof\textsuperscript{a}

- Recall that the majority condition holds if and only if no party gets \( n + 1 \) or more seats.
- So each party can hold up to \( n \) seats.
- Give each party \( n \) slots to hold real seats.
- As there are \( 2n + 1 \) seats, there will be
  \[
  3n - (2n + 1) = n - 1
  \]
  empty slots in the end.

\textsuperscript{a}Contributed by Mr. Weicheng Lee (B01902065) on March 14, 2013.
Political Majority: An Alternative Proof (concluded)

• So the answer to the desired number is the number of ways to distribute the $n - 1$ empty slots to 3 parties.

• The count is (p. 86)

$$\binom{3 + (n - 1) - 1}{n - 1} = \binom{n + 1}{n - 1} = \binom{n + 1}{2}.$$
Integer Solutions of a Linear Inequality

- Consider

\[ x_1 + x_2 + \cdots + x_n \leq r, \]

where \( x_i \geq 0 \) for \( 1 \leq i \leq n \).

- It is equivalent to

\[ x_1 + x_2 + \cdots + x_n + x_{n+1} = r, \]

where \( x_i \geq 0 \) for \( 1 \leq i \leq n + 1 \).

- The number of integer solutions of the original inequality is therefore (p. 86)

\[
\binom{(n + 1) + r - 1}{r} = \binom{n + r}{r}.
\]  

(17)
The Hockeystick Identity (P. 39) Reproved

• By Eq. (17) on p. 100, there are \( \binom{n+1+m}{m} \) nonnegative integer solutions to

\[ x_1 + x_2 + \cdots + x_{n+1} \leq m, \quad m \geq 0. \]

• By p. 86, there are \( \binom{n+k}{k} \) nonnegative integer solutions to

\[ x_1 + x_2 + \cdots + x_{n+1} = k. \]

• Any solution to \( x_1 + x_2 + \cdots + x_{n+1} \leq m \) is a solution to \( x_1 + x_2 + \cdots + x_{n+1} = k \) for some \( 0 \leq k \leq m \).
The Proof (concluded)

- The opposite is also true.
- It is also clear the correspondence is one-to-one.
- So
  \[ \sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}. \]
- This is exactly the hockeystick identity (p. 39).
Compositions of Positive Integers

- Let $m$ be a positive integer.
- A composition for $m$ is a sum of positive integers whose order is relevant and which sum to $m$.
- For $m = 3$, the number of compositions is 4:
  \[ 3, 2+1, 1+2, 1+1+1. \]
- For $m = 4$, the number of compositions is 8:
  \[ 4, 3+1, 2+2, 1+3, 1+1+2, 1+2+1, 2+1+1, 1+1+1+1. \]
- Is the number of compositions for general $m$ equal to $2^{m-1}$?
The Number of Compositions

Theorem 19 The number of compositions for $m > 0$ is $2^{m-1}$.

- Every composition with $i$ summands corresponds to a positive integer solution to
  
  \[ x_1 + x_2 + \cdots + x_i = m. \]

- So the number of solutions is $\binom{m-1}{m-i}$ by Eq. (14) on p. 90.

- The total number of compositions is therefore
  
  \[ \sum_{i=1}^{m} \binom{m-1}{m-i} = 2^{m-1} \]

  by Eq. (8) on p. 57.
An Alternative Proof for Theorem 19 (p. 104)\textsuperscript{a}

- Let $f(m)$ denote the number of compositions for $m > 0$.
- A composition for $m$ is either (1) $m$ or (2) $i$ plus a composition for $m - i$ ("$i + \cdots$") for $i = 1, 2, \ldots, m - 1$.
- Then

\[
f(m) = 1 + \sum_{i=1}^{m-1} f(m - i) = 1 + \sum_{i=1}^{m-1} f(i).
\]

- The above implies that $f(m + 1) - f(m) = f(m)$ so

\[
f(m + 1) = 2f(m).
\]

\textsuperscript{a}Contributed by Mr. Chih-Ning Chou (B01902046) on March 7, 2013.
The Proof (concluded)

• As a result,

\[ f(m) = 2^{m-1} f(1) \]

by induction.

