The Method of Generating Functions (Recall p. 565)

• Consider the relation $a_n - 3a_{n-1} = n$ with $a_0 = 1$.

• Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for $a_0, a_1, \ldots$.

• From the recurrence equation,

\[
\sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} 3a_{n-1} x^n = \sum_{n=1}^{\infty} nx^n.
\]

• $f(x) - a_0 - 3xf(x) = \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ from p. 474.

• Hence

\[
f(x) = \frac{x}{(1-x)^2} + 1
\]

\[
= \frac{1}{1 - 3x}.
\]
The Method of Generating Functions (continued)

• Now,

\[ f(x) = \frac{1}{1 - 3x} + \frac{x}{(1 - x)^2(1 - 3x)} \]

\[ = \frac{7/4}{1 - 3x} + \frac{-1/4}{1 - x} + \frac{-1/2}{(1 - x)^2} \]

by a partial fraction decomposition.

– The following equivalent form is not a partial fraction decomposition:

\[ \frac{7/4}{-3x + 1} + \frac{x - 3}{(1 - x)^2}. \]
The Method of Generating Functions (continued)

- Now,

\[
\frac{7/4}{1 - 3x} = (7/4) \frac{1}{1 - 3x}
\]

\[
= (7/4) \sum_{n=0}^{\infty} (3x)^n,
\]

\[
\frac{-1/4}{1 - x} = -(1/4) \frac{1}{1 - x}
\]

\[
= -(1/4) \sum_{n=0}^{\infty} x^n,
\]

\[
\frac{-1/2}{(1 - x)^2} = -(1/2) \frac{1}{(1 - x)^2}
\]

\[
= -(1/2) \sum_{n=0}^{\infty} (n + 1) x^n, \quad \text{from p. 473.}
\]
The Method of Generating Functions (concluded)

- Now,

\[ f(x) = \frac{7}{4} \sum_{n=0}^{\infty} 3^n x^n - \frac{1}{4} \sum_{n=0}^{\infty} x^n - \frac{1}{2} \sum_{n=0}^{\infty} (n + 1) x^n. \]

- So

\[ a_n = \frac{7}{4} 3^n - \frac{1}{4} - \frac{1}{2} (n + 1). \]

- The methodology should be clear.
The Method of Generating Functions for 
\[ a_{n+1} - a_n = 3^n \text{ with } a_0 = 1 \]

- Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) be the generating function for \( a_0, a_1, \ldots \).

- From the recurrence equation,
  \[
  \sum_{n=0}^{\infty} a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} 3^n x^{n+1}.
  \]

- Hence \( a_n = (3^n + 1)/2 \).
The Method of Generating Functions for
\[ a_{n+1} - Aa_n = B \] Again

- Assume \( A \neq 1 \).
- We want to obtain Eq. (96) on p. 611 by the method of generating functions.
- Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) be the generating function for \( a_0, a_1, \ldots \).
The Proof (continued)

• Then

\[ \sum_{n=0}^{\infty} a_{n+1}x^n - \sum_{n=0}^{\infty} Aa_nx^n = \sum_{n=0}^{\infty} Bx^n. \]

• So

\[ \frac{f(x) - a_0}{x} - Af(x) = \frac{B}{1 - x} \]

from p. 470.
The Proof (continued)

- Simplify the identity to yield

\[
    f(x) = \frac{a_0}{1 - Ax} + \frac{Bx}{(1 - x)(1 - Ax)}
    \]

\[
    = \frac{a_0}{1 - Ax} + \frac{B}{1 - A} \left( \frac{1}{1 - x} - \frac{1}{1 - Ax} \right)
    \]

\[
    = \frac{a_0}{1 - Ax} + a_n^{(p)} \left( \frac{1}{1 - x} - \frac{1}{1 - Ax} \right)
    \]

\[
    = \left[ a_0 - a_n^{(p)} \right] \frac{1}{1 - Ax} + a_n^{(p)} \frac{1}{1 - x},
    \]

where \( a_n^{(p)} \triangleq B/(1 - A) \).
The Proof (concluded)

• From p. 470,

\[ f(x) = [a_0 - a_n^{(p)}] \sum_{n=0}^{\infty} A^n x^n + a_n^{(p)} \sum_{n=0}^{\infty} x^n. \]

- Note that \(a_n^{(p)}\) is independent of \(n\).

