Recurrence Relations
(Difference Equations)
Pure mathematics is the subject in which we do not know what we are talking about, or whether what we are saying is true.

— Bertrand Russell (1872–1970)
Recurrence Relations Arise Naturally

- When a problem has a recursive nature, recurrence relations often arise.
  - A problem can be solved by solving 2 subproblems of the same nature.

- When an algorithm is of the divide-and-conquer type, a recurrence relation describes its running time.
  - Sorting, fast Fourier transform, etc.

- Certain combinatorial objects are constructed recursively such as hypercubes (p. 610).
First-Order Linear Homogeneous Recurrence Relations

• Consider the recurrence relation

\[ a_{n+1} = da_n, \]

where \( n \geq 0 \) and \( d \) is a constant.

• The **general solution** is given by

\[ a_n = Cd^n \]

for any constant \( C \).

  – It satisfies the relation: \( Cd^{n+1} = dCd^n \).

• There are infinitely many solutions, one for each choice of \( C \).
First-Order Linear Homogeneous Recurrence Relations (concluded)

- Now suppose we impose the initial condition $a_0 = A$.
- Then the (unique) particular solution is $a_n = Ad^n$.
  - Because $A = a_0 = Cd^0 = C$.
- Note that $a_n = na_{n-1}$ is not a first-order linear homogeneous recurrence relation.
  - Its solution is $n!$ when $a_0 = 1$. 
First-Order Linear Nonhomogeneous Recurrence Relations

• Consider the recurrence relation

\[ a_{n+1} + da_n = f(n). \]

- \( n \geq 0. \)
- \( d \) is a constant.
- \( f(n) : \mathbb{N} \rightarrow \mathbb{N}. \)

• A general solution no longer exists.
Consider the $k$th-order recurrence relation

$$C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = 0, \quad (68)$$

where $C_n, C_{n-1}, \ldots, C_{n-k} \in \mathbb{R}$, $C_n \neq 0$, and $C_{n-k} \neq 0$.

• Add $k$ initial conditions for $a_0, a_1, \ldots, a_{k-1}$.

• Clearly, $a_n$ is well-defined for each $n = k, k + 1, \ldots$.

• Indeed, $a_n$ can be calculated with $O(kn)$ operations.
\[ a_{n-k} a_{n-k+1} \cdots a_{n-1} \]

\[ a_n \]
$k$th-Order Linear Homogeneous Recurrence Relations with Constant Coefficients (concluded)

- A solution $y$ for $a_n$ is general if for any particular solution $y^*$, the undetermined coefficients of $y$ can be found so that $y$ is identical to $y^*$.

- Any general solution for $a_n$ that satisfies the $k$ initial conditions and Eq. (68) is a particular solution.

- In fact, it is the unique particular solution because any solution agreeing at $n = 0, 1, \ldots, k - 1$ must agree for all $n \geq 0$. 
Conditions for the General Solution

Theorem 83  Let $a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(k)}$ be $k$ particular solutions of Eq. (68). If

$$\begin{vmatrix}
  a_0^{(1)} & a_0^{(2)} & \cdots & a_0^{(k)} \\
  a_1^{(1)} & a_1^{(2)} & \cdots & a_1^{(k)} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{k-1}^{(1)} & a_{k-1}^{(2)} & \cdots & a_{k-1}^{(k)}
\end{vmatrix} \neq 0, \quad (69)$$

then $a_n = c_1 a_n^{(1)} + c_2 a_n^{(2)} + \cdots + c_k a_n^{(k)}$ is the general solution, where $c_1, c_2, \ldots, c_k$ are arbitrary constants.\(^{a}\)

Fundamental Sets

• The particular solutions of Eq. (68) on p. 510,

\[ a_n^{(1)}, a_n^{(2)}, \ldots, a_n^{(k)}, \]

that also satisfy inequality (69) in Theorem 83 (p. 513) are said to form a fundamental set of solutions.

• Solving a linear homogeneous recurrence equation thus reduces to finding a fundamental set!
$k$th-Order Linear Homogeneous Recurrence Relations with Constant Coefficients: Distinct Roots

- Let $r_1, r_2, \ldots, r_k$ be the (characteristic) roots of the characteristic equation

$$C_n x^k + C_{n-1} x^{k-1} + \cdots + C_{n-k} = 0. \quad (70)$$

- If $r_1, r_2, \ldots, r_k$ are distinct, then the general solution has the form

$$a_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n,$$

for constants $c_1, c_2, \ldots, c_k$ determined by the initial conditions.
The Proof

• Assume $a_n$ has the form $cr^n$ for nonzero $c$ and $r$.

