Problem 1 (10 points) Recall that a directed graph is called strongly connected if there is a directed path from $a$ to $b$ for all vertices $a, b$, where $a \neq b$. Prove or disprove the following claims.

1) (5 points) If a directed graph has a directed Euler circuit, then it is strongly connected.

2) (5 points) The converse holds.

Ans: If $G = (V, E)$ is a directed graph with a directed Euler circuit then for all $x, y \in V, x \neq y$, there is a directed path from $x$ to $y$ and one from $y$ to $x$, so the graph is strongly connected. On the other hand, the converse is false. The directed graph shown here is strongly connected.

![Graph Diagram]

However, since the out degree of $b$ is not equal to the in degree of $b$, the graph does not have a directed Euler circuit.

Problem 2 (10 points) Let $G = (V, E)$ be a loop-free connected undirected graph with $|V| \geq 11$. Prove that $G$ and its complement $\overline{G}$ cannot both be planar. (Recall that a loop-free connected planar graph $G = (V, E)$ with more than 2 edges has $|E| \leq 3|V| - 6$.)

Ans: Suppose that $G = (V, E)$ with $|V| = 11$. Then $\overline{G} = (V, E_1)$ where $\{a, b\} \in E_1$ if and only if $\{a, b\} \notin E$. Let $e = |E|$, $e_1 = |E_1|$, and $v = |V|$. If both $G$ and $\overline{G}$ are planar, then $e \leq 3v - 6 = 27$ and $e_1 \leq 3v - 6 = 27$. However, $e + e_1 = \binom{11}{2} = 55$. Hence either $e \geq 28$ or $e_1 \geq 28$, a contradiction.

Problem 3 (10 points) We define the total path length $f(T)$ of a binary tree $T$ as the sum of the depth of node $x$ for all nodes $x$ in $T$. Let $T_L$ and $T_R$ denote the left and right subtrees of $T$, respectively. Prove that if $T$ has $n$ nodes, then $f(T) = f(T_L) + f(T_R) + n - 1$. 
Ans: Let \( d(x, T) \) denote the depth of node \( x \) in binary tree \( T \). Because the distance to the root of \( T_L \) is one less than the distance to the root of \( T \), we have \( d(x, T_L) = d(x, T) - 1 \) for any node \( x \) in \( T_L \). Similarly, we have \( d(x, T_R) = d(x, T) - 1 \) for any node \( x \) in \( T_R \). Thus, if \( T \) has \( n \) nodes, we have \( f(T) = f(T_L) + f(T_R) + n - 1 \).

**Problem 4 (5 points)** Let \( \pi \) be the permutation,

\[
\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} = (1 \, 6)(2 \, 5)(3 \, 4).
\]

Obviously \( \pi \circ \pi = I \), the identity permutation. Consider the permutation group \( G = \langle \pi \rangle = \{ I, \pi \} \) on \( X = \{1, 2, 3, 4, 5, 6\} \). Calculate the orbits \( O_1, O_2, \ldots, O_6 \), the stabilizers \( G_1, G_2, \ldots, G_6 \), and the permutation characters \( F(I) \) and \( F(\pi) \).

Ans:
1) \( O_1 = O_6 = \{1, 6\}, O_2 = O_5 = \{2, 5\}, O_3 = O_4 = \{3, 4\} \).
2) \( G_1 = G_2 = G_3 = G_4 = G_5 = G_6 = \{I\} \).
3) \( F(I) = \{1,2,3,4,5,6\}, F(\pi) = \phi \).

**Problem 5 (15 points)** Let \( f : (G, \circ) \to (H, \circ') \) be a homomorphism.

1) (5 points) Prove that their identities correspond under \( f \), i.e. \( f(e_G) = e_H \).
2) (5 points) Prove that \( f(x^{-1}) = f(x)^{-1} \) for all \( x \in G \).
3) (5 points) Prove that \( f(a^n) = [f(a)]^n \) for all \( a \in G \) and all \( n \geq 1 \).

Ans:
1) Let \( f \) be a homomorphism. Then for any \( x \in G \),

\[
e_H \circ' f(x) = f(x) = f(e_G \circ x) = f(e_G) \circ' f(x).
\]

By the right-cancellation property, \( e_H = f(e_G) \). Finally, recall that the identities are unique.
2) By (1),

\[
e_H = f(e_G) = f(x \circ x^{-1}) = f(x) \circ' f(x^{-1}).
\]

Hence \( f(x^{-1}) \) is \( f(x) \)'s inverse.
3) For \( n = 1 \), \( f(a^1) = f(a) = [f(a)]^1 \). Assume that the result true for \( n = k \geq 1 \). Consider \( n = k + 1 \). Then

\[
f(a^{k+1}) = f(a^k \circ a) = f(a^k) \circ' f(a) = [f(a)]^k \circ' f(a) = [f(a)]^{k+1}.
\]

Thus the claims holds by induction.
Problem 6 (10 points) Solve the recurrence equation using the generating functions method:

\[ a_{n+2} - 3a_{n+1} + 2a_n = 0, \]

for \( n \geq 0 \) where \( a_0 = 1 \) and \( a_1 = 6 \). (Note: Solutions that only show the answer or not use the generating functions method will get 0 point.)

**Ans:** \( a_n = 5(2^n) - 4, \) with \( n \geq 0. \)

Problem 7 (10 points) Solve the recurrence equation using the generating functions method:

\[ a_{n+2} - 2a_{n+1} + a_n = 2^n, \]

for \( n \geq 0 \) where \( a_0 = 1 \) and \( a_1 = 2 \). (Note: Solutions that only show the answer or not use the generating functions method will get 0 point.)

**Ans:** \( a_n = 2^n, \) with \( n \geq 0. \)

Problem 8 (7 points) Draw the 14 rooted binary trees with 4 nodes.

(Unclear drawings and missing trees will get 0 point.)

**Ans:** The 14 trees are drawn below.

![Rooted ordered binary trees of 4 nodes](image)

Figure 1: Rooted ordered binary trees of 4 nodes.

Problem 9 (10 points) Prove that every group of prime order is cyclic.

**Ans:** Let \( G \) be a group of prime order \( p \) and let \( a \neq e \) be an element in \( G \). Because \( a \neq e \), we know that \( 1 < o(a) \). By the Lagrange Theorem, we know that \( o(a) \) divides \( |G| \) which is a prime, hence \( o(a) = p \). This implies that \( a \) generates \( G \), i.e., \( G \) is cyclic.

Problem 10 (8 points) Find \( 15^{-1} \) in \((\mathbb{Z}_{26}, \times)\).

**Ans:** \( 15^{-1} \equiv 7 \pmod{26}. \)

Problem 11 (5 points) Show that \( x^2 + x + 1 \) is irreducible over \( \mathbb{Z}_2[x] \).

**Ans:** Since \( x \nmid (x^2 + x + 1) \) and \( (x + 1) \nmid (x^2 + x + 1) \), we conclude that \( x^2 + x + 1 \) is irreducible over \( \mathbb{Z}_2[x] \).