• Finally, as \( f(1) = 1 = 2^0 \),

\[ f(m) = 2^{m-1}. \]
A Third Proof for Theorem 19 (p. 104)\textsuperscript{a}

- Start with $m$ x’s and $m - 1$ |’s.
- Consider this arrangement:

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
 & & & & & & & & & \\
\hline
x & x & x & \cdots & x \\
\hline
\end{array}
\]

- Think of the |’s as dividers.
- Now remove some of the |’s.

\textsuperscript{a}Contributed by Mr. Jerry Lin (B01902113) on March 6, 2014.
The Proof (concluded)

• For example,

\[ xx | xxx | x | x \]

means the composition

\[ 2 + 3 + 1 + 1 \]

for 7.

• Each removal of some |’s leads to a unique composition.

• As there are

\[ 2^{m-1} \]

ways to remove the |’s, this is the number of compositions for \( m \).
Palindromes of Positive Integers

Let $m$ be a positive integer.

A palindrome for $m$ is a composition for $m$ that reads the same left to right as right to left.

- For $m = 4$, the number of palindromes is 4:
  \[ 4, 1 + 2 + 1, 2 + 2, 1 + 1 + 1 + 1. \]

- For $m = 5$, the number of palindromes is 4:
  \[ 5, 1 + 3 + 1, 2 + 1 + 1 + 1 + 1 + 1. \]

- The center elements are boxed above.
Palindromes of Positive Integers (concluded)

- The numbers to the left of the center element mirror those to the right, and with the same sum.

- Palindrome is possibly the hardest form of wordplay.\(^a\)

- For example,\(^b\)

  A man, a plan, a canal, Panama!

\(^a\)Bryson (2001, p. 228).
\(^b\)Skip the blanks and punctuation marks.
The Number of Palindromes

**Theorem 20** The number of palindromes for \( m > 0 \) is \( 2^\left\lfloor m/2 \right\rfloor \).

- Assume \( m \) is even first.

- The central element of a composition of \( m \) can be \( m, m-2, \ldots, 2 \) or “+” (we will think of it as 0).\(^a\)

- When the central element is \( m \), the number of palindromes is clearly 1.

- Suppose the central element is some even number \( 0 \leq i < m \).

\(^a\)The central element must be even (why)!
The Proof (concluded)

- Then the numbers to its left sum to \((m - i)/2\).\(^a\)

- Hence the number of palindromes is \(2^{(m-i)/2-1}\) by Theorem 19 (p. 104).

- The total number of palindromes for \(m\) is thus

\[
1 + \left(1 + 2 + 2^2 + \cdots + 2^{(m-2)/2-1} + 2^{m/2-1}\right) = 2^{m/2}.
\]

- Follow the same argument when \(m\) is odd to obtain a count of \(2^{(m-1)/2}\).

\(^a\)By symmetry, the numbers to its right automatically sum to \((m - i)/2\).
Runs

- Consider a permutation of 10 Os and 5 Es:
  
  \[ \text{OOEOOOOEEEOOOEO} \]

- It has 7 runs:
  
  \[ \text{OOE} \quad \text{000} \quad \text{000E} \quad \text{EEE} \quad \text{000E} \quad \text{0} \]

- In general, a run is a \textit{maximal} consecutive list of identical objects.
The Number of Runs

Theorem 21  There are
\[
\left( \frac{m-1}{m - \lceil r/2 \rceil} \right) \left( \frac{n-1}{n - \lfloor r/2 \rfloor} \right) + \left( \frac{n-1}{n - \lfloor r/2 \rfloor} \right) \left( \frac{m-1}{m - \lfloor r/2 \rfloor} \right)
\]
ways that \(m\) identical objects of type 1 and \(n\) identical objects of type 2 can give rise to \(r\) runs.

- Suppose the run starts with a type-1 object.
- Let \(x_i\) denote the number of type-1 objects in run \(i = 1, 3, \ldots, 2\lfloor r/2 \rfloor - 1\).
The Proof (continued)

• The number of runs with the said counts $x_1, x_3, \ldots$ equals the number of positive-integer solutions to

$$x_1 + x_3 + \cdots + x_{2\lceil r/2 \rceil - 1} = m.$$ 

  - There are $\lceil r/2 \rceil$ terms.

