Majority Decision

In a court with $2n + 1$ judges, in how many ways can a majority “yes” decision be handed down?

- There are $\binom{2n+1}{i}$ ways such that $i$ judges vote “yes.”
- From Eq. (12) on p. 60, the desired answer is

$$\sum_{i=n+1}^{2n+1} \binom{2n + 1}{i} = 2^{2n}.$$
Ways To Merge Sets

What is the number of ways to merge members of

\{ \{1\}, \{2\}, \ldots, \{n\} \}

to form

\{ \{1,2,\ldots,n\} \}

in \(n-1\) steps?

- Each merge involves two members.
- For example, the number is 3 when \(n = 3\):

\[
\begin{align*}
\{\{1\}, \{2\}, \{3\}\} & \rightarrow \{\{1,2\}, \{3\}\} \rightarrow \{\{1,2,3\}\}, \\
\{\{1\}, \{2\}, \{3\}\} & \rightarrow \{\{1,3\}, \{2\}\} \rightarrow \{\{1,2,3\}\}, \\
\{\{1\}, \{2\}, \{3\}\} & \rightarrow \{\{2,3\}, \{1\}\} \rightarrow \{\{1,2,3\}\}.
\end{align*}
\]
Ways To Merge Sets (continued)

- The 1st step begins with $n$ members.
- In general, the $i$th step begins with $n - i + 1$ members.
- There are 
  \[
  \binom{n - i + 1}{2}
  \]
  ways to pick the two members.
Ways To Merge Sets (concluded)

- The desired number is thus

\[
\prod_{i=1}^{n-1} \binom{n-i+1}{2} = \binom{n}{2} \binom{n-1}{2} \cdots \binom{2}{2}
\]

\[
= \frac{n! (n-1)! \cdots 2!}{2^{n-1} (n-2)! (n-3)! \cdots 1!}
\]

\[
= \frac{n! (n-1)!}{2^{n-1}}.
\]
The Multinomial Theorem

Theorem 14

\[(x_1 + x_2 + \cdots + x_t)^n\]

\[= \sum_{0 \leq n_1, n_2, \ldots, n_t \leq n} \frac{n!}{n_1! n_2! \cdots n_t!} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}.\]

- Expand \((x_1 + x_2 + \cdots + x_t)^n\).
- Each term in the expansion must have the form

  \((\text{coefficient}) \times x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t},\)

  where \(0 \leq n_1, n_2, \ldots, n_t \leq n\) and \(n_1 + n_2 + \cdots + n_t = n\).
The Proof (concluded)

• The coefficient of

\[ x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t} \]

equals the number of ways to pick \( n_1 \) \( x_1 \)'s, \( n_2 \) \( x_2 \)'s, and so on.

• By Eq. (2) on p. 16, there are

\[
\binom{n}{n_1, n_2, \ldots, n_t} \triangleq \frac{n!}{n_1! n_2! \cdots n_t!}
\]

ways.
Coefficient of $a^2b^3c^2d^5$ in $(a + 2b - 3c + 2d + 5)^{16}$

- Make $x_1 = a$, $x_2 = 2b$, $x_3 = -3c$, $x_4 = 2d$, and $x_5 = 5$ symbolically.

- The coefficient of $a^2(2b)^3(-3c)^2(2d)^55^4$ is

$$\binom{16}{2, 3, 2, 5, 4} = \frac{16!}{2!3!2!5!4!} = 302,702,400$$

by the multinomial theorem with $n = 16$.

- The desired coefficient is then

$$302,702,400 \times 2^3 \times (-3)^2 \times 2^5 \times 5^4 = 435,891,456,000,000.$$
Distinct Objects into Identical Containers

Corollary 15  There are \( \frac{(rn)!}{(r!)^n n!} \) ways to distribute \( rn \) distinct objects into \( n \) identical containers so that each container contains exactly \( r \) objects.

- Consider \( (x_1 + x_2 + \cdots + x_n)^{rn} \).
  - Let \( x_i \) denote the containers (distinct, for now).
  - Each object is associated with one \( x_1 + x_2 + \cdots + x_n \).
  - It means an object can be assigned to one of the \( n \) containers.

