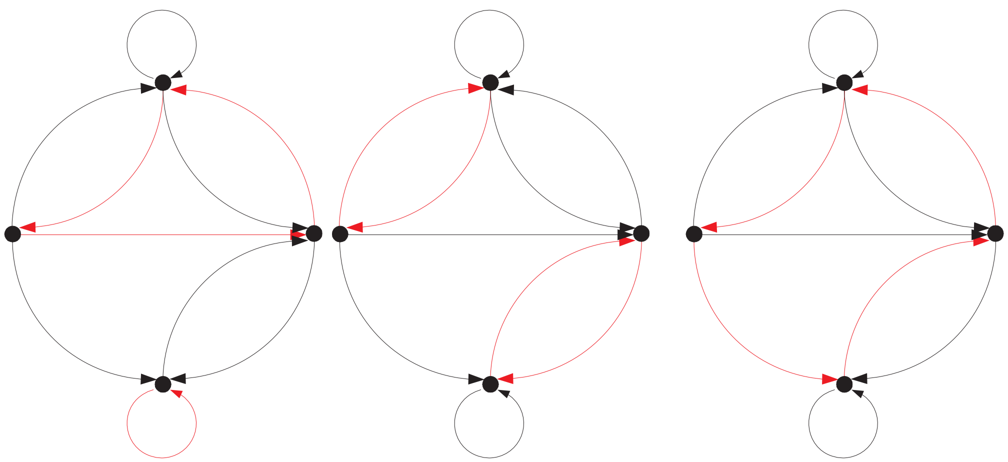
CYCLE COVER

- A set of node-disjoint cycles that cover all *nodes* in a *directed* graph is called a **cycle cover**.



- There are 3 cycle covers (in red) above.

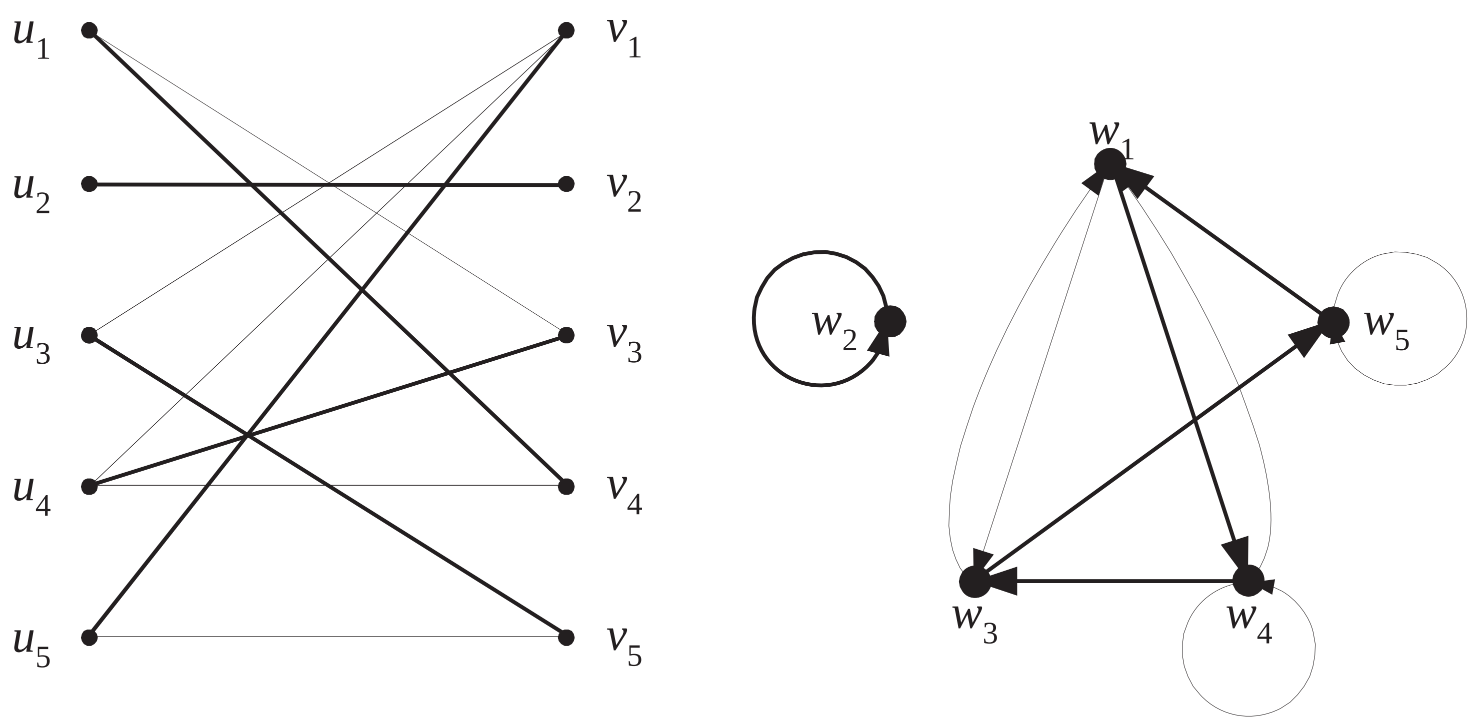
CYCLE COVER and BIPARTITE PERFECT MATCHING

Proposition 96 CYCLE COVER *and* BIPARTITE PERFECT MATCHING *(p. 532) are parsimoniously reducible to each other.*

- A polynomial-time algorithm creates a bipartite graph G' from any directed graph G (see next page).
- The number of cycle covers for G equals the number of bipartite perfect matchings for G'.
- And vice versa.