• So

\[ a_n = A^n [a_0 - a_n^{(p)}] + a_n^{(p)}, \]

matching the earlier solution on p. 611 as desired.
Convolutions

• Consider the following recurrence equation,

\[ b_{n+1} = b_0 b_n + b_1 b_{n-1} + \cdots + b_{n-1} b_1 + b_n b_0. \]

• Let \( f(x) = \sum_{n=0}^{\infty} b_n x^n. \)

• Then

\[ \sum_{n=0}^{\infty} b_{n+1} x^{n+1} = \sum_{n=0}^{\infty} (b_0 b_n + b_1 b_{n-1} + \cdots + b_n b_0) x^{n+1}. \]

• So \( f(x) - b_0 = x f^2(x) \) from p. 480.
The Proof (continued)

- When \( b_0 = 1 \),
  
  \[
  f(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.
  \]

- Pick
  
  \[
  f(x) = \frac{1 - \sqrt{1 - 4x}}{2x}
  \]
  
  to match \( b_0 \).\(^a\)

- By Eq. (67) on p. 491,
  
  \[
  \sqrt{1 - 4x} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n = \sum_{n=0}^{\infty} \binom{1/2}{n} (-4)^n x^n.
  \]

\(^a f(0) = \infty \) if one picks + (Graham, Knuth, & Patashnik, 1989).
The Proof (concluded)

- Now, by Eq. (63) on p. 489,
  \[
  \frac{1}{2} \left( \frac{1}{2} - 1 \right) \cdots \left( \frac{1}{2} - n + 1 \right) \frac{(-4)^n}{n!} = - \frac{1}{2n-1} \binom{2n}{n}.
  \]

- So
  \[
  f(x) = \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2(2n-1)} x^{n-1} = \sum_{n=1}^{\infty} \frac{\binom{2n-2}{n-1}}{n} x^{n-1} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{n+1} x^n, \tag{97}
  \]
  the Catalan numbers (recall Eq. (18) on p. 119)!
An Example

- It is easy to verify that
  \[ f(x) = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + \cdots. \]

- The coefficients are indeed
  \[
  \frac{0}{1}, \frac{2}{1}, \frac{4}{3}, \frac{6}{4}, \frac{8}{5}, \frac{10}{6}, \cdots.
  \]
A Binary Tree\textsuperscript{a}

\textsuperscript{a}Gustav Kirchhoff (1824–1887).
Number of Rooted Binary Trees

- There is a distinct node called the root.
- A rooted binary tree is ordered if the left and right branches are considered distinct.
- What is the number $b_n$ of rooted ordered binary trees on $n$ nodes?
Illustration: $b_3 = 5$
Number of Rooted Binary Trees: The Formula

- $b_0 = 1$, as it is the empty tree.
- Recursively,
  \[ b_{n+1} = b_0b_n + b_1b_{n-1} + \cdots + b_{n-1}b_1 + b_nb_0. \]
  - $b_ib_{n-i}$: $i$ nodes on the left and $n - i$ nodes on the right, $0 \leq i \leq n$.
- So $b_n$ is the $n$th Catalan number by Eq. (97) on p. 632:
  \[ b_n = \frac{{2n \choose n}}{n+1}. \]
An Introduction to Graph Theory
If 50 million people believe a foolish thing,
it’s still a foolish thing.
— George Bernard Shaw (1856–1950)
Graphs\textsuperscript{a}