• After substitution into recurrence equation (68) on p. 510, $r$ satisfies characteristic equation (70).

• Let $r_1, r_2, \ldots, r_k$ be the $k$ distinct (nonzero) roots.

• Hence $a_n = r_i^n$ is a solution for $1 \leq i \leq k$.

• Solutions $r_i^n$ form a fundamental set because
The Proof (continued)

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
1 & r_2 & \cdots & r_k \\
\vdots & \vdots & \ddots & \vdots \\
r_1^{k-1} & r_2^{k-1} & \cdots & r_k^{k-1}
\end{vmatrix} \neq 0.
\]

• The \( k \times k \) matrix is called a \textbf{Vandermonde matrix},
which is nonsingular whenever \( r_1, r_2, \ldots, r_k \) are distinct.\(^a\)

\(^a\)This is a standard result in linear algebra.
• Hence

\[ a_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n \]

is the general solution.

• The \( k \) coefficients \( c_1, c_2, \ldots, c_k \) are determined uniquely by the \( k \) initial conditions \( a_0, a_1, \ldots, a_{k-1} \):

\[
\begin{bmatrix}
  a_0 \\
  a_1 \\
  \vdots \\
  a_{k-1}
\end{bmatrix}
= \begin{bmatrix}
  1 & 1 & \cdots & 1 \\
  r_1 & r_2 & \cdots & r_k \\
  \vdots & \vdots & \ddots & \vdots \\
  r_1^{k-1} & r_2^{k-1} & \cdots & r_k^{k-1}
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_k
\end{bmatrix}
\] (71)
The Fibonacci Relation

- Consider $a_{n+2} = a_{n+1} + a_n$.
- The initial conditions are $a_0 = 0$ and $a_1 = 1$.\(^a\)
- The characteristic equation is $r^2 - r - 1 = 0$, with two roots $(1 \pm \sqrt{5})/2$.
- The fundamental set is hence

$$\left\{ \left( \frac{1 + \sqrt{5}}{2} \right)^n, \left( \frac{1 - \sqrt{5}}{2} \right)^n \right\}.$$ 

\(^a\)So $a_n$ can be calculated with $O(n)$ operations.
The Fibonacci Relation (continued)

• For example, \( \left( \frac{1+\sqrt{5}}{2} \right)^n \) satisfies the Fibonacci relation, as

\[
\left( \frac{1 + \sqrt{5}}{2} \right)^{n+2} = \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} + \left( \frac{1 + \sqrt{5}}{2} \right)^n.
\]

• The general solution is hence

\[
a_n = c_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n.
\]
The Fibonacci Relation (concluded)

• Solve

\[
0 = a_0 = c_1 + c_2 \\
1 = a_1 = c_1 \frac{1 + \sqrt{5}}{2} + c_2 \frac{1 - \sqrt{5}}{2}
\]

for \( c_1 = 1/\sqrt{5} \) and \( c_2 = -1/\sqrt{5} \).

• The particular solution is finally

\[
a_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n,
\]

known as the Binet formula.\(^a\)

\(^a\)So \( a_n \) can now be calculated with \( O(\log n) \) operations!
Don’t Believe It?

\[ a_2 = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^2 - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^2 \]

\[ = \frac{1}{\sqrt{5}} \frac{1 + 2\sqrt{5} + 5}{4} - \frac{1}{\sqrt{5}} \frac{1 - 2\sqrt{5} + 5}{4} = 1. \]

\[ a_3 = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^3 - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^3 \]

\[ = \frac{1}{\sqrt{5}} \frac{1 + 3\sqrt{5} + 15 + 5\sqrt{5}}{8} - \frac{1}{\sqrt{5}} \frac{1 - 3\sqrt{5} + 15 - 5\sqrt{5}}{8} = 2. \]
Initial Conditions

- Different initial conditions give rise to different solutions.
- Suppose $a_0 = 1$ and $a_1 = 2$.
- Then solve

\[
\begin{align*}
1 &= a_0 = c_1 + c_2, \\
2 &= a_1 = c_1 \frac{1 + \sqrt{5}}{2} + c_2 \frac{1 - \sqrt{5}}{2},
\end{align*}
\]

for $c_1 = \left[ (1 + \sqrt{5})/2 \right]^2 / \sqrt{5}$ and $c_2 = -\left[ (1 - \sqrt{5})/2 \right]^2 / \sqrt{5}$ to obtain

\[
a_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+2} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+2}. \tag{73}
\]
Initial Conditions (concluded)

• Suppose \( a_0 = a_1 = 1 \) instead.