• By Eq. (14) on p. 90, the number of solutions equals

$$\binom{m - 1}{\lceil r/2 \rceil - 1} = \binom{m - 1}{m - \lceil r/2 \rceil}.$$
The Proof (continued)

• Now let $x_i$ denote the number of type-2 objects in run $i = 2, 4, \ldots, 2\lfloor r/2 \rfloor$.

• The number of runs with the said counts $x_2, x_4, \ldots$ equals that of positive-integer solutions to

$$x_2 + x_4 + \cdots + x_{2\lfloor r/2 \rfloor} = n.$$

  – There are $\lfloor r/2 \rfloor$ terms.

• By Eq. (14) on p. 90, the number of solutions equals

$$\binom{n - 1}{\lfloor r/2 \rfloor - 1} = \binom{n - 1}{n - \lfloor r/2 \rfloor}.$$
The Proof (concluded)

• Therefore the number of runs that start with a type-1 object equals

\[
\binom{m - 1}{m - \lfloor r/2 \rfloor} \binom{n - 1}{n - \lfloor r/2 \rfloor}.
\]

• Repeat the argument for the case where the 1st run starts with a type-2 object.

• The count is

\[
\binom{n - 1}{n - \lfloor r/2 \rfloor} \binom{m - 1}{m - \lfloor r/2 \rfloor}
\]

(by swapping \( m \) and \( n \)).
The Catalan\textsuperscript{a} Numbers (1838)

- A binomial random walk starts at the origin (p. 43).
- What is the number of ways it can end at the origin in $2n$ steps \textit{without} being in the negative territory?
- A left move lowers the position, whereas a right move increases the position.
- So it is equivalent to the number of ways

\[
\underbrace{RR\cdots R}_{n} \quad \underbrace{LL\cdots L}_{n}
\]

can be permuted so that no prefix has more $L$s than $R$s.

\textsuperscript{a}Eugène Charles Catalan (1814–1894). But it was known to Euler (1707–1783) and, even earlier, Mongolian mathematician Minggatu (1730).
The Catalan Numbers (concluded)

- For example,

\[
\begin{array}{c}
0 \\
1 \\
2 \\
1 \\
0 \\
1 \\
0 \\
1 \\
R \ LRLRRLLL.
\end{array}
\]
Formula for the Catalan Number\(^a\)

The number is\(^b\)

\[
b_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n}, \quad n \geq 1. \tag{18}
\]

with \(b_0 = 1\).

\[\begin{array}{c} RR \cdots R \\ LL \cdots L \end{array} \]

• \(\begin{array}{c} RR \cdots R \\ LL \cdots L \end{array}\) can be permuted in \(\binom{2n}{n}\) ways by formula (2) on p. 16.\(^c\)

• Some of the permutations are illegal, such as

\(\text{RLLLRR}\).

\(^a\)Attributed to Jacques Touchard (1885–1968).

\(^b\)The subscript in \(b_n\) is \(n\) not \(2n!\).

\(^c\)Alternatively, recall formula (4) on p. 44.
The Proof (continued)

- We now prove that \( \binom{2n}{n-1} \) of the permutations are illegal.
- For every illegal permutation, we consider the first \( L \) move that makes the particle land at \(-1\).
  - Such as \( RL[L]LRR \).
- Swap \( L \) and \( R \) for this offending \( L \) and all earlier moves.
  - Such as \( [L]R[R]LRR \).
- The result is a permutation of

\[
\begin{array}{c}
\underbrace{RR \cdots R}^{n+1} \underbrace{LL \cdots L}^{n-1}
\end{array}
\]
The Proof (concluded)

• There are \( \binom{2n}{n-1} \) ways to permute

\[
\underbrace{RR \cdots R}_{n+1} \underbrace{L L \cdots L}_{n-1}
\]

by Eq. (2) on 16.

• But the correspondence is one-to-one between the permutations of

\[
\underbrace{RR \cdots R}_{n+1} \underbrace{L L \cdots L}_{n-1}
\]

and illegal permutations (see next page).