Corollary 97 CYCLE COVER $\in P$.

Illustration of the Proof



Permanent

- The **permanent** of an $n \times n$ integer matrix A is

$$\text{perm}(A) = \sum_{\pi} \prod_{i=1}^{n} A_{i,\pi(i)}.$$

  - π ranges over all permutations of n elements.
- 0/1 PERMANENT computes the permanent of a 0/1 (binary) matrix.
  - The permanent of a binary matrix is at most $n!$.
- Simpler than determinant (9) on p. 536: no signs.
- Surprisingly, much harder to compute than determinant!

Permanent and Counting Perfect Matchings

- BIPARTITE PERFECT MATCHING is related to determinant.[a]
- #BIPARTITE PERFECT MATCHING is related to permanent.

Proposition 98 0/1 PERMANENT *and* BIPARTITE PERFECT MATCHING *are parsimoniously reducible to each other.*

---

[a]Recall p. 537.

The Proof

- Given a bipartite graph G, construct an $n \times n$ binary matrix A.
  - The (i,j)th entry A_{ij} is 1 if $(i,j) \in E$ and 0 otherwise.
- Then $\mathrm{perm}(A) =$ number of perfect matchings in G.

Illustration of the Proof Based on p. 872 (Left)

$$A = \begin{bmatrix} 0 & 0 & 1 & \boxed{1} & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \boxed{1} \\ 1 & 0 & \boxed{1} & 1 & 0 \\ \boxed{1} & 0 & 0 & 0 & 1 \end{bmatrix}.$$

- $\text{perm}(A) = 4$.
- The permutation corresponding to the perfect matching on p. 872 is marked.

Permanent and Counting Cycle Covers

Proposition 99 0/1 PERMANENT *and* CYCLE COVER *are parsimoniously reducible to each other.*

- Let A be the adjacency matrix of the graph on p. 872 (right).
- Then $\text{perm}(A) =$ number of cycle covers.

Three Parsimoniously Equivalent[a] Problems

We summarize Propositions 96 (p. 871) and 98 (p. 874) in the following.

Lemma 100 *0/1* PERMANENT, BIPARTITE PERFECT MATCHING, *and* CYCLE COVER *are* **parsimoniously equivalent***.*

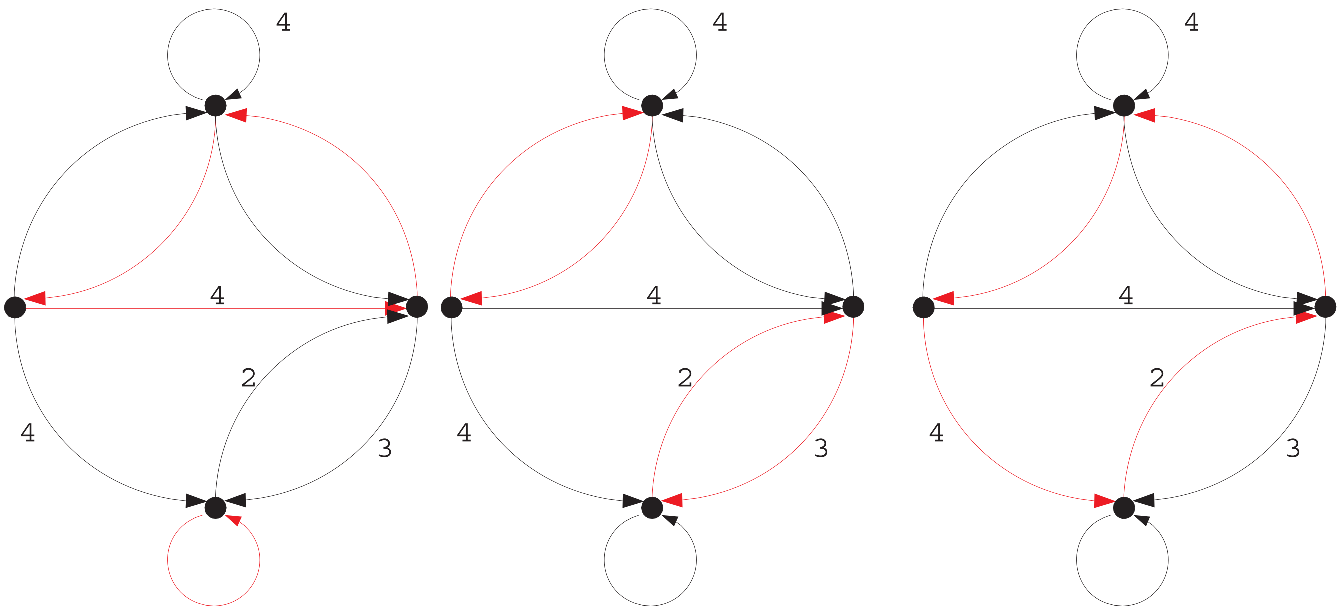
We will show that the counting versions of all three problems are in fact #P-complete.

[a] Meaning the numbers of solutions are equal in a reduction.

WEIGHTED CYCLE COVER

- Consider a directed graph G with integer weights on the edges.
- The weight of a cycle cover is the product of its edge weights.
- The **cycle count** of G is sum of the weights of all cycle covers.
  - Let A be G's adjacency matrix but $A_{ij} = w_i$ if the edge (i, j) has weight w_i.
  - Then $\text{perm}(A) = G$'s cycle count (same proof as Proposition 99 on p. 877).
- #CYCLE COVER is a special case: All weights are 1.

An Example[a]



There are 3 cycle covers, and the