• Then solve

\[
\begin{align*}
1 &= a_0 = c_1 + c_2, \\
1 &= a_1 = c_1 \frac{1 + \sqrt{5}}{2} + c_2 \frac{1 - \sqrt{5}}{2},
\end{align*}
\]

for \( c_1 = \left[ \frac{1 + \sqrt{5}}{2} \right] / \sqrt{5} \) and \( c_2 = -\left[ \frac{1 - \sqrt{5}}{2} \right] / \sqrt{5} \)

to obtain

\[
a_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1}. \tag{74}
\]
Generating Function for the Fibonacci Numbers

- From $a_{n+2} = a_{n+1} + a_n$, we obtain

$$
\sum_{n=0}^{\infty} a_{n+2}x^{n+2} = \sum_{n=0}^{\infty} \left(a_{n+1}x^{n+2} + a_nx^{n+2}\right).
$$

- Let $f(x)$ be the generating function for $\{a_n\}_{n=0,1,2,\ldots}$.

- Then

$$
f(x) - a_0 - a_1x = x[f(x) - a_0] + x^2f(x).
$$

- Hence

$$
f(x) = \frac{-a_0x + a_0 + a_1x}{1 - x - x^2}.
$$

(75)
A Formula for the Fibonacci Numbers

\[ a_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{n - \lfloor n/2 \rfloor}{\lfloor n/2 \rfloor - 1}. \]

• From Eq. (75) on p. 525, the generating function is \(^a\)

\[
\begin{align*}
-a_0 x + a_0 + a_1 x & \quad \frac{-a_0 x + a_0 + a_1 x}{1 - x - x^2} \\
& = \frac{x}{1 - x(1 + x)} \\
& = x + x^2(1 + x) + x^3(1 + x)^2 + \cdots \\
& \quad + x^{n-1}(1 + x)^{n-2} + x^n(1 + x)^{n-1} + \cdots \\
& = \cdots + \left[ \binom{n - \lfloor n/2 \rfloor}{\lfloor n/2 \rfloor - 1} + \cdots + \binom{n - 2}{1} + \binom{n - 1}{0} \right] x^n + \cdots.
\end{align*}
\]

\(^a\)Recall that \(a_0 = 0\) and \(a_1 = 1\).
Number of Binary Sequences without Consecutive 0s

- Let \( a_n \) denote the number of binary sequences of length \( n \) without consecutive 0s.

- There are \( a_{n-1} \) valid sequences with the \( n \)th symbol being 1.

- There are \( a_{n-2} \) valid sequences with the \( n \)th symbol being 0 because any such sequence must end with 10.

- Hence \( a_n = a_{n-1} + a_{n-2} \), a Fibonacci sequence.

- Because \( a_1 = 2 \) and \( a_2 = 3 \), we must have \( a_0 = 1 \) to retrofit the Fibonacci sequence.

- The formula is Eq. (73) on p. 523.
Number of Subsets without Consecutive Numbers

- A binary sequences \( b_1 b_2 \cdots b_n \) of length \( n \) can be interpreted as the set \( \{ i : b_i = 0 \} \subseteq \{ 1, 2, \ldots, n \} \).

- Hence there are \( a_n \) subsets of \( \{ 1, 2, \ldots, n \} \) that contain no 2 consecutive integers, where
  - \( a_n \) is the Fibonacci number with \( a_0 = 1 \) and \( a_1 = 2 \) (formula is Eq. (73) on p. 523).

- It can be shown that
  \[
  a_n = \binom{n+1}{0} + \binom{n}{1} + \cdots + \binom{n - \lfloor n/2 \rfloor + 1}{\lfloor n/2 \rfloor}.
  \]

- This formula can also be proved by Eq. (11) on p. 77.
Number of Subsets without Consecutive Numbers (continued)

• How many subsets of \( \{1, 2, \ldots, n\} \) contain no 2 consecutive integers when 1 and \( n \) are considered consecutive?

• Let \( a_n \) be the solution on p. 528.