- Let $V$ be a finite nonempty set of nodes.
- Let $E \subseteq V \times V$ be a set of edges.
- $G = (V, E)$ is the directed graph (or digraph) made up of the node set $V$ and the edge set $E$.
- When $E$ is considered to consist of unordered pairs, $(V, E)$ is called an \textit{undirected graph}.\textsuperscript{b}

\textsuperscript{a}Founded by Leonhard Euler in 1736.
\textsuperscript{b}Assumed unless stated otherwise.
Graphs (continued)

- A graph is **loop-free** if it contains no (self-)loops.
- A **multigraph** allows parallel edges between nodes.
Graphs (concluded)

- A loop-free undirected graph without parallel edges between nodes is said to be **simple**.
- For an undirected graph, we typically use \( \{x, y\} \) to represent an edge.
- For a digraph, we always use \((x, y)\) to represent an edge.
Illustration of Graphs

- In the following graph $G$,

\[
V = \{ a, b, c, d, e, f, g, h \}
\]

\[
E = \{ \{ a, b \}, \{ a, e \}, \{ a, f \}, \{ b, c \}, \{ b, g \}, \{ b, f \}, \\
\{ f, g \}, \{ f, h \}, \{ c, d \}, \{ c, h \}, \{ c, g \}, \\
\{ d, e \}, \{ d, h \}, \{ g, h \}, \{ h, e \} \}.
\]
Applications of Graph Theory

• Representation of networks, both structured ones like interconnection networks and unstructured ones like telephone networks or social networks.

• Natural representation of relations (p. 362).

• Practically any computation can be described as a graph.

• Optimization problems such as circuit layout.

• Physical systems such as ferromagnetism.

• …
Additional Notions

• Let $G = (V, E)$ be a graph (directed or otherwise).

• $G_1 = (V_1, E_1)$ is called a subgraph of $G$ if
  - $\emptyset \neq V_1 \subseteq V$.
  - $E_1 \subseteq V_1 \times V_1$.
  - $E_1 \subseteq E$.

• $G_1$ is an induced subgraph of $G$ if it is a subgraph of $G$ and $E_1 = E \cap (V_1 \times V_1)$.

• An undirected graph $G$ is connected if there is a path between any two distinct nodes of $G$.

• A component is a maximal subgraph that is connected.
Illustration of Subgraphs
All Kinds of Walks on Undirected Graphs

- A **walk** from $x$ to $y$ is a finite sequence of non-loop edges connecting $x$ and $y$.
- The **length** of a walk is the number of **edges** in it.
- A walk from $x$ to $y$ where $x \neq y$ is called an **open walk**.
- A walk from $x$ to itself is called a **closed walk**.
- A walk without repeated **edges** is called a **trail**.
- A closed trail is called a **circuit**.
All Kinds of Walks on Undirected Graphs (concluded)

- A walk without repeated nodes is a \textit{(simple)} path.
- A closed path is called a \textbf{cycle}.
  - A cycle must be a circuit, but not vice versa.
- By convention, a cycle has at least 3 distinct edges.
- A cycle of even length is called an \textbf{even cycle}; a cycle of odd length is called an \textbf{odd cycle}.
- These definitions apply to digraphs with minimal changes.
- A digraph that has no cycles is called \textbf{acyclic}. 
Illustration of Walks

- $(b, c, g, b, f)$ is a trail of length 4.
- $(a, b, c)$ is a path of length 2.
- $(a, b, c, d, e, a)$ is a cycle of length 5.
- $(g, b, c, g, h, e, a, f, g)$ is a circuit but not a cycle.
Partial Order and Its Digraph Representation

- The digraph representation of a partial order (p. 369) must be acyclic.
  - Recall p. 374.\textsuperscript{a}

- Any acyclic digraph entails a partial order.
  - Take the transitive closure of the digraph.
  - The resulting digraph clearly remains acyclic.
  - Add a loop to every node.
  - It is not hard to check that the digraph’s associated relation satisfies the definition of partial order.

