Discrete Mathematics Lecture Notes

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Class Information

• Grimaldi. *Discrete and Combinatorial Mathematics.*
  – An excellent book for undergraduate students.
  – All odd-numbered exercises have an answer.
  – We more or less follow the topics of the book.
  – More “advanced” materials may be added.

• Subjects on probability theory, algorithms, boolean circuits, and information theory will be skipped as they are covered by other classes.
Class Information (concluded)

• More information and lecture notes (in PDF format) can be found at

  www.csie.ntu.edu.tw/~lyuu/dm.html

  – Exams, solutions, and teaching assistants will be announced there.

• Please ask many questions in class.

  – The best way for me to remember you in a large class.\(^a\)

\(^a\)“[A] science concentrator […] said that in his eighth semester of [Harvard] college, there was not a single science professor who could identify him by name.” (New York Times, September 3, 2003.)
Grading

• Two to three exams.

• You must show up for the exams in person.

• If you cannot make it to an exam, please email me or a TA beforehand (there should be a legitimate reason).
Fundamental Principles of Counting
All those holes and pebbles.
Who could count them?
— James Joyce, *Ulysses* (1922)

And though the holes were rather small,
they had to count them all.
Counting

• Capable of solving difficult problems.

• Sometimes useful in reproving difficult theorems in mathematics in an elementary way.

• Very useful in establishing the existence of solutions.

• Occasionally helps design efficient algorithms.

• Essential for the analysis of algorithms.

• Exact counts may not be necessary in many applications.

• Maybe the only method available to solve some open problems in complexity theory.
Founder of Combinatorics

- Archimedes (287BC–212BC) is the founder.

- Archimedes in his *Stomachion* tried to find the number of ways we put the 14 pieces together to make a square (see next page).

- The answer was 17,152.

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\(^b\)Reviel Netz (2003).
Permutation and Combination

• $n! = n \cdot (n - 1) \cdots 1$.

• $0! = 1$.

• $C(n, k) \triangleq \binom{n}{k} = \frac{n!}{k! (n - k)!}$ for $0 \leq k \leq n$.

  – Note that

  $C(0, 0) = \frac{0!}{0! 0!} = 1$. 
Permutations with Repetition

- Suppose there $n$ distinct objects and $r$ is an integer, $0 \leq r$.

- The number of permutations (linear arrangements of these objects) of size $r$ is

  $$n^r$$

  when repetitions are allowed.
  - There are $n$ choices for the 1st position, $n$ for the 2nd, etc.
The Proof (concluded)

\[
\begin{array}{cccccc}
\text{n choices} & \text{n choices} & \text{n choices} & \text{n choices} & \ldots & \text{n choices} \\
\downarrow & \downarrow & \downarrow & \downarrow & \ldots & \downarrow \\
\square & \square & \square & \square & \square & \square 
\end{array}
\]
Permutations (without Repetition)

- Suppose there are \( n \) distinct objects and \( r \) is an integer, \( 1 \leq r \leq n \).

- A permutation (without repetition) is a linear arrangement of these objects.

- The number of permutations of size \( r \) is\(^a\)

\[
P(n, r) = n(n - 1) \cdots (n - r + 1) = \frac{n!}{(n - r)!}. \quad (1)
\]

- In particular,

\[
P(n, n) = n!.
\]

\(^a\)Alternatively, the Pochhammer symbol \((n)_r\).
The Proof (concluded)
A Common Numerical Mistake!

• For some reason, students sometimes calculate

\[ \frac{n!}{(n-r)!} \]

as if it were

\[ \frac{n!}{r!(n-r)!} \]

• So \( \frac{4!}{2!} \) becomes \( \frac{4 \cdot 3}{2!} = 6 \) instead of the correct 12.

• It may be that people subconsciously calculate \( \frac{n!}{(n-r)!} \) as if it were the binomial coefficient \( \binom{n}{r} \).
Permutations with Repeated Objects

• Suppose there are $n$ objects with $n_1$ of a first type, $n_2$ of a second type, ..., and $n_r$ of an $r$th type.
  \[- \sum_{i=1}^{r} n_i = n, \ 1 \leq r \leq n.\]
  \[- \text{Objects of the same type are indistinguishable.}\]

• The number of permutations of these objects is

    $\frac{n!}{n_1! n_2! \cdots n_r!}$. (2)
Number of Permutations of MASSASAUAGA

- There are 4 As and 3 Ss.
- The other alphabets have only 1 occurrence.
- So there are
  \[ \frac{10!}{4!3!1!1!1!} = 25,200 \]
  permutations of MASSASAUAGA by formula (2).
Combinatorial Proofs

Lemma 1 \( \frac{(2k)!}{2^k} \) is an integer.