• So \( a_n \) is the Fibonacci number with \( a_0 = 1 \) and \( a_1 = 2 \) (formula is Eq. (73) on p. 523).

• Now assume \( n \geq 3 \).

• There are \( a_{n-1} \) acceptable subsets that do not contain \( n \).
Number of Subsets without Consecutive Numbers (continued)

• If $n$ is included, an acceptable subset cannot contain 1 or $n - 1$.

• Hence there are $a_{n-3}$ such subsets.

• The total is therefore $L_n \equiv a_{n-1} + a_{n-3}$, the **Lucas number**.\(^a\)

• It can be easily checked that

$$L_n = a_{n-1} + a_{n-3}$$

\[
= a_{n-2} + a_{n-3} + a_{n-4} + a_{n-5}
\]

\[
= L_{n-1} + L_{n-2}.
\]

\(^a\)Corrected by Mr. Gong-Ching Lin (B00703082) on May 19, 2012.
Number of Subsets without Consecutive Numbers (continued)

- Furthermore, \( L_0 = 2 \) and \( L_1 = 1 \).
  - \( L_3 = a_2 + a_0 = 3 + 1 = 4 \) and
    \( L_4 = a_3 + a_1 = 5 + 2 = 7 \).
  - So
    \[ 
    \begin{align*}
    L_2 &= L_4 - L_3 = 3, \\
    L_1 &= L_3 - L_2 = 1, \\
    L_0 &= L_2 - L_1 = 2.
    \end{align*}
    \]
Number of Subsets without Consecutive Numbers (continued)

• The general solution is

\[ L_n = c_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n \]

by Eq. (72) on p. 520.

• Solve

\[ 2 = L_0 = c_1 + c_2, \]
\[ 1 = L_1 = c_1 \frac{1 + \sqrt{5}}{2} + c_2 \frac{1 - \sqrt{5}}{2}, \]

for \( c_1 = 1 \) and \( c_2 = 1 \).
Number of Subsets without Consecutive Numbers (concluded)

• The solution is finally

\[ L_n = \left( \frac{1 + \sqrt{5}}{2} \right)^n + \left( \frac{1 - \sqrt{5}}{2} \right)^n. \]
Number of Palindromes Revisited

- A palindrome is a composition for $m \in \mathbb{Z}^+$ that reads the same left to right as right to left (p. 84).

- Let $a_n$ denote the number of palindromes for $n$.

- Clearly, $a_1 = 1$ and $a_2 = 2$.

- Given each palindrome for $n - 2$, we can do two things.
  - Add 1 to the first and last summands to obtain a palindrome for $n$.
  - Insert summand 1 to the start and end to obtain a palindrome for $n$.

- Hence $a_{n+2} = 2a_n$, $n \geq 1$. 
The Proof (continued)

• The characteristic equation \( r^2 - 2 = 0 \) has two roots \( \pm \sqrt{2} \).

• The general solution is hence

\[
a_n = c_1 (\sqrt{2})^n + c_2 (-\sqrt{2})^n.
\]

• Solve\(^a\)

\[
\begin{align*}
1 &= a_1 = \sqrt{2} (c_1 - c_2), \\
2 &= a_2 = 2 (c_1 + c_2),
\end{align*}
\]

for \( c_1 = (1 + \frac{1}{\sqrt{2}})/2 \) and \( c_2 = (1 - \frac{1}{\sqrt{2}})/2 \).

\(^a\)This time, we are not retrofitting.
The Proof (concluded)

- The number of palindromes for $n$ therefore equals

$$a_n = \frac{1 + \sqrt{2}}{2} (\sqrt{2})^n + \frac{1 - \sqrt{2}}{2} (-\sqrt{2})^n$$

$$= \begin{cases} 
\frac{1 + \sqrt{2}}{2} 2^{n/2} + \frac{1 - \sqrt{2}}{2} 2^{n/2} & \text{if } n \text{ is even,} \\
\frac{1 + \sqrt{2}}{2} \sqrt{2} 2^{(n-1)/2} - \frac{1 - \sqrt{2}}{2} \sqrt{2} 2^{(n-1)/2} & \text{if } n \text{ is odd,}
\end{cases}$$

$$= \begin{cases} 
2^{n/2} & \text{if } n \text{ is even,} \\
2^{(n-1)/2} & \text{if } n \text{ is odd,}
\end{cases}$$

$$= 2^\lfloor n/2 \rfloor .$$

- This matches Theorem 22 (p. 85).
An Example: A Third-Order Relation

- Consider
  \[2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n\]
  with \(a_0 = 0\), \(a_1 = 1\), and \(a_2 = 2\).