• So there are \( \binom{2n}{n-1} \) illegal walks.
The Reflection Principle$^a$

$^a$André (1887).
A Simple Corollary

Corollary 22  For $n \geq 1$,

$$b_n = \frac{\sum_{i=0}^{n} \binom{n}{i}^2}{n + 1}.$$  

• See Eq. (13) on p. 62.
Application: No Return to Origin until End

What is the number of ways a binomial random walk that is never in the negative territory *and* returns to the origin the *first* time after $2n$ steps?

- Let $n \geq 1$.
- The answer is $b_{n-1}$. 
Application: No Return to Origin until End (concluded)

What is the number of ways a binomial random walk returns to the origin the first time after $2n$ steps?

- Let $n \geq 1$.
- The answer is

$$2b_{n-1} = \frac{1}{2n-1} \binom{2n}{n}.$$  \hspace{1cm} (19)

- It may return to the origin by way of the negative territory.
- It may return to the origin by way of the positive territory.
Application: Nonnegative Partial Sums

What is the number of ways we can arrange $n$ “+1” and $n$ “−1” such that all $2n$ partial sums are nonnegative?

• For example, the six partial sums of $(1, 1, −1, 1, −1, −1)$ are $(1, 2, 1, 2, 1, 0)$.

• Let $n \geq 1$.

• The answer is $b_n$.

• The number remains $b_n$ if we have only $n − 1$ “−1”.
  – In the original problem, the last number must be −1.
  – So it is “redundant.”
Application: Nonpositive Partial Sums

What is the number of ways we can arrange $n$ “+1” and $n$ “−1” such that all $2n$ partial sums are nonpositive?

• For example, the six partial sums of $(-1, -1, 1, -1, 1, 1)$ are $(-1, -2, -1, -2, -1, 0)$.

• Let $n \geq 1$.

• The answer is $b_n$.

• The number remains $b_n$ if we have only $n - 1$ “+1”.
  − In the original problem, the last number must be 1.
  − So it is “redundant.”
Combinatorics and “Higher” Mathematics
For relaxation,
General Bradley did algebra problems,
and he worked at integral calculus
when he was flying an airplane
— or flying in his airplane.
He said it relaxed him, made him think.
— Chet Hansen, Major,
aide to 5-star General Omar Bradley (1893–1981)
## Growth of Factorials

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A Logplot (Base Two)

Logplot of $n!$
A Useful Lower Bound for $n!$

**Lemma 23** $n! > (n/e)^n$.

Proof:

$$\ln(n!) = \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n$$

$$= \sum_{k=1}^{n} \ln k$$

$$> \sum_{k=1}^{n} \int_{k-1}^{k} \ln x \, dx \quad \text{as} \ \ln x \ \text{is increasing}$$

$$= \int_{0}^{n} \ln x \, dx$$

$$= [x \ln x - x]_{x=0}^{n}$$

$$= n \ln n - n.$$
How Good Is the Bound?

Conclusion: good but probably not of the same order as $n!$. 
A Marginally Better Lower Bound

**Lemma 24** \( n! > e^{(n/e)^n} \).

Proof:

\[
\ln(n!) = \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n = \sum_{k=2}^{n} \ln k
\]

\[
> \sum_{k=2}^{n} \int_{k-1}^{k} \ln x \, dx
\]

\[
\geq \int_{1}^{n} \ln x \, dx = [x \ln x - x]_{x=1}^{n} = n \ln n - n + 1.
\]
A Useful Upper Bound for $C(n, m)$

Lemma 25  $C(n, m) < (ne/m)^m$ for any $0 < m < n$.\(^a\)

Proof:

\[
C(n, m) = \frac{n!}{(n - m)! \cdot m!} \\
= \frac{n(n - 1) \cdots (n - m + 1)}{m!} \\
\leq \frac{n^m}{m!} \\
< \frac{n^m}{(m/e)^m} \quad \text{by Lemma 23 (p. 133)} \\
= (ne/m)^m.
\]