- Consider \( 2k \) symbols \( x_1, x_1, x_2, x_2, \ldots, x_k, x_k \).

- By formula (2) on p. 16, the number of ways they can be permuted is

\[
\frac{(2k)!}{2! \cdot 2! \cdots 2!} = \frac{(2k)!}{2^k}.
\]

- It must be an integer.\(^a\)

\(^a\)Algebraic proof: \( (2k)! = (2k)(2k-1) \cdots 1 = 2(k)(2k-1)2(k-1)(2k-3) \cdots 1 = 2^k (k)(2k-1)(k-1)(2k-3) \cdots 1. \)
Arrangement around a Circle

- Consider \( n \) distinct objects.
- Consider two circular arrangements to be equivalent if one can be obtained from the other by rotation.
- The number of circular arrangements is

\[
\frac{n!}{n} = (n - 1)!.
\]
Two Proofs

• First proof:
  – There are $n!$ linear arrangements.
  – For each linear arrangement, $n - 1$ others can be rotated to give the same linear arrangement.
  – So they are equivalent.

• Second proof:
  – Fix say the first object to a particular position.
  – The problem becomes that of permuting $n - 1$ distinct objects.
  – There are $(n - 1)!$ such permutations.
Circular Arrangement with Restrictions

- There are 2 types of objects.
- Consider \( n/2 \) distinct objects of type one and \( n/2 \) distinct objects of type two, where \( n \) is even.
- The number of circular arrangements where object types alternate is
  \[ (n/2)! \left( (n/2) - 1 \right)! \].
The Proof

- Fix say the first object of type one to a particular position.

- Clockwise:
  - There are \( n/2 \) ways to fill the next position with a type-two object.
  - There are \( (n/2) - 1 \) ways to fill the next position with a type-one object.
  - And so on.

- So the desired number is

\[
(n/2)[(n/2) - 1][(n/2) - 1] \cdots 1 \cdot 1 = (n/2)![(n/2) - 1]!.
\]
Galileo’s Dice$^a$

- Consider rolling three fair 6-sided dice.
- There are 6 ways to obtain a sum of 10:
  \[
  6 + 3 + 1, 6 + 2 + 2, 5 + 4 + 1, 5 + 3 + 2, 4 + 4 + 2, 4 + 3 + 3.
  \]
- There are also 6 ways to obtain a sum of 9:
  \[
  6 + 2 + 1, 5 + 3 + 1, 5 + 2 + 2, 4 + 4 + 1, 4 + 3 + 2, 3 + 3 + 3.
  \]
- Are they then equally probable?

$^a$Galileo (1564–1642).
Galileo’s Dice (concluded)

• For example, by formula (2) on p. 16:
  - $6 + 3 + 1$ is the result of $3! = 6$ different outcomes.
  - $6 + 2 + 2$ is the result of $\frac{3!}{2!} = 3$ different outcomes.

• So the number of ways to roll a 10 is

$$3! + \frac{(3!/2!)}{2} + 3! + 3! + \frac{(3!/2!)}{2} + \frac{(3!/2!)}{2} = 27.$$ 

• But the number of ways to roll a 9 is

$$3! + 3! + \frac{(3!/2!)}{2} + \frac{(3!/2!)}{2} + 3! + \frac{(3!/3!)}{3} = 25.$$ 

• Galileo concludes that 10 is slightly more likely than 9.
Combinations

- Suppose there are $n$ distinct objects and $r$ is an integer, $1 \leq r \leq n$.

- A combination is a selection (without reference to order) of some of these objects.

- The number of combinations of $r$ of these objects is

$$C(n, r) = \binom{n}{r} = \frac{n(n-1) \cdots (n-r+1)}{r(r-1) \cdots 1}.$$  

- $C(n, r) = P(n, r)/r!$ because order is irrelevant.