- What does the coefficient of
  \[ x_1^r x_2^r \cdots x_n^r \]
  mean?
Distinct Objects into Identical Containers (continued)

- It is the number of ways \( rn \) distinct objects can be distributed into \( n \) distinct containers, each of which contains \( r \) objects.

- By Theorem 14 (p. 71), it is

\[
\binom{rn}{r, r, \ldots, r} \triangleq \frac{(rn)!}{r! r! \cdots r!}.
\]

- Finally, divide the above count by \( n! \) to remove the identities of the containers.
Distinct Objects into Identical Containers (concluded)

Corollary 16 \( \frac{(rn)!}{(r!)^n n!} \) is an integer.

- Immediate from Corollary 15 (p. 74).
An Alternative Proof of Corollary 16 (p. 76)\(^a\)

\[
\frac{(rn)!}{(r!)^nn!} = \frac{1}{n!} \frac{(rn)!}{[r(n-1)]!r!} \frac{[r(n-1)]!}{[r(n-2)]!r!} \cdots \frac{[r(1)]!}{[r(n-n)]!r!} = \prod_{k=0}^{n-1} \left( \frac{(r(n-k))}{r(n-k)} \right) = \prod_{k=0}^{n-1} \frac{[r(n-k)]!}{(n-k)r![r(n-k-1)]!} = \prod_{k=0}^{n-1} \frac{r(n-k)[r(n-k)-1]!}{(n-k)r[r-1]![r(n-k-1)]!} = \prod_{k=0}^{n-1} \frac{r(n-k)-1}{r-1}.
\]

\(^a\)Contributed by Mr. Ansel Lin (B93902003) on September 20, 2004.
Distinct Objects into Identical Containers (continued)

- Take \( n = 3 \) and \( r = 2 \).
- So we have

\[
(x_1 + x_2 + x_3)^6 = (x_1^6 + \cdots + x_3^6) \\
+ 6 \left( x_1^5 x_2 + \cdots + x_2 x_3^5 \right) \\
+ 15 \left( x_1^4 x_2^2 + \cdots + x_2^2 x_3^4 \right) \\
+ 20 \left( x_1^3 x_2^3 + \cdots + x_2^3 x_3^3 \right) \\
+ 30 \left( x_1^4 x_2 x_3 + \cdots + x_1 x_2 x_3^4 \right) \\
+ 60 \left( x_1^3 x_2^2 x_1 + \cdots + x_1 x_2^2 x_3^3 \right) \\
+ 90 x_1^2 x_2^2 x_3.
\]
An Example (concluded)

- Indeed, the coefficients are
  \[
  \binom{6}{6}, \binom{6}{6}, \binom{6}{5,1}, \binom{6}{4,2}, \binom{6}{3,3}, \binom{6}{4,1,1}, \binom{6}{3,2,1}, \binom{6}{2,2,2},
  \]
  consistent with the multinomial theorem (p. 71).

- The coefficient of \( x_1^2x_2^2x_3^3 \) is 90.

- Thus the desired count is
  \[
  \frac{90}{3!} = 15.
  \]
Combinations (Selections) with Repetition

**Theorem 17** Suppose there are $n$ distinct objects and $r \geq 0$ is an integer. The number of selections of $r$ of these objects, with repetition, is

$$C(n + r - 1, r) = \binom{n + r - 1}{r}.$$  

- Note that the order of selection is not important.
- Imagine there are $n$ distinct types of objects.
The Proof (continued)

• Permute

\[
\begin{array}{c|c|c|c|c}
\{ r \} & \{ n-1 \} \\
xx \cdots x & | & | & | & |
\end{array}
\]

• Think of the \( i \)th interval as containing the \( i \)th type of objects.

• So

\[
xx | xxx | x | | | |
\]

means, out of 7 distinct objects, we pick 2 type-1 objects, 3 type-2 objects, and 1 type-3 object.
The Proof (concluded)

• Our goal equals the number of permutations of

\[
\underbrace{xx \cdots x}_{r} \underbrace{\mid \mid \cdots \mid}_{n-1}.
\]

• By Eq. (2) on p. 16, it is

\[
\frac{(r + n - 1)!}{r!(n-1)!} = \binom{n + r - 1}{r} = C(n + r - 1, r).
\]
Combinatorial Proof of the Hockeystick Identity (P. 36)

**Corollary 18**  For $m, n \geq 0$, $\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}$.

- The number of ways to select $m$ objects out of $n + 2$ types is $\binom{n+m+1}{m}$ by Theorem 17 (p. 80).
- Alternatively, let us focus on how the objects of the first $n + 1$ types are chosen.
- There are $\binom{n+m}{m}$ ways to select $m$ objects out of the first $n + 1$ types.
- There are $\binom{n+m-1}{m-1}$ ways to select $m - 1$ objects out of the first $n + 1$ types and 1 object out of the last type.