\textsuperscript{a}We called cycles “loops” there.
Transitive Closure of a Digraph
Diameter

- Let $G(V, E)$ be an undirected graph.

- The **distance** between nodes $x, y \in V$ (or $d(x, y)$) is the minimum length of all the paths between $x$ and $y$.

- The **diameter** $d(G)$ of $G$ is the maximum distance over all pairs of nodes of $G$.
  
  - So any two nodes must have distance at most $d(G)$ between them.

- Diameter can be computed by an efficient all-pair-shortest-paths algorithm.\(^a\)

\(^a\)Roy (1959); Floyd (1962); Warshall (1962).
Complete Graphs

- Let $V$ be a set of $n$ nodes.

- The complete graph on $V$, denoted $K_n$, is a loop-free undirected graph.
  - There is an edge between any pair of distinct nodes.
  - $K_n$ has $\binom{n}{2}$ edges.
  - Depending on applications, sometimes (self-)loops are allowed.

- The diameter of $K_n$ is clearly one.
$K_{17}$
Complete Graphs (concluded)

- There are \( \binom{n}{i} \) ways to pick \( i \) nodes from \( K_n \).\(^a\)

- As there are \( \binom{i}{2} \) pairs of nodes, there are \( 2^{\binom{i}{2}} \) ways to pick the edges.

- Hence \( K_n \) has

\[
\sum_{i=1}^{n} \binom{n}{i} 2^{\binom{i}{2}}
\]

subgraphs.

- Can you simplify it?

\(^a\)Recall that \( K_n \) is labeled.
An Inequality Relating $|V|$ and $|E|$

Lemma 72 Let $G = (V, E)$ be an undirected graph. Then $|V| \geq \frac{1+\sqrt{1+8\times |E|}}{2}$.

- $G$ has at most $\left(\frac{|V|}{2}\right)$ edges (the complete graph).
- So $V$ must be big enough such that $\left(\frac{|V|}{2}\right) \geq |E|$.
- This results in $|V|^2 - |V| \geq 2 \times |E|$, or

\[
\left( |V| - \frac{1}{2} \right)^2 \geq \frac{1}{4} + 2 \times |E| \geq \frac{1 + 8 \times |E|}{4}.
\]
Complements

- The complement of graph $G$, denoted $\overline{G}$, is the subgraph of $K_n$ consisting of the nodes in $G$ and all edges that are not in $G$.

- $\overline{K}_n$, consisting of $n$ nodes and no edges is called a null graph.
Degrees

• Let $G = (V, E)$ be an undirected graph.

• For each node $v \in G$, the **degree** of $v$, or $\text{deg}(v)$, is the number of edges in $G$ that are incident with $v$.

• A loop is considered as *two* incident edges.
A Useful Identity

Lemma 73 (The handshaking theorem)

$$\sum_{v \in V} \deg(v) = 2 \times |E|.$$  \hfill (98)

- An edge is counted twice, once at each end.

Corollary 74  *For finite graphs, the number of nodes of odd degree must be even.*
Existence of Nodes with Identical Degree

- Let $G = (V, E)$ be a loop-free connected undirected graph with $n = |V| \geq 2$.
- Observe that $1 \leq \deg(v) \leq n - 1$.
- But there are $n$ nodes.
- By the pigeonhole principle (p. 305), there must be 2 nodes with the same degree.
Regular Graphs

• A $d$-regular graph is an undirected graph such that every node has degree $d$.

• An $d$-regular graph $G = (V, E)$ must have an even number of nodes if $d$ is odd.
  