- By convention, $C(n, 0) = 1$ for all $n \geq 0$. 
Combinations (concluded)

• It is easy to check that

\[ C(n, r) = C(n, n - r) \quad (3) \]

for all \( n \geq 0 \).

• We shall adopt the convention that

\[ \binom{n}{i} = 0 \]

for \( i \leq 0 \) or \( i > n \), where \( n \) is a positive integer.
Basic Properties of Combinations

• A finite sequence $a_1, a_2, \ldots, a_n$ of real numbers is **unimodal** if

\[ a_1 < a_2 < \cdots < a_{j-1} \leq a_j > a_{j+1} > \cdots > a_n \]

for some positive integer $1 < j < n$.

• $C(n, 0), C(n, 1), \ldots, C(n, n)$ is unimodal.
  - Note that $C(n, r + 1)/C(n, r) = (n - r)/(r + 1)$.
  - $C(n, n/2)$ is the maximum element when $n$ is even.
  - $C(n, (n - 1)/2)$ and $C(n, (n + 1)/2)$ are the maximum elements when $n$ is odd.
An Example

How many ways are there to arrange TALLAHASSEE with no adjacent As?

- Rearrange the characters as AAAEEHLLSST.
- AAAEEHLLSST has 11 characters, among which there are 3 As.
- There are \( \frac{8!}{2!1!2!1!} = 5,040 \) ways to arrange the 8 non-A characters by formula (2) on p. 16.
- For each such arrangement, there are 9 places to insert the 3 As:
  \[
  \square \ T \square \ E \square \ E \square \ H \square \ L \square \ L \square \ S \square \ S \square
  \]
- The desired number is hence \( 5,040 \times \binom{9}{3} = 423,360 \).
A Comment$^a$

- Technically, we still miss two things in the previous proof.
  - Each legal arrangement of TALLAHASSEE has one representative in our proof.
  - No legal arrangements of TALLAHASSEE are counted twice.

- We will skip proving exhaustion and mutual exclusiveness when they are obvious.

$^a$Thanks to a lively class discussion on February 25, 2016.
Pascal’s Identity\(^\text{a}\)

**Lemma 2** \(\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}\).

- An algebraic proof.
- A combinatorial proof.
- A generating-function proof.\(^\text{b}\)

\(^{\text{a}}\)Shih-chieh Chu (1249–1314) in 1303; Blaise Pascal (1623–1662) in 1653.

\(^{\text{b}}\)See p. 485.
Newton’s\textsuperscript{a} Identity

Lemma 3 \( \binom{n}{r} \binom{r}{k} = \binom{n}{k} \binom{n-k}{r-k} \).

- Here is a combinatorial proof.
- A university has \( n \) professors.
- The faculty assembly requires \( r \) professors.
- Among the members of the assembly, \( k \) serve the executive committee.
- Let us count the number of ways the executive committee and the faculty assembly can be formed.

\textsuperscript{a}Isaac Newton (1643–1727).
The Proof (continued)

• We can first form the assembly in \( \binom{n}{r} \) ways.
• Then we pick the executive committee members from the assembly in \( \binom{r}{k} \) ways.
• The total count is \( \binom{n}{r} \binom{r}{k} \).
The Proof (concluded)

- Or we can pick the executive committee first, in \( \binom{n}{k} \) ways.

- Then we pick the remaining \( r - k \) members of the assembly in \( \binom{n-k}{r-k} \) ways.

- The total count is

\[
\binom{n}{k} \cdot \binom{n-k}{r-k}.
\]

- As we count the same thing in two different ways, the results must match.
Combinatorial Proof Again

**Lemma 4** For $0 \leq m \leq n$, \( \sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1} \).

- We want to pick \( m + 1 \) tickets from a set of \( n + 1 \) tickets.
- There are \( \binom{n+1}{m+1} \) ways.
- Alternatively, label the \( n + 1 \) tickets from 0 to \( n \).
- There are \( \binom{k}{m} \) ways to do the selection when the ticket with the largest number is \( k \) (\( m \leq k \leq n \)).
  - That is, pick the \( m \) remaining ticket numbers from \( \{ 0, 1, \ldots, k - 1 \} \).
- Alternative proof: Apply Lemma 2 (p. 31) iteratively.
Combinatorial Proof Again (continued)

Lemma 5 \( \binom{m+n}{2} - \binom{m}{2} - \binom{n}{2} = mn \).
• Consider $m$ men and $n$ women.\textsuperscript{a}