---

*Contributed by Mr. Jerry Lin (B01902113) on March 13, 2014.*
The Proof (concluded)

• There are \( \binom{n+m-2}{m-2} \) ways to select \( m-2 \) objects out of the first \( n+1 \) types and 2 objects of the last type.

• \ldots

• So,

\[
\binom{n+m}{m} + \binom{n+m-1}{m-1} + \binom{n+m-2}{m-2} + \cdots + \binom{n+0}{0} = \binom{n+m+1}{m}.
\]
Integer Solutions of a Linear Equation

The following three problems are equivalent:

1. The number of nonnegative integer solutions of

\[ x_1 + x_2 + \cdots + x_n = r. \]

2. The number of selections, with repetition, of size \( r \) from a collection of \( n \) distinct objects (Theorem 17 on p. 80).

3. The number of ways \( r \) identical objects can be distributed among \( n \) distinct containers.\(^a\)

They all equal \( \binom{n+r-1}{r} \).\(^b\)

\(^a\)The case of distinct objects and identical containers will be covered on p. 271 (see p. 74 for a special case).

\(^b\)See p. 496 and p. 500 for alternative proofs.
Application: The Multinomial Theorem (P. 71)

• It concerned the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$ in the expansion of

$$ (x_1 + x_2 + \cdots + x_t)^r. $$

• But let us ask how many distinct forms of summands are there?

• Each term has the form $x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$ such that
  - $n_1 + n_2 + \cdots + n_t = r$, and
  - $0 \leq n_1, n_2, \ldots, n_t$.

• For example, consider

$$ r = n_1 + n_2 + n_3 = 2. $$
Application: The Multinomial Theorem (continued)

• Now,

\[(x_1 + x_2 + x_3)^2 = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3.\]

- E.g., the solution “\(n_1 = 1, n_2 = 1, n_3 = 0\)” to \(n_1 + n_2 + n_3 = 2\) contributes 1 to the term \(x_1^1x_2^1x_3^0 = x_1x_2\).

- So there are 6 nonnegative integer solutions to \(n_1 + n_2 + n_3 = 2\) because there are 6 terms.
Application: The Multinomial Theorem (concluded)

• The desired number of terms is therefore
  \[ \binom{r + t - 1}{r} \]
  from the equivalencies on p. 85.

• Indeed, \( \binom{2+3-1}{2} = 6 \).
Positive Integer Solutions of a Linear Equation

- Consider

\[ x_1 + x_2 + \cdots + x_n = r, \]

where \( x_i > 0 \) for \( 1 \leq i \leq n \).

- Define \( x'_i \equiv x_i - 1 \).

- The original problem becomes

\[ x'_1 + x'_2 + \cdots + x'_n = r - n, \]

where \( x'_i \geq 0 \) for \( 1 \leq i \leq n \).

- The number of solutions is therefore (p. 85)

\[
\binom{n + (r - n) - 1}{r - n} = \binom{r - 1}{r - n} = \binom{r - 1}{n - 1}.
\] (14)
Application: Subsets with Restrictions

How many $n$-element subsets of $\{1, 2, \ldots, r\}$ contain no consecutive integers?

- Say $r = 4$ and $n = 2$.
- Then the valid 2-element subsets of $\{1, 2, 3, 4\}$ are
  \[
  \{1, 3\}, \{1, 4\}, \{2, 4\}.
  \]
The Proof (continued)

- For each valid subset \( \{i_1, i_2, \ldots, i_n\} \), where \( 1 \leq i_1 < i_2 < \cdots < i_n \leq r \), define
  \[
  d_k = i_{k+1} - i_k.
  \]

- As “placeholders,” introduce
  \[
  i_0 = 1, \\
  i_{n+1} = r.
  \]