  – By Eq. (98) on p. 659, $2 \times |E| = d \times |V|$.  
  – As $d$ is odd, $|V|$ must be even.
The Hypercube

- The nodes of the $n$-dimensional hypercube $Q_n$ are represented as $n$-bit numbers (see p. 594).
  - There are $2^n$ nodes.
- Two nodes are connected if they differ in one dimension.
  - For example, there is an edge between 00100 and 00110.
  - The diameter is $n$.
  - It is $n$-regular.
  - There are
    \[
    \frac{n2^n}{2} = n2^{n-1}
    \]
    undirected edges.
The Hypercube (concluded)

- The hypercube was once a popular topology for massively parallel processors (MPPs).

- The record is $n = 16$ set by Thinking Machine Corp.’s Connection Machine CM-2.\textsuperscript{a}

\textsuperscript{a}Hillis (1985).
Illustration with $Q_3$
Bipartite Graphs

- A graph $G = (V, E)$ is called **bipartite** if:
  - $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.
  - Every edge is of the form $\{x, y\}$ with $x \in V_1$ and $y \in V_2$.

- Express the above bipartite graph as

$$G = (V_1, V_2, E).$$
Bipartite Graphs (continued)

• If each node in $V_1$ is joined with every node in $V_2$, we have a complete bipartite graph.

• If $|V_1| = m$ and $|V_2| = n$, the complete bipartite graph is denoted by $K_{m,n}$. 
$K_{5,5}$
Bipartite Graphs (concluded)

- Let graph $G = (V, E) = (V_1, V_2, E)$ be bipartite.
- Then $G$ has at most $|V_1| \times |V_2|$ edges.
- Let $|V| = n$, $|E| = e$, and $|V_1| = m$.
- Then $e \leq (n - m) m$, which is maximized at (1) $m = n/2$ when \( n \) is even and (2) $m = (n \pm 1)/2$ when \( n \) is odd.
- In either case,
  \[
e \leq (n/2)^2.
  \]
- Hence a graph with $e > (n/2)^2$ cannot be bipartite.
Euler Circuits and Trails

- Let $G = (V, E)$ be an undirected graph or multigraph with no isolated nodes.
  - Isolated nodes are nodes without incident edges.
- $G$ is said to have an Euler circuit if there is a circuit in $G$ that traverses every edge of the graph exactly once.
  - You can draw the edges without lifting the pen.
- If there is an open trail from $x$ to $y$ in $G$ and this trail traverses every edge of the graph exactly once, the trail is called an Euler trail.

---

\(^a\)Euler in 1736, the year graph theory was born.
An Euler Circuit
Characterization of Having Euler Circuits

**Theorem 75 (Euler, 1736)** Let $G = (V, E)$ be an undirected graph or multigraph with no isolated nodes. Then $G$ has an Euler circuit if and only if $G$ is connected and every node in $G$ has an even degree.

- Testing if a graph is Eulerian hence is trivial.
- The proof will be constructive.
- Let $n = |E|$. 
The Proof (⇒)

- Clearly $G$ is connected.
- Each time the Euler circuit enters a non-starting node $v$, it must exit it before coming back again, if ever.
- This contributes a count of 2 to $\deg(v)$.
- Because every edge is traversed, $\deg(v)$ must be even.
- The Euler circuit must start from the starting node $s$ and end at the same starting node.
- Each exit is matched by one entry.
- So $\deg(s)$ is even.
The Proof ($\Leftarrow$)

- The $n = 1, 2$ cases are easy, by inspection.
- Assume the result is true when there are $< n$ edges.
- If $G$ has $n$ edges, select a node $s \in G$ as the starting and ending node.
- Construct a circuit $C$ from $s$.
  - Start from $s$.
  - Traverse any hitherto untraversed edge, and repeat.
  - We must eventually return to $s$ because every node has an even degree and hence the last visit to it must be an exit, except $s$. 
The Proof ($\iff$) (continued)

- If $C$ traverses every edge, we are done.
- Otherwise, remove the edges of $C$ and isolated nodes to yield a new graph $K$.
- The degree of each node in $K$ remains even.
  - This observation is key to induction.
The Proof ($\iff$) (continued)$^a$

- Suppose $K$ is connected and $s$ is not isolated.
  - Construct an Euler circuit $c$ of $K$ (doable by the induction hypothesis).
  - Node $s$ is on this Euler circuit because $s \in K$ and $K$ is connected.
  - The desired Euler circuit: Start from $s$ and travel on $C$ until we end at $s$ and then traverse $c$ until we end at $s$ again.