\textsuperscript{a}Facebook’s (2014) 58 gender options include: agender, androgyne, androgynous, bigender, cis, cisgender, cis female, cis male, cis man, cis woman, cisgender female, cisgender male, cisgender man, cisgender woman, female to male, FTM, gender fluid, gender nonconforming, gender questioning, gender variant, genderqueer, intersex, male to female, MTF, neither, neutrois, non-binary, other, pangender, trans, trans*, trans female, trans* female, trans male, trans* male, trans man, trans* man, trans person, trans* person, trans woman, trans* woman, transfeminine, transgender, transgender female, transgender male, transgender man, transgender person, transgender woman, transmasculine, transsexual, transsexual female, transsexual male, transsexual man, transsexual person, transsexual woman, two-spirit. (See http://techland.time.com/2014/02/14/a-comprehensive-guide-to-facebooks-new-options-for-gender-identity/ if you are confused.)
Combinatorial Proof Again (concluded)

- The number of possible heterosexual marriages is $mn$.
- On the other hand, there are $\binom{m+n}{2}$ ways to choose 2 persons.
- Among them, $\binom{m}{2} + \binom{n}{2}$ are same-sex and must be excluded.

Corollary 6 \( \binom{2n}{2} = n^2 + 2\binom{n}{2} \).
The Hockeystick Identity$^a$

**Corollary 7** For $m, n \geq 0$, \( \sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m} \).

Note that
\[
\sum_{k=0}^{m} \binom{n+k}{k} = \sum_{k=0}^{m} \binom{n+k}{n} \quad \text{by Eq. (3) on p. 27}
\]
\[
= \sum_{k=n}^{n+m} \binom{k}{n} \quad \text{by Lemma 4 (p. 35)}
\]
\[
= \binom{n+m+1}{n+1} = \binom{n+m+1}{m}.
\]

$^a$See p. 84 and p. 101 for alternative proofs.
The Hockeystick Identity (concluded)

**Corollary 8** \(1 + 2 + \cdots + n = n(n + 1)/2\).

- Set \(m = 1\) in Lemma 4 (p. 35) to obtain
  \[
  \sum_{k=1}^{n} \binom{k}{1} = \binom{n+1}{2}.
  \]

- But this is equivalent to
  \[
  \sum_{k=1}^{n} k = n(n + 1)/2,
  \]
  as desired.
Algebraic Proofs

**Corollary 9**  $1^2 + 2^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$.

- From Corollary 6 (p. 38),
  \[
  \sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} \binom{2k}{2} - 2 \sum_{k=2}^{n} \binom{k}{2}.
  \]

- By Lemma 4 (p. 35), the above equals
  \[
  \sum_{k=1}^{n} k(2k - 1) - 2 \left( \binom{n + 1}{3} \right).
  \]
Algebraic Proofs (concluded)

• So
\[ \sum_{k=1}^{n} k^2 = 2 \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k - 2 \binom{n+1}{3}. \]

• We conclude that
\[
\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} k + 2 \binom{n+1}{3}
= \frac{n(n+1)}{2} + \frac{n(n+1)(n-1)}{3}
= \frac{n(n+1)(2n+1)}{6}.
\]
Binomial Random Walk

- A particle starting at the origin can move right (up) or left (down) in each step.

- It is a standard model for stock price movements called the binomial option pricing model.\(^b\)

\(^a\)Or the Rademacher random walk.

\(^b\)Cox, Ross, & Rubinstein (1979).
Dynamics of the Binomial Random Walk

**Lemma 10**  The number of ways the particle can move from the origin to position $k$ in $n$ steps is

$$\begin{cases} 
\binom{n}{\frac{n+k}{2}}, & \text{if } n+k \text{ is even,} \\
0, & \text{otherwise.}
\end{cases} \quad (4)$$

- To reach position $k$, the number of up moves must exceed the number of down moves by exactly $k$.
- UUDUUDUUDUDD reaches position 3 in 9 steps as there are 6 Us and 3 Ds.
The Proof (continued)

• Now, solve

\[ n_u + n_d = n, \]
\[ n_u - n_d = k, \]

for \( n_u \) and \( n_d \).