- Then, by telescoping,
  \[
  d_0 + d_1 + \cdots + d_n = i_{n+1} - i_0 = r - 1.
  \]
The Proof (continued)

- Observe that

\[ 0 \leq d_0, d_n \]
\[ 2 \leq d_1, d_2, \ldots, d_{n-1}. \]

- Define

\[ d'_0 \triangleq d_0, \]
\[ d'_k \triangleq d_k - 2, \quad k = 1, 2, \ldots, n - 1, \]
\[ d'_n \triangleq d_n. \]
The Proof (concluded)

• So equivalently,

\[ d'_0 + d'_1 + \cdots + d'_n = r - 1 - 2(n - 1) \]

with \( 0 \leq d'_0, d'_1, \ldots, d'_n \).

• The answer to the desired number is (p. 85)

\[
\begin{pmatrix}
(n + 1) + (r - 1 - 2(n - 1)) - 1 \\
r - 1 - 2(n - 1)
\end{pmatrix}
\begin{pmatrix}
r - n + 1 \\
r - 2n + 1
\end{pmatrix}
= \begin{pmatrix}
r - n + 1 \\
n
\end{pmatrix}.
\]

(15)
Application: Political Majority$^a$

In how many ways can $2n + 1$ seats in a parliament be divided among 3 parties so that the coalition of any 2 parties form a majority?

- If $n = 2$, there are 5 seats.
- Clearly, no party should have 3 or more seats.
- The only valid distribution of the 5 seats to 3 parties is: 2, 2, 1.
- The number of ways is therefore 3.

$^a$Recall p. 67.
The Proof (continued)

• This is a problem of distributing identical objects (the seats) among distinct containers (the parties) (p. 85).

• So without the majority condition, the number is

\[
\binom{3 + (2n + 1) - 1}{2n + 1} = \binom{2n + 3}{2}.
\]

• Observe that the majority condition is violated if and only if a party gets \(n + 1\) or more seats (why?).
The Proof (concluded)

• If a given party gets \( n + 1 \) or more seats, the number of ways of distributing the seats is

\[
\binom{3 + n - 1}{n} = \binom{n + 2}{2}.
\]

– Allocate \( n + 1 \) seats to that party before allocating the remaining \( n \) seats to the 3 parties.

– Then refer to p. 85 for the formula.

• The desired number of no dominating party is

\[
\binom{2n + 3}{2} - 3 \binom{n + 2}{2} = \frac{n}{2} (n + 1) = \binom{n + 1}{2}. \quad (16)
\]
Political Majority: An Alternative Proof\textsuperscript{a}

- Recall that the majority condition holds if and only if no party gets $n + 1$ or more seats.
- So each party can hold up to $n$ seats.
- Give each party $n$ slots to hold real seats.
- As there are $2n + 1$ seats, there will be
  \[3n - (2n + 1) = n - 1\]
  empty slots in the end.

\textsuperscript{a}Contributed by Mr. Weicheng Lee (B01902065) on March 14, 2013.
Political Majority: An Alternative Proof (concluded)

• So the answer to the desired number is the number of ways to distribute the \( n - 1 \) empty slots to 3 parties.

• The count is (p. 85)

\[
\binom{3 + (n - 1) - 1}{n - 1} = \binom{n + 1}{n - 1} = \binom{n + 1}{2}.
\]
Integer Solutions of a Linear Inequality

• Consider

\[ x_1 + x_2 + \cdots + x_n \leq r, \]

where \( x_i \geq 0 \) for \( 1 \leq i \leq n \).

• It is equivalent to

\[ x_1 + x_2 + \cdots + x_n + x_{n+1} = r, \]

where \( x_i \geq 0 \) for \( 1 \leq i \leq n + 1 \).

• The number of integer solutions of the original inequality is therefore (p. 85)

\[
\binom{(n + 1) + r - 1}{r} = \binom{n + r}{r}.
\]  

(17)
The Hockeystick Identity (P. 36) Reproved

• By Eq. (17) on p. 99, there are \( \binom{n+1+m}{m} \) nonnegative integer solutions to

\[
x_1 + x_2 + \cdots + x_{n+1} \leq m, \quad m \geq 0.
\]

• By p. 85, there are \( \binom{n+k}{k} \) nonnegative integer solutions to

\[
x_1 + x_2 + \cdots + x_{n+1} = k.
\]

• Any solution to \( x_1 + x_2 + \cdots + x_{n+1} \leq m \) is a solution to \( x_1 + x_2 + \cdots + x_{n+1} = k \) for some \( 0 \leq k \leq m \).
The Proof (concluded)

- The opposite is also true.
- It is also clear the correspondence is one-to-one.
- So
  \[ \sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}. \]
- This is exactly the hockeystick identity (p. 36).
Compositions of Positive Integers

• Let $m$ be a positive integer.