---

$^a$With input from Mr. Cheng-Yu Lee (B91902103) on December 1, 2003.
The Proof ($\leftrightarrow$) (concluded)

- Suppose $K$ is disconnected or $s$ is isolated.
  - Construct an Euler circuit $c_i$ in each component of $K$ (doable by the induction hypothesis).
  - Each component must have at least one node in common with $C$ because originally $G$ is connected.
  - Let $s_i$ be the first node with which $C$ visits $c_i$.\(^a\)
  - The desired Euler circuit: Start from $s$ and travel on $C$ until we reach $s_1$, traverse $c_1$, return to $s_1$, continue on $C$ until we reach $s_2$, and so on.

\(^a\)C may visit many nodes of $c_i$ (but not a single edge by definition).

Thanks to a lively class discussion on May 31, 2012.
Constructing an Euler Circuit
Characterization of Having Euler Trails

Corollary 76 Let $G = (V, E)$ be an undirected graph or multigraph with no isolated nodes. Then $G$ has an Euler trail if and only if $G$ is connected and has exactly two nodes of odd degree.

- Let $x, y$ be the two nodes of odd degree.
- Add edge $\{x, y\}$ to $G$.
- Construct an Euler circuit, which exists by Theorem 75.
- Remove the edge $\{x, y\}$ from the circuit to arrive at an Euler trail.
In and Out Degrees

• Let $G$ be a directed graph.

• The **in degree** of $v \in V$ is the number of edges in $G$ that are incident *into* $v$.

• The **out degree** of $v \in V$ is the number of edges in $G$ that are incident *from* $v$.
  
  – The in and out degrees of a node may not equal.

• Similar to the definition of (undirected) regular graphs (p. 661), a directed $d$-regular graph is a directed graph such that every node has in-degree *and* out-degree $d$. 
Characterization of Having Directed Euler Circuits

**Theorem 77** Let $G = (V, E)$ be a digraph. Then $G$ has a directed Euler circuit if and only if $G$ is connected and the in degree equals the out degree at every node.

- Follow the same proof as Theorem 75 (p. 671).

- The only difference is that, whereas we maintained even node degrees, we now maintain the equality of in and out degrees.
Euler Circuits: Additional Remarks

- Counting the number of Euler circuits for digraphs can be solved efficiently.\(^b\)
- Counting the number of Euler circuits for undirected graphs is computationally hard—it is \(\#P\)-complete.\(^c\)
- Asymptotic formulas exist for the number of Euler circuits on \(K_n\) when \(n\) is odd.\(^d\)
- Very useful in approximation algorithms.\(^e\)

\(^a\)Contributed by Mr. Eric Ruei-Min Lee (B00902106) on June 4, 2012.
\(^b\)Harary & Palmer (1973).
\(^c\)Brightwell & Winkler (2004).
\(^d\)McKay & Robinson (1995).
\(^e\)Vazirani (2003).
Planar Graphs

• A graph or multigraph \( G \) is called **planar** if it can be drawn in the plane with the edges intersecting only at nodes of \( G \).