• The unique solutions are

\[ n_u = \frac{(n + k)}{2}, \]
\[ n_d = \frac{(n - k)}{2}. \]

• So \( n + k \) must be even for position \( k \) to be reachable.
The Proof (concluded)

- So the desired number equals the number of ways
  \[ \binom{(n+k)/2}{(n-k)/2} \]
  \text{UU} \cdots \text{UD} \text{DD} \cdots \text{D}
  
  can be permuted.

- Finally, the desired number is
  \[
  \frac{n!}{[(n+k)/2]! [(n-k)/2]!} = \binom{n}{\frac{n+k}{2}}
  \]
  by formula (2) on p. 16.
Probability of Reaching a Position

• Suppose the binomial random walk has a probability of $p$ of going up and $1 - p$ of going down.

• The number of ways it is at position $k$ after $n$ steps is

$$\binom{n+k}{\frac{n+k}{2}}$$

by formula (4) on p. 44.

• The probability for this to happen is therefore

$$\binom{n+k}{\frac{n+k}{2}} p^{\frac{n+k}{2}} (1 - p)^{\frac{n-k}{2}}.$$
Probability of Reaching a Position (concluded)

• Alternatively, suppose a position is the result of $i$ up moves and $n - i$ down moves.
  
  – Clearly, the final position is

$$i - (n - i) = 2i - n.$$  

• The number of ways of reaching it after $n$ steps is

$$\binom{n}{i}.$$  \hspace{1cm} (5)

• The probability for this to happen is therefore

$$\binom{n}{i} p^i (1 - p)^{n-i}.$$  \hspace{1cm} (6)
Vandermonde’s Convolution\textsuperscript{a}

\[
\binom{n}{i} = \sum_{l=0}^{k} \binom{k}{l} \binom{n-k}{i-l}.
\]

\textbullet\ Let state \((i, j)\) be the result of \(j\) up moves and \(i - j\) down moves.\textsuperscript{b}

\textbullet\ Suppose the walk starts at state \((0, 0)\) and ends at \((n, i)\).\textsuperscript{c}

\textbullet\ There are \(\binom{n}{i}\) such walks.

\textsuperscript{a}Chu (1303); Alexandre-Théophile Vandermonde (1735–1796) in 1772.

\textsuperscript{b}State \((i, j)\) corresponds to \(j\) successes out of \(i\) trials or position \((i, 2j - i)\) in binomial random walks.

\textsuperscript{c}State \((n, i)\) is the result of \(i\) up moves and \(n - i\) down moves.
Vandermonde’s Convolution (continued)

• Fix a time step $k$, where $0 \leq k \leq n$.

• Every walk that reaches $(n, i)$ must go through a state $(k, l)$ for some $0 \leq l \leq k$.

• By formula (5) on p. 48, state $(k, l)$ is on

$$\binom{k}{l} \binom{n-k}{i-l}$$

walks that reach $(n, i)$, where $0 \leq k \leq n$ and $0 \leq l \leq k$. 
The labels are for states, however, not positions.
Vandermonde’s Convolution (concluded)

- Add up those walks that go through state \((k, l)\) over \(0 \leq l \leq k\) to obtain\(^{a}\)

\[
\binom{n}{i} = \sum_{l=0}^{k} \binom{k}{l} \binom{n-k}{i-l}.
\]

- Applications in artificial neural networks.\(^{b}\)

\(^{a}\)Technically, the summation should be over \(0 \leq l \leq \min(k, i)\). But recall that \(\binom{n}{i} = 0\) for \(i < 0\) or \(i > n\), where \(n\) is a positive integer.

\(^{b}\)Baum & Lyuu (1991); Lyuu & Rivin (1992).
Vandermonde’s Convolution: An Alternative Proof

- Consider $n$ distinct objects.
- There are \( \binom{n}{i} \) ways to select $i$ objects.
- Alternatively, paint $k$ of the $n$ objects red and $n - k$ of them black.
  - Objects of the same color remain distinguishable.
- Select $i$ objects again.

\(^a\)Contributed by Mr. Weicheng Lee (B01902065) on March 14, 2013.
Vandermonde’s Convolution: An Alternative Proof (concluded)

- Clearly, some \( l \) of them can be red and \( i - l \) of them black, for \( 0 \leq l \leq k \).

- The possibilities are \( \binom{k}{l} \binom{n-k}{i-l} \).