• A **composition** for $m$ is a sum of positive integers whose order is *relevant* and which sum to $m$.

• For $m = 3$, the number of compositions is 4:

  $3, 2 + 1, 1 + 2, 1 + 1 + 1$.

• For $m = 4$, the number of compositions is 8:

  $4, 3 + 1, 2 + 2, 1 + 3, 1 + 1 + 2, 1 + 2 + 1, 2 + 1 + 1, 1 + 1 + 1 + 1$.

• Is the number of compositions for general $m$ equal to $2^{m-1}$?
The Number of Compositions

Theorem 19 *The number of compositions for* $m > 0$ *is* $2^{m-1}$.

- Every composition with $i$ summands corresponds to a positive integer solution to
  
  $$x_1 + x_2 + \cdots + x_i = m.$$ 

- So the number of solutions is $\binom{m-1}{m-i}$ by Eq. (14) on p. 89.

- The total number of compositions is therefore

  $$\sum_{i=1}^{m} \binom{m-1}{m-i} = 2^{m-1}$$

  by Eq. (8) on p. 56.
An Alternative Proof for Theorem 19 (p. 103)

- Let \( f(m) \) denote the number of compositions for \( m > 0 \).
- A composition for \( m \) is either (1) \( m \) or (2) \( i \) plus a composition for \( m - i \) (“\( i + \cdots \)” ) for \( i = 1, 2, \ldots, m - 1 \).
- Then

\[
f(m) = 1 + \sum_{i=1}^{m-1} f(m - i) = 1 + \sum_{i=1}^{m-1} f(i).
\]

- The above implies that \( f(m + 1) - f(m) = f(m) \) so

\[
f(m + 1) = 2f(m).
\]

\[\text{Contributed by Mr. Chih-Ning Chou (B01902046) on March 7, 2013.}\]
The Proof (concluded)

- As a result,
  \[ f(m) = 2^{m-1} f(1) \]
  by induction.

- Finally, as \( f(1) = 1 = 2^0 \),
  \[ f(m) = 2^{m-1}. \]
A Third Proof for Theorem 19 (p. 103)\textsuperscript{a}

- Start with $m$ x’s and $m - 1$ ’s.
- Consider this arrangement:

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c}
\hline
& & & & & & & & & & & & \\
\hline
x & x & x & | & | & | & | & | & | & | & | & x \\
\hline
\end{array}
\]

- $2m - 1$

- Think of the ’s as dividers.
- Now remove some of the ’s.

\textsuperscript{a}Contributed by Mr. Jerry Lin (B01902113) on March 6, 2014.
The Proof (concluded)

• For example,

\[ xx \mid xxx \mid x \mid x \]

means the composition

\[ 2 + 3 + 1 + 1 \]

for 7.

• Each removal of some |’s leads to a unique composition.

• As there are

\[ 2^{m-1} \]

ways to remove the |’s, this is the number of compositions for \( m \).
Palindromes of Positive Integers

- Let $m$ be a positive integer.
- A palindrome for $m$ is a composition for $m$ that reads the same left to right as right to left.
  - For $m = 4$, the number of palindromes is 4:
    $\begin{bmatrix} 4 \end{bmatrix}, 1 + \begin{bmatrix} 2 \end{bmatrix} + 1, 2 + \begin{bmatrix} 2 \end{bmatrix}, 1 + 1 + 1 + 1 + 1$.
  - For $m = 5$, the number of palindromes is 4:
    $\begin{bmatrix} 5 \end{bmatrix}, 1 + \begin{bmatrix} 3 \end{bmatrix} + 1, 2 + \begin{bmatrix} 1 \end{bmatrix} + 2, 1 + 1 + \begin{bmatrix} 1 \end{bmatrix} + 1 + 1$.
  - The center elements are boxed above.
Palindromes of Positive Integers (concluded)

- The numbers to the left of the center element mirror those to the right, and with the same sum.
- Palindrome is possibly the hardest form of wordplay.\(^a\)
- For example,\(^b\)

  A man, a plan, a canal, Panama!