• Planarity can be tested efficiently.\(^a\)

\(^a\)Hopcroft & Tarjan (1974).
A Planar Graph

Such a drawing of $G$ is called an **embedding** of $G$ in the plane.
Euler’s Theorem\textsuperscript{a}

- Let $G = (V, E)$ be a connected planar graph or multigraph with $|V| = v$ and $|E| = e$.
- Let $r$ be the number of regions in the plane determined by a planar embedding of $G$.
- One of these regions has infinite area.
  - It is called the \textbf{infinite region}.
- Then
  
  \[ v - e + r = 2. \]  

\textsuperscript{a}Euler (1752).
A Planar Graph with $v = 16$, $e = 35$, $r = 21$
The Proof\textsuperscript{a}

- The theorem holds if $e = 0, 1$.\textsuperscript{b}
- Assume the theorem holds for any connected planar graph with $e$ edges, where $0 \leq e \leq k$.
- Let $G = (V, E)$ be a connected planar graph with $v$ nodes, $r$ regions, and $e = k + 1$ edges.
- Let $\{x, y\} \in E$.
- Delete $\{x, y\}$ to obtain graph $H$:

\[ G = H + \{x, y\}. \]


\textsuperscript{b}See p. 545 of the textbook (5th ed.).
The Proof When $H$ Is Connected

- The dotted edge on p. 688 is $\{x, y\}$.
- So $H$ has $v$ nodes, $k$ edges, and $r - 1$ regions.
- $H$ is also planar.
- The induction hypothesis applied to $H$ says
  \[
  v - k + (r - 1) = 2.
  \]
- Hence
  \[
  v - (k + 1) + r = 2.
  \]
- The theorem is proved because $G$ has $v$ nodes, $e = k + 1$ edges, and $r$ regions.
A Planar $G$ from a Planar $H$
The Proof When $H$ Is Not Connected

- The dotted edge on p. 690 is $\{x, y\}$.
- So $H$ has $v$ nodes, $k = e - 1$ edges, and $r$ regions.
- $H$ has two components $H_1$ and $H_2$, both planar.
- Let $H_i$ have $v_i$ nodes, $e_i$ edges, and $r_i$ regions.
- The induction hypothesis applied to $H_i$ says
  \[ v_i - e_i + r_i = 2. \]

Therefore,

\[ (v_1 + v_2) - (e_1 + e_2) + (r_1 + r_2) = 4. \quad (100) \]

\[ ^a \text{Thanks to a lively class discussion on December 1, 2003.} \]
A Planar $G$ from Planar $H_1$ and $H_2$
The Proof When \( H \) Is Not Connected (concluded)

• Now,

\[
\begin{align*}
v_1 + v_2 &= v, \\
e_1 + e_2 &= k = e - 1, \\
r_1 + r_2 &= r + 1.
\end{align*}
\]

– Note that the infinite region is counted twice.

• Hence Eq. (100) on p. 689 becomes

\[
v - (e - 1) + (r + 1) = 4.
\]

• So, again, \( v - e + r = 2 \).
A Useful Corollary

**Corollary 78** Let $G = (V, E)$ be a loop-free connected planar graph with $|V| = v$ and $|E| = e > 2$. Then

$$(3/2) r \leq e \leq 3v - 6.$$ 

- Let there be $r$ regions.

- Each edge is shared by $\leq 2$ regions.
  - The edge $\{x, y\}$ on p. 690 is shared by one region.
  - One can replace the above with “$= 2$” if that edge is considered to be shared by 2 regions.a

- The boundary of each region (including the infinite region) contains at least 3 edges ($G$ is not a multigraph).

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aSee p. 546 of the textbook.
The Proof (concluded)

• Hence

\[ 2e \geq \sum_{\text{region } R} |R's \text{ boundary}| \geq 3r. \]  

(101)

– This proves the first inequality of the corollary.