- So the total number of possible selections is

\[
\sum_{l=0}^{k} \binom{k}{l} \binom{n-k}{i-l}.
\]
Mrs. Poppleton no more understood the nature of a pun than of the binomial theorem.

― George Gissing, *The Odd Women* (1893)
The Binomial Theorem\textsuperscript{a}

**Theorem 11**

\[(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.\]

- \((x + y)^n = (x + y)(x + y) \cdots (x + y).\)
- Each term must have the form \(x^i y^{n-i}.\)
- There are \(\binom{n}{i}\) ways to pick \(i\ x's\) and \(n - i\ y's.\)

\textsuperscript{a}Attributed to Newton. First appeared in a book by Colin Maclaurin (1698–1746).
Corollaries of the Binomial Theorem

\[ 2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}. \quad (8) \]

- Set \( x = y = 1 \) in the binomial theorem.

**Corollary 12** \( \binom{n}{k} \leq 2^n \) with \( 0 \leq k \leq n \).

- A part cannot be greater than the total.
Corollaries of the Binomial Theorem (continued)

**Corollary 13** \( \binom{n}{\lfloor n/2 \rfloor} \geq 2^n / n \) for \( n \geq 2 \).\(^a\)

- Note that

\[
2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2 + \binom{n}{1} + \cdots + \binom{n}{n-1}.
\]

- Now \( \binom{n}{\lfloor n/2 \rfloor} \geq \binom{n}{k} \geq 2 \) with \( 0 < k < n \) (p. 28).

- Hence

\[
2^n \leq n \binom{n}{\lfloor n/2 \rfloor}.
\]

\(^a\)Corrected by Mr. Connor J. Shinn (T03203102) on March 5, 2015.
Corollaries of the Binomial Theorem (continued)

For odd $n$,

$$2^{n-1} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\frac{n-1}{2}}$$

$$= \binom{n}{\frac{n+1}{2}} + \binom{n}{\frac{n+3}{2}} + \cdots + \binom{n}{n}. \quad (9)$$

• Because $\binom{n}{r} = \binom{n}{n-r}$. 
Corollaries of the Binomial Theorem (continued)

- Set $x = 1$ and $y = -1$ in the binomial theorem to obtain

$$\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = 0.$$  \hfill (10)

- As a by-product, when $n > 0$,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}.$$  \hfill (11)
Corollaries of the Binomial Theorem (continued)

\[
\sum_{i=n+1}^{2n+1} \binom{2n+1}{i} = 2^{2n} = 4^n. \tag{12}
\]

- It is just Eq. (9) on p. 59!
Corollaries of the Binomial Theorem (continued)

\[
\left( \begin{array}{c} 2n \\ n \end{array} \right) = \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right)^2.
\]

(13)

• Consider

\[
f(x) = (1 + x)^n (1 + x^{-1})^n = \left( (1 + x) \cdots (1 + x) \right)^n \left( (1 + x^{-1}) \cdots (1 + x^{-1}) \right)^n
\]

\[
= \sum_{i=-n}^{n} f_i x^i.
\]

• Concentrate on the constant term \( f_0 \) of \( f(x) \).
Corollaries of the Binomial Theorem (concluded)

• \( \binom{n}{i}^2 \) is the number of ways to pick \( i \) \( x \)'s and \( i \) \( x^{-1} \)'s.

• So

\[
f_0 = \sum_{i=0}^{n} \binom{n}{i}^2.
\]

• Rewrite \( f(x) \) as

\[
f(x) = (1 + x)^n (1 + x)^n x^{-n} = x^{-n} (1 + x)^{2n}.
\]

• The constant term in \( f(x) \) is the coefficient of \( x^n \) in \( (1 + x)^{2n} \).

• So \( f_0 = \binom{2n}{n} \).

\[\text{aCorollary 13 (p. 58) gives a lower bound on } \binom{2n}{n}. \text{ See Lemma 28 (p. 150) for an upper bound.}\]
A Second Proof for Eq. (13) on P. 62

- Consider a $2n$-step binomial random walk that ends at the origin.
- There are $\binom{2n}{n}$ such walks by formula (4) on p. 44.
- Among them, consider walks that go through position $i$ at step $n$, where $n + i$ is even.
- There are $\binom{n}{(n+i)/2}^2$ such walks by formula (4) on p. 44.
- So

$$\binom{2n}{n} = \sum_{i=-n,-n+2,...,n} \binom{n}{(n+i)/2}^2 = \sum_{i=0}^{n} \binom{n}{i}^2.$$
A Third Proof for Eq. (13) on P. 62

• There are \( \binom{2n}{n} \) ways to pick \( n \) objects out of \( 2n \) distinct objects.