\(^a\)Obtain the slightly tighter bound $(ne/m)^m/e$ with Lemma 24 (p. 136).
Stirling’s Formula\textsuperscript{a} (1730)

• The notation \( f(x) \sim g(x) \) means

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1,
\]

i.e.,

\[
f(x) = g(x) + o(g(x))
\]

as \( x \to \infty \).\textsuperscript{b}

• Stirling’s formula says:

**Theorem 26** \( n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \).

**Corollary 27** \( e = \lim_{n \to \infty} n/(n!)^{1/n} \).

\textsuperscript{a}James Stirling (1692–1770); but actually due to Abraham DeMoivre (1667–1754)!

\textsuperscript{b}It does not imply \( f(x) - g(x) \to 0 \).
Goodness of Approximation to $n!$

$n!$ over approximation
Approximation of $C(n, m)$

- Stirling’s formula can be used to approximate $C(n, m)$ better than Lemma 25 (p. 137) under some conditions.

- For that purpose, a more refined Stirling’s formula is stated below without proof:\(^a\)

\[
\sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{\frac{1}{12n}}. \quad (20)
\]

\(^a\)Robbins (1955).
The Proof (concluded)

• Now from inequalities (20) on p. 140,

\[
\begin{align*}
C(n, m) &= \frac{n!}{(n-m)! m!} \\
&< \frac{\sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{\frac{1}{12n}}}{\sqrt{2\pi (n-m)} \left( \frac{n-m}{e} \right)^{n-m} e^{\frac{1}{12(n-m)+1}} \sqrt{2\pi m} \left( \frac{m}{e} \right)^m e^{\frac{1}{12m+1}}} \\
&= \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n-m} \right)^{n-m} \sqrt{\frac{n}{m(n-m)}} \\
&\quad \times e^{\frac{1-12n-144(m-n)^2-144mn}{(\cdots)(\cdots)(\cdots)}} \\
&< \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n-m} \right)^{n-m} \sqrt{\frac{n}{m(n-m)}}. \quad (21)
\end{align*}
\]
Approximation of $C(n, m)$, $1 \leq m \leq n/2$

\[
C(n, m)
\]

\[
\begin{align*}
&> \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n - m} \right)^{n-m} \sqrt{\frac{n}{m(n - m)}} \sqrt{\frac{n}{m(n - m)}} e^{\frac{1}{12(n+1)} - \frac{1}{12(n-m)} - \frac{1}{12m}} \\
&= \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n - m} \right)^{n-m} \sqrt{\frac{n}{m(n - m)}} e^{\frac{-12m-1}{12(n-m)(12n+1)} - \frac{1}{12m}} \\
&\geq \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n - m} \right)^{n-m} \sqrt{\frac{n}{m(n - m)}} e^{\frac{-12m-1}{12m(24m+1)} - \frac{1}{12m}} \\
&= \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n - m} \right)^{n-m} \sqrt{\frac{n}{m(n - m)}} e^{-\frac{1}{6m} + \frac{1}{(24m+1)}} \\
&> \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n - m} \right)^{n-m} \sqrt{\frac{n}{m(n - m)}} e^{-\frac{1}{6m}}. \quad (22)
\end{align*}
\]
The Proof (continued)

• Combining inequalities (21) on p. 141 and (22) on p. 142 under $1 \leq m \leq n/2$, we conclude that

$$e^{-\frac{1}{6m}} \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n-m} \right)^{n-m} \sqrt{\frac{n}{m(n-m)}} < C(n,m)$$

$$< \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n-m} \right)^{n-m} \sqrt{\frac{n}{m(n-m)}}.$$
The Proof (concluded)

• So

$$C(n, m) \sim \frac{1}{\sqrt{2\pi}} \left( \frac{n}{m} \right)^m \left( \frac{n}{n-m} \right)^{n-m} \sqrt{\frac{n}{m(n-m)}}$$ (23)

as \( m \to \infty \) and \( n - m \to \infty \).