• Euler’s theorem implies

\[ 2 = v - e + r \leq v - e + \left( \frac{2}{3} \right) e = v - \left( \frac{1}{3} \right) e. \]
$K_5$ Is Not Planar

- $K_5$ has $v = 5$ nodes and $e = 10$ edges.
- Suppose it is planar.
- By Corollary 78,

\[ 10 = e \leq 3v - 6 = 9, \]

a contradiction.
$K_{3,3}$ Is Not Planar

- $K_{3,3}$ has $v = 6$ nodes and $e = 9$ edges.
- Suppose it is planar.
- By Euler’s formula (99) on p. 684, the number of regions is

$$r = 2 + e - v = 5.$$
The Proof (concluded)

• But $K_{3,3}$ has no 3 nodes forming a complete subgraph.
• So the border of a region must contain at least 4 edges.
• The sum of those edges is at least $4r = 20$.
• By inequalities (101) on p. 693,

$$2e \geq \sum_{\text{region } R} | R's \text{ boundary } | \geq 20,$$

contradicting $e = 9$. 
Kuratowski’s\textsuperscript{a} Theorem

Theorem 79 (Kuratowski, 1930) A graph is nonplanar if and only if it contains a subgraph that is “homeomorphic” to either $K_5$ or $K_{3,3}$.

Corollary 80 (1) Shrinking any edge of a planar graph to a single node preserves planarity. (2) Shrinking any connected component of a planar graph to a single node preserves planarity.

\textsuperscript{a}Kasimir Kuratowski (1896–1980).
Hamiltonian\textsuperscript{a} Paths and Cycles

• Let $G = (V, E)$ be a graph with $|V| \geq 3$.

• A Hamiltonian cycle is a cycle in $G$ that contains every node (exactly once) in $V$.

• A Hamiltonian path is a path in $G$ that contains every node (exactly once) in $V$.

• Testing if $G$ has a Hamiltonian path or cycle is computationally hard—it is NP-complete.\textsuperscript{b}

\textsuperscript{a}William Rowan Hamilton (1805–1865).
\textsuperscript{b}Karp (1972).
William Rowan Hamilton (1805–1865)
Richard Karp\textsuperscript{a} (1935–)

\textsuperscript{a}Turing Award (1985).
Application: Tournaments

• Let $K_n^*$ be a directed graph with $n$ nodes.

• If for each distinct pair $x, y$ of nodes, either $(x, y) \in K_n^*$ or $(y, x) \in K_n^*$ but not both, then $K_n^*$ is called a tournament.\(^a\)

• A tournament is not necessarily transitive.
  - A digraph $(V, E)$ is transitive if
    
    $$(a, b) \in E \land (b, c) \in E \Rightarrow (a, c) \in E.$$  

• But the next theorem says that players can be ranked in at least one way.

\(^a\)Recall p. 342.
Tournaments Are Hamiltonian\textsuperscript{a}

**Theorem 81 (Redei, 1934)** A tournament always contains a directed Hamiltonian path.

- Let $p_m = (v_1, v_2, \ldots, v_m)$ be a path of maximum length.
- Assume $m < n = |V|$ and proceed to derive a contradiction.
- Let $v$ be a node not on $p_m$.
- If $(v, v_1) \in K_n^*$, then $p_m$ can be lengthened to $(v, v_1, v_2, \ldots, v_m)$.
- Hence $(v, v_1) \notin K_n^*$ and $(v_1, v) \in K_n^*$.

\textsuperscript{a}Similar results appear on p. 344 and p. 381.
The Proof (continued)

• If there exists a $2 \leq j \leq m$ such that $(v_{j-1}, v) \in K_n^*$ and $(v, v_j) \in K_n^*$, then the path $(v_1, \ldots, v_{j-1}, v, v_j, \ldots, v_m)$ is longer than $p_m$, a contradiction.$^a$

• As $(v_1, v) \in K_n^*$, we conclude that for each $2 \leq j \leq m$, $(v_{j-1}, v) \in K_n^*$ but $(v, v_j) \notin K_n^*$ by induction.

$^a$Improved by a lively discussion on June 5, 2014.
The Proof (concluded)

- In particular, \((v, v_m) \notin K_n^*,\) so \((v_m, v) \in K_n^*\).

- We can add \((v_m, v)\) to \(p_m\) to make it longer, a contradiction.

- Remark: Now that \(K_n^*\) is Hamiltonian, how to find a Hamiltonian path efficiently?