- The characteristic equation \(2r^3 - r^2 - 2r + 1 = 0\) has three distinct real roots: 1, -1, and 0.5.

- The general solution is
  \[a_n = c_11^n + c_2(-1)^n + c_3(1/2)^n\]
  \[= c_1 + c_2(-1)^n + c_3(1/2)^n.\]
An Example: A Third-Order Relation (concluded)

• Solve the three initial conditions with Eq. (71) on p. 518,

\[
\begin{bmatrix}
0 \\
1 \\
2
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 \\
1 & -1 & 0.5 \\
1^2 & (-1)^2 & 0.5^2
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}.
\]

• The solutions are \(c_1 = 2.5\), \(c_2 = 1/6\), and \(c_3 = -8/3\).
The Case of Complex Roots

- Consider

\[ a_n = 2(a_{n-1} - a_{n-2}) \]

with \( a_0 = 1 \) and \( a_1 = 2 \).

- The characteristic equation \( r^2 - 2r + 2 = 0 \) has two distinct complex roots \( 1 \pm i \).

- The general solution is

\[ a_n = c_1(1 + i)^n + c_2(1 - i)^n. \]
The Case of Complex Roots (concluded)

- Solve the two initial conditions for $c_1 = (1 - i)/2$ and $c_2 = (1 + i)/2$.

- The particular solution becomes

\[
a_n = (1 + i)^{n-1} + (1 - i)^{n-1} = (\sqrt{2})^n [\cos(n\pi/4) + \sin(n\pi/4)].
\]

---

\[a\] An equivalent one is $a_n = (\sqrt{2})^{n+1} \cos((n - 1)\pi/4)$ by Mr. Tunglin Wu (B00902040) on May 17, 2012.
Consider the recurrence relation

\[ C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = 0, \]

where \( C_n, C_{n-1}, \ldots \) are real constants, \( C_n \neq 0, C_{n-k} \neq 0. \)

Let \( r \) be a characteristic root of \textbf{multiplicity} \( m \), where \( 2 \leq m \leq k \), of the characteristic equation

\[ f(x) = C_n x^k + C_{n-1} x^{k-1} + \cdots + C_{n-k} = 0. \]

The general solution that involves \( r \) has the form

\[ (A_0 + A_1 n + A_2 n^2 + \cdots + A_{m-1} n^{m-1}) r^n, \]

with \( A_0, A_1, \ldots, A_{m-1} \) are constants to be determined.
The Proof

• If \( f(x) \) has a root \( r \) of multiplicity \( m \), then
  \[ f(r) = f'(r) = \cdots = f^{(m-1)}(r) = 0. \]

• Because \( r \neq 0 \) is a root of multiplicity \( m \),
  \[
  0 = r^{n-k}f(r), \\
  0 = r(r^{n-k}f(r))', \\
  0 = r(r(r^{n-k}f(r)))', \\
  \vdots \\
  0 = r(\cdots r(r^{n-k}f(r)))' \cdots )'.
  \]
The Proof (continued)

• We differentiate and then multiply by \( r \) before iterating.

• These give

\[
0 = C_n r^n + C_{n-1} r^{n-1} + \cdots + C_{n-k} r^{n-k},
\]
\[
0 = C_n n r^n + C_{n-1} (n-1) r^{n-1} + \cdots + C_{n-k} (n-k) r^{n-k},
\]
\[
0 = C_n n^2 r^n + C_{n-1} (n-1)^2 r^{n-1} + \cdots + C_{n-k} (n-k)^2 r^{n-k},
\]
\[
\vdots
\]

• Now, \( n^k r^n \) for \( 0 \leq k \leq m - 1 \) is indeed a solution for the \( k \)th row above:

\[
0 = C_n n^k r^n + C_{n-1} (n-1)^k r^{n-1} + \cdots + C_{n-k} (n-k)^k r^{n-k}.
\]
The Proof (continued)