---

\(^a\)Bryson (2001, p. 228).
\(^b\)Skip the blanks and punctuation marks.
The Number of Palindromes

**Theorem 20** The number of palindromes for $m > 0$ is $2^{\lfloor m/2 \rfloor}$.

- Assume $m$ is even first.

- The central element of a composition of $m$ can be $m, m - 2, \ldots, 2$ or “+” (we will think of it as 0).\(^a\)

- When the central element is $m$, the number of palindromes is clearly 1.

- Suppose the central element is some even number $0 \leq i < m$.

\(^a\)The central element must be even (why?)!
The Proof (concluded)

• Then the numbers to its left sum to \((m - i)/2\).\(^a\)

• Hence the number of palindromes is \(2^{(m-i)/2-1}\) by Theorem 19 (p. 103).

• The total number of palindromes for \(m\) is thus

\[
1 + \left(1 + 2 + 2^2 + \cdots + 2^{(m-2)/2-1} + 2^{m/2-1}\right) = 2^{m/2}.
\]

• Follow the same argument when \(m\) is odd to obtain a count of \(2^{(m-1)/2}\).

\(^a\)By symmetry, the numbers to its right automatically sum to \((m - i)/2\).
Runs

• Consider a permutation of 10 Os and 5 Es:

\[0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ E \ E \ E \ O \ O \ O \ E \ O.\]

• It has 7 runs:

\[
\underbrace{0 \ 0 \ E \ 0 \ 0 \ 0 \ 0 \ E} \underbrace{E \ E \ E} \underbrace{O \ O \ O \ E} \underbrace{E \ O}.
\]

• In general, a run is a maximal consecutive list of identical objects.
The Number of Runs

**Theorem 21** There are

\[
\binom{m - 1}{m - \lfloor r/2 \rfloor} \binom{n - 1}{n - \lceil r/2 \rceil} + \binom{n - 1}{n - \lfloor r/2 \rfloor} \binom{m - 1}{m - \lceil r/2 \rceil}
\]

ways that \( m \) identical objects of type 1 and \( n \) identical objects of type 2 can give rise to \( r \) runs.

- Suppose the run starts with a type-1 object.
- Let \( x_i \) denote the number of type-1 objects in run \( i = 1, 3, \ldots, 2\lfloor r/2 \rfloor - 1 \).
The Proof (continued)

- The number of runs with the said counts \( x_1, x_3, \ldots \) equals the number of positive-integer solutions to

\[
x_1 + x_3 + \cdots + x_{2\lceil r/2 \rceil - 1} = m.
\]

- There are \( \lceil r/2 \rceil \) terms.

- By Eq. (14) on p. 89, the number of solutions equals

\[
\binom{m-1}{\lceil r/2 \rceil - 1} = \binom{m-1}{m - \lfloor r/2 \rfloor}.
\]
The Proof (continued)

• Now let $x_i$ denote the number of type-2 objects in run $i = 2, 4, \ldots, 2\lfloor r/2 \rfloor$.

• The number of runs with the said counts $x_2, x_4, \ldots$ equals that of positive-integer solutions to

$$x_2 + x_4 + \cdots + x_{2\lfloor r/2 \rfloor} = n.$$ 

  – There are $\lfloor r/2 \rfloor$ terms.

• By Eq. (14) on p. 89, the number of solutions equals

$$\binom{n - 1}{\lfloor r/2 \rfloor - 1} = \binom{n - 1}{n - \lfloor r/2 \rfloor}.$$
The Proof (concluded)

• Therefore the number of runs that start with a type-1 object equals

\[
\binom{m-1}{m-\lceil r/2 \rceil} \binom{n-1}{n-\lfloor r/2 \rfloor}.
\]

• Repeat the argument for the case where the 1st run starts with a type-2 object.