• Now, divide the \( 2n \) objects into two groups equally.

• There are \( \binom{n}{i} \binom{n}{n-i} = \binom{n}{i}^2 \) ways to pick \( i \) objects from the first group and the remaining \( n - i \) objects from the second.

• As \( i \) can vary from 0 to \( n \),

\[
\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2.
\]

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\(^a\)Contributed by Mr. Gong-Ching Lin (B00703082) on February 27, 2012.
A Fourth Proof for Eq. (13) on P. 62

• Recall Vandermonde’s convolution (p. 49):

\[
\binom{m}{n} = \sum_{i=0}^{k} \binom{k}{i} \binom{m-k}{n-i}.
\]

• Now choose \(m = 2n\) and \(k = n\) to obtain

\[
\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i} \binom{2n-n}{n-i} = \sum_{i=0}^{n} \binom{n}{i} \binom{n}{i} = \sum_{i=0}^{n} \binom{n}{i}^2.
\]
Corollaries of the Binomial Theorem (concluded)

\[ \sum_{i=1}^{n} i \binom{n}{i} = n2^{n-1}. \]

- Differentiate

\[ (1 + x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \]

to obtain

\[ n(1 + x)^{n-1} = \sum_{i=1}^{n} i \binom{n}{i} x^{i-1}. \]

- Now set \( x = 1. \)\(^a\)

\(^a\)An alternative proof to avoid calculus is to observe that \( i \binom{n}{i} = n \binom{n-1}{i-1}. \) So \( \sum_{i=1}^{n} i \binom{n}{i} = \sum_{i=0}^{n} \binom{n}{i} n \binom{n-1}{i-1} = n \sum_{i=1}^{n} \binom{n-1}{i-1} = n2^{n-1}. \) Contributed by Mr. Gong-Ching Lin (B00703082) on February 27, 2012.
Majority Decision

In a court with $2n + 1$ judges, in how many ways can a majority “yes” decision be handed down?

- There are $\binom{2n+1}{i}$ ways for $i$ judges to vote “yes.”
- From Eq. (12) on p. 61, the desired answer is

$$\sum_{i=n+1}^{2n+1} \binom{2n+1}{i} = 2^{2n}.$$
Ways To Merge Sets

What is the number of ways to merge members of

\[ \{ \{ 1 \}, \{ 2 \}, \ldots, \{ n \} \} \]

pairwise to form

\[ \{ \{ 1, 2, \ldots, n \} \} \]

in \( n - 1 \) steps?

- Each merge involves two members.
- For example, the number is 3 when \( n = 3 \):

\[
\begin{align*}
\{ \{ 1 \}, \{ 2 \}, \{ 3 \} \} & \rightarrow \{ \{ 1, 2 \}, \{ 3 \} \} \rightarrow \{ \{ 1, 2, 3 \} \}, \\
\{ \{ 1 \}, \{ 2 \}, \{ 3 \} \} & \rightarrow \{ \{ 1, 3 \}, \{ 2 \} \} \rightarrow \{ \{ 1, 2, 3 \} \}, \\
\{ \{ 1 \}, \{ 2 \}, \{ 3 \} \} & \rightarrow \{ \{ 2, 3 \}, \{ 1 \} \} \rightarrow \{ \{ 1, 2, 3 \} \}.
\end{align*}
\]
Ways To Merge Sets (continued)

- The 1st step begins with $n$ members.
- In general, the $i$th step begins with $n - i + 1$ members.
- There are \( \binom{n - i + 1}{2} \) ways to pick the two members.
Ways To Merge Sets (concluded)

• The desired number is thus

\[\prod_{i=1}^{n-1} \binom{n - i + 1}{2} = \binom{n}{2} \binom{n-1}{2} \cdots \binom{2}{2} \]

\[= \frac{n!(n-1)! \cdots 2!}{2^{n-1}(n-2)!(n-3)!(\cdots)1!} \]

\[= \frac{n!(n-1)!}{2^{n-1}}.\]