• An alternative formulation is

$$C(n, m) \sim \frac{1}{\sqrt{2\pi pqn}} (pq)^{-n},$$

where \( p \triangleq m/n \) and \( q \triangleq 1 - p \).
Application: Probability of Return to Origin

• Suppose the binomial random walk has a probability of $2^{-1} = 0.5$ of going in either direction (p. 47).
  – This is called a symmetric random walk.

• The number of ways it is at the origin after $2n$ steps is $\binom{2n}{n}$ by formula (4) on p. 44.$^a$

• The probability for this to happen is

$$\frac{\binom{2n}{n}}{2^{2n}} \approx \frac{1}{\sqrt{2\pi}} \frac{2^n}{2^{2n}} \sqrt{\frac{2}{n}} \approx \sqrt{\frac{1}{\pi n}} = O \left( \frac{1}{\sqrt{n}} \right)$$  \hspace{1cm} (24)

by Eq. (23) on p. 144.

$^a$We have seen $\binom{2n}{n}$ many times before (e.g., p. 58, p. 62, p. 120, and p. 124). We will continue to encounter it.
Application: Probability of Return to Origin (concluded)

- Suppose 100 U.S. Senators vote on a bill randomly.\(^a\)

- What is the probability of a tie (which has to be broken by the Vice President)?

- By Eq. (24), it equals

\[
\frac{\binom{100}{50}}{2^{100}} = 0.0795892 \approx \frac{1}{12}.
\]

- The probability is surprisingly high.

- It rises to 0.176197 with 20 Senators in late 18th century.

\(^a\)Dixit & Nalebuff (1993).
Application: Deviation

- Consider the symmetric random walk again.
- Its average position at the end is 0.
- Assume $n$ is even.
- Given $c > 0$, after $n$ steps what is the probability for the walk to end at a position $\geq c\sqrt{n}$ for $n$ sufficiently large?
Application: Deviation (continued)

• The probability that the walk ends at position $k$ after $n$ steps is
  \[
  \left( \frac{n}{n+k} \right)^2 2^{-n}
  \]
  by formula (4) on p. 44, where $k$ is even.

• The probability that the position is at least $c \sqrt{n}$ is
  \[
  \sum_{k=\lceil c \sqrt{n} \rceil}^{n} \left( \frac{n}{n+k} \right)^2 2^{-n} \approx \frac{1}{2} - \sum_{k=2}^{\lfloor c \sqrt{n} \rfloor} \left( \frac{n}{n+k} \right)^2 2^{-n}
  \]
  by Eq. (9) on p. 59.
  - The integer $k$ must also be even.
Application: Deviation (concluded)

• But

\[
\frac{1}{2} - \sum_{k=2}^{\lfloor c\sqrt{n} \rfloor} \left( \frac{n}{n+k} \right) 2^{-n} \geq \frac{1}{2} - 2^{-n} \frac{c\sqrt{n}}{2} \left( \frac{n}{2} \right)
\]

according to the unimodal property (p. 28).\(^a\)

– That \( k \) is even accounts for the 2 in the denominator.

• Finally, the desired probability is

\[
\frac{1}{2} - 2^{-n} \frac{c\sqrt{n}}{2} \left( \frac{n}{2} \right) \geq \frac{1}{2} - c\sqrt{\frac{1}{2\pi}}
\]

by Eq. (24) on p. 145 for \( n \) sufficiently large.

\(^a\)Corrected by Mr. Gong-Ching Lin (B00703082) on March 8, 2012 and Mr. Rajon Geng (B03902010) on March 5, 2016.
An Upper Bound for $C(2n, n)$

**Lemma 28** $\binom{2n}{n} < \frac{4^n}{\sqrt{n\pi}}$.

Proof: From inequality (21) on p. 141,

\[
\binom{2n}{n} < \frac{1}{\sqrt{2\pi}} \left( \frac{2n}{n} \right)^n \left( \frac{2n}{2n-n} \right)^{2n-n} \sqrt{\frac{2n}{n(2n-n)}}
\]

\[
= \frac{1}{\sqrt{n\pi}} 4^n.
\]

Note that Lemma 25 (p. 137) gives a much looser upper bound of $(2e)^n \sim 5.43656^n$. 
A Tight Bound for $C(2n, n)$

**Lemma 29** $\binom{2n}{n} \sim 4^n / \sqrt{n\pi}$.\(^a\)

- From inequality (22) on p. 142,

$$
\binom{2n}{n} > \frac{1}{\sqrt{2\pi}} \left(\frac{2n}{n}\right)^n \left(\frac{2n}{2n-n}\right)^{2n-n} \sqrt{\frac{2n}{n(2n-n)}} e^{-\frac{1}{6n}}
$$

$$
= \frac{1}{\sqrt{n\pi}} 4^n e^{-\frac{1}{6n}}.
$$

- Finally, recall Lemma 28 (p. 150).