• The count is

\[
\binom{n-1}{n-\lceil r/2 \rceil} \binom{m-1}{m-\lfloor r/2 \rfloor}
\]

(by swapping \(m\) and \(n\)).
The Catalan\textsuperscript{a} Numbers (1838)

- A binomial random walk starts at the origin (p. 43).
- What is the number of ways it can end at the origin in $2n$ steps \textit{without} being in the negative territory?
- A left move lowers the position, whereas a right move increases the position.
- So it is equivalent to the number of ways
  \[
  \underbrace{RR \cdots R}_{n} \underbrace{LL \cdots L}_{n}
  \]
  can be permuted so that no prefix has more $L$s than $R$s.

\textsuperscript{a}Eugène Charles Catalan (1814–1894). But it was known to Euler (1707–1783) and, even earlier, Mongolian mathematician Minggatu (1730).
The Catalan Numbers (concluded)

- For example,

```
R LRLRRLLL.
```
Formula for the Catalan Number

The number is

\[ b_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n}, \quad n \geq 1. \]  

(18)

with \( b_0 = 1 \).

- \( \underbrace{RR \cdots RR}_{n} \underbrace{LL \cdots L}_{n} \) can be permuted in \( \binom{2n}{n} \) ways by Eq. (2) on p. 16.\(^b\)

- Some of the permutations are illegal, such as \( RLLLRR \).

\(^a\) The subscript in \( b_n \) is \( n \) not \( 2n! \)

\(^b\) Alternatively, recall Eq. (4) on p. 44.
The Proof (continued)

- We now prove that \( \binom{2n}{n-1} \) of the permutations are illegal.
- For every illegal permutation, we consider the first \( L \) move that makes the particle land at \(-1\).
  - Such as \( RL\underline{L}LRR \).
- Swap \( L \) and \( R \) for this offending \( L \) and all earlier moves.
  - Such as \( \underline{L}R\underline{R}LRR \).
- The result is a permutation of

\[
\underbrace{RR\cdots R}_{n+1} \underbrace{LL\cdots L}_{n-1}.
\]
The Proof (concluded)

- There are \( \binom{2n}{n} \) ways to permute

\[
\underbrace{RR \cdots R}_{n+1} \underbrace{LL \cdots L}_{n-1}
\]

by Eq. (2) on 16.

- But the correspondence is one-to-one between the permutations of

\[
\underbrace{RR \cdots R}_{n+1} \underbrace{LL \cdots L}_{n-1}
\]

and illegal permutations (see next page).

- So there are \( \binom{2n}{n-1} \) illegal walks.
The Reflection Principle

\[ a \]

André (1887).
A Simple Corollary

Corollary 22 For $n \geq 1$,

$$b_n = \frac{\sum_{i=0}^{n} \binom{n}{i}^2}{n + 1}.$$ 

• See Eq. (13) on p. 61.
Application: No Return to Origin until End

What is the number of ways a binomial random walk that is never in the negative territory \textit{and} returns to the origin the \textit{first} time after $2n$ steps?

- Let $n \geq 1$.
- The answer is $b_{n-1}$. 
Application: No Return to Origin until End (concluded)

What is the number of ways a binomial random walk returns to the origin the first time after \(2n\) steps?

- Let \(n \geq 1\).
- The answer is

\[
2b_{n-1} = \frac{1}{2n - 1} \binom{2n}{n}.
\]  

(19)

- It may return to the origin by way of the negative territory.
- It may return to the origin by way of the positive territory.
Application: Nonnegative Partial Sums

What is the number of ways we can arrange \( n \) “+1” and \( n \) “−1” such that all \( 2n \) partial sums are nonnegative?

- For example, the six partial sums of \((1, 1, -1, 1, -1, -1)\) are \((1, 2, 1, 2, 1, 0)\).

- Let \( n \geq 1 \).

- The answer is \( b_n \).

- The number remains \( b_n \) if we have only \( n - 1 \) “−1”.
  - In the original problem, the last number must be −1.
  - So it is “redundant.”
Application: Nonpositive Partial Sums

What is the number of ways we can arrange \( n \) “+1” and \( n \) “−1” such that all \( 2n \) partial sums are nonpositive?

- For example, the six partial sums of \((-1, -1, 1, -1, 1, 1)\) are \((-1, -2, -1, -2, -1, 0)\).
- Let \( n \geq 1 \).
- The answer is \( b_n \).
- The number remains \( b_n \) if we have only \( n - 1 \) “+1”.
  - In the original problem, the last number must be 1.
  - So it is “redundant.”