• \( r^n, nr^n, n^2r^n, \ldots, n^{m-1}r^n \) form a fundamental set if

\[
\begin{vmatrix}
1 & 0 & \cdots & 0 \\
r & r & \cdots & r \\
r^2 & 2r^2 & \cdots & 2^{m-1}r^2 \\
\vdots & \vdots & \ddots & \vdots \\
r^{m-1} & (m-1)r^{m-1} & \cdots & (m-1)^{m-1}r^{m-1}
\end{vmatrix} \neq 0.
\]

• But it is a Vandermonde matrix in disguise.
The Proof (concluded)

• Specifically, the determinant equals

\[(m - 1)! \cdot r^{1+2+\cdots+(m-1)}\]

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 2^{m-2} \\
\vdots & \vdots & \ddots & \vdots \\
1 & (m-1) & \cdots & (m-1)^{m-2}
\end{vmatrix} \neq 0.
\]
Nonhomogeneous Recurrence Relations

• Consider

\[ C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = f(n). \] (76)

• If \( a_n = a_{n-1} + f(n) \), then the solution is

\[ a_n = a_0 + \sum_{i=1}^{n} f(i). \]

  - A closed-form formula exists if one for \( \sum_{i=1}^{n} f(i) \)
    does.

• In general, no failure-free methods exist except for specific \( f(n) \)s.

  - Consult pp. 441–2 of the textbook (4th ed.).
Examples \((c, c_1, c_2, \ldots \text{ Are Arbitrary Constants})\)

<table>
<thead>
<tr>
<th>Equation</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_{n+1} - a_n = 0)</td>
<td>(a_n = c)</td>
</tr>
<tr>
<td>(a_{n+1} - a_n = 1)</td>
<td>(a_n = n + c)</td>
</tr>
<tr>
<td>(a_{n+1} - a_n = n)</td>
<td>(a_n = n(n - 1)/2 + c)</td>
</tr>
<tr>
<td>(a_{n+2} - 3a_{n+1} + 2a_n = 0)</td>
<td>(a_n = c_1 + c_22^n)</td>
</tr>
<tr>
<td>(a_{n+2} - 3a_{n+1} + 2a_n = 1)</td>
<td>(a_n = c_1 + c_22^n - n)</td>
</tr>
<tr>
<td>(a_{n+2} - a_n = 0)</td>
<td>(a_n = c_1 + c_2(-1)^n)</td>
</tr>
<tr>
<td>(a_{n+1} = a_n/(1 + a_n))</td>
<td>(a_n = c/(1 + cn))</td>
</tr>
</tbody>
</table>
Trial and Error

• Consider \( a_{n+1} = 2a_n + 2^n \) with \( a_1 = 1 \).

• Calculations show that \( a_2 = 4 \) and \( a_3 = 12 \).

• Conjecture:

\[
a_n = n2^{n-1}.
\]

(77)

• Verify that, indeed,

\[
(n + 1)2^n = 2(n2^{n-1}) + 2^n,
\]

and \( a_1 = 1 \).
Application: Number of Edges of a Hasse Diagram

- Let \( a_n \) be the number of edges of the Hasse diagram for the partial order \((2\{1,2,\ldots,n\}, \subseteq)\).

- Consider the Hasse diagrams \( H_1 \) for \((2\{1,2,\ldots,n\}, \subseteq)\) and \( H_2 \) for \((\{T \cup \{n+1\} : T \subseteq \{1,2,\ldots,n\}\}, \subseteq)\).
  - \( H_1 \) and \( H_2 \) are “isomorphic.”

- The Hasse diagram for \((2\{1,2,\ldots,n+1\}, \subseteq)\) is constructed by adding an edge from each node \( T \) of \( H_1 \) to node \( T \cup \{n+1\} \) of \( H_2 \).

- Hence \( a_{n+1} = 2a_n + 2^n \) with \( a_1 = 1 \).

- The desired number has been solved in Eq. (77) on p. 548.
Illustration with \((2^{\{1,2,3\}}, \subseteq)\)
Trial and Error Again

• Consider $a_{n+1} - Aa_n = B$.

• Calculations show that

\[
\begin{align*}
    a_1 &= Aa_0 + B, \\
    a_2 &= Aa_1 + B = A^2a_0 + B(A + 1), \\
    a_3 &= Aa_2 + B = A^3a_0 + B(A^2 + A + 1).
\end{align*}
\]

• Conjecture (easily verified by substitution):

\[
a_n = \begin{cases} 
  A^n a_0 + B \frac{A^n-1}{A-1}, & \text{if } A \neq 1 \\
  a_0 + Bn, & \text{if } A = 1
\end{cases}.
\]  

(78)
Financial Application: Compound Interest\textsuperscript{a}

• Consider \( a_{n+1} = (1 + r) a_n \).
  - Deposit grows at a period interest rate of \( r > 0 \).
  - The initial deposit is \( a_0 \) dollars.