\(^a\)In fact, $e^{-1/(8n)} < \sqrt{n\pi} \binom{2n}{n} / 4^n < 1$ (Hipp & Mattner, 2008).
A Tight Bound for $C'(2n, n)$ (concluded)

$\binom{2n}{n}/\left(4^n/\sqrt{n\pi}\right)$
First Return to Origin\(^a\)

What is the probability a symmetric binomial random walk returns to the origin the *first* time at step \(2n\)?

- From Eq. (19) on p. 126, the probability is
  \[
  2^{-2n} \frac{1}{2n - 1} \binom{2n}{n}.
  \]
- The above probability is asymptotically
  \[
  \sim \frac{1}{2\sqrt{n^3 \pi}}
  \]
  by Lemma 29 (p. 151).

\(^a\)Recall p. 125.
Analytic Number Theory
A proof is that which convinces a reasonable man;
a rigorous proof is that which convinces an unreasonable man.
— Mark Kac (1914–1984)
There Are an Infinite Number of Primes\textsuperscript{a}

Theorem 30 (Euclid, 300 B.C.) There are infinitely many primes.

- A prime is a positive integer larger than 1 whose only divisors are itself and 1.
- Suppose $p_1, p_2, \ldots, p_k$ are all the primes.
- Let $B = p_1 p_2 \cdots p_k + 1$.
- Because $B > p_i$ for all $i$, $B$ cannot be a prime.

\textsuperscript{a}Euclid (325 B.C.–265 B.C.). Some claim this is the most important result in all mathematics, such as Calude (1994).
The Proof (concluded)

- So there must be a prime $p_j$ such that $p_j$ divides $B = p_1p_2 \cdots p_k + 1$.

- But that implies $p_j$ must divide 1, a contradiction.
There Are an Infinite Number of Primes: An Alternative Proof\textsuperscript{a}

- Every number \( n \) can be uniquely factorized into prime factors \( p_1^{k_1} p_2^{k_2} \cdots \).

- So

\[
\left( \sum_{k=0}^{\infty} \frac{1}{2^k} \right) \left( \sum_{k=0}^{\infty} \frac{1}{3^k} \right) \left( \sum_{k=0}^{\infty} \frac{1}{5^k} \right) \cdots
\]

\[
= \sum \frac{1}{2^{k_1} 3^{k_2} 5^{k_3} \cdots}
\]

\[
= \sum_{n \geq 1} \frac{1}{n}.
\]

\textsuperscript{a}Leonhard Euler (1707–1783) in 1737.
The Proof (concluded)

• The right-hand side is an infinite number (why?).

• The left-hand side equals

\[
\frac{1}{1 - (1/2)} \frac{1}{1 - (1/3)} \frac{1}{1 - (1/5)} \cdots
\]

• It is an infinite number only if the number of primes is infinite.
Leonhard Euler (1707–1783)
The Prime Number Theorem

Let \( \pi(n) \) stand for the number of primes up to \( n \).

**Theorem 31** \( \pi(n) \sim n / \ln n \).

**Corollary 32** The average density of primes from 1 to \( n \) is \( 1 / \ln n \).

**Corollary 33** The \( n \)th prime number is about \( n \ln n \).

---

\(^a\)Jacques Salomon Hadamard (1865–1963) and Charles De la Vallée Poussin (1866–1962) in 1896. John Derbyshire (2003), “[Hadamard’s] daughter claimed he could not count beyond four, ‘After that came \( n \).’”
\( \pi(n) \text{ vs. } n / \ln n \)