• By Eq. (78) on p. 551, the solution is
  \[ a_n = (1 + r)^n a_0. \]

• The deposit therefore grows exponentially with time.

\textsuperscript{a}“In the fifteenth century mathematics was mainly concerned with questions of commercial arithmetic and the problems of the architect,” wrote Joseph Alois Schumpeter (1883–1950) in \textit{Capitalism, Socialism and Democracy}. 
Financial Application: Amortization

- Consider $a_{n+1} = (1 + r) a_n - M$.
  - The initial loan amount is $a_0$ dollars.
  - The monthly payment is $M$ dollars.
  - The outstanding loan principal after the $n$th payment is $a_{n+1}$.

- By Eq. (78) on p. 551, the solution is

$$a_n = (1 + r)^n a_0 - M \frac{(1 + r)^n - 1}{r}.$$
The Proof (concluded)

• What is the unique monthly payment \( M \) for the loan to be closed after \( k \) months?

• Set \( a_k = 0 \) to obtain

\[
a_k = (1 + r)^k a_0 - M \frac{(1 + r)^k - 1}{r} = 0.
\]

• Hence

\[
M = \frac{(1 + r)^k a_0 r}{(1 + r)^k - 1}.
\]

• This is standard calculation for home mortgages and annuities.\(^a\)

\(^a\)Lyuu (2002).
Trial and Error a Third Time

- Consider the more general $a_{n+1} - Aa_n = BC^n$.

- Calculations show that

\[
\begin{align*}
a_1 &= Aa_0 + B, \\
a_2 &= Aa_1 + BC = A^2a_0 + B(A + C), \\
a_3 &= Aa_2 + BC^2 = A^3a_0 + B(A^2 + AC + C^2).
\end{align*}
\]

- Conjecture (easily verified by substitution):

\[
a_n = \begin{cases} 
A^n a_0 + B \frac{A^n - C^n}{A - C} & \text{if } A \neq C \\
A^n a_0 + B A^{n-1} n & \text{if } A = C
\end{cases} .
\] (79)
Application: Runs of Binary Strings

• A run is a maximal consecutive list of identical objects (p. 87).
  – Binary string “0 0 1 1 1 0” has 3 runs.

• Let $r_n$ denote the total number of runs determined by the $2^n$ binary strings of length $n$.

• First, $r_1 = 2$.
  – Each of “0” and “1” has 1 run.

• In general, suppose we append a bit to an $(n - 1)$-bit string $b_1 b_2 \cdots b_{n-1}$ to make $b_1 b_2 \cdots b_{n-1} b_n$. 
The Proof (continued)

- For those with $b_{n-1} = b_n$, the total number of runs does not change.
  - The total number of runs remains $r_{n-1}$.

- For those with $b_{n-1} \neq b_n$, the total number of runs increases by 1 for each $(n - 1)$-bit string.
  - There are $2^{n-1}$ of them.
  - The total number of runs becomes $r_{n-1} + 2^{n-1}$.

- Hence

$$r_n = 2r_{n-1} + 2^{n-1}, n \geq 2.$$
The Proof (concluded)

- By Eq. (79) on p. 555,

\[ r_n = 2^n r_0 + 2^{n-1} n. \]

- To make sure that \( r_1 = 2 \), it is easy to see that \( r_0 = 1/2 \).

- Hence

\[ r_n = 2^{n-1} + 2^{n-1} n = 2^{n-1} (n + 1). \]

  - The recurrence is identical to that for the number of edges of a Hasse diagram (p. 549) except for the initial condition, whose solution is in Eq. (77) on p. 548, \( a_n = n 2^{n-1} \).
Method of Undetermined Coefficients

- Recall Eq. (76) on p. 546:

\[ C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = f(n). \]

- Let \( a_n^{(h)} \) denote the general solution of the associated homogeneous relation (with \( f(n) = 0 \)).

- Let \( a_n^{(p)} \) denote a particular solution of the nonhomogeneous relation.

- Then

\[ a_n = a_n^{(h)} + a_n^{(p)}. \]

- All the entries in the table on p. 547 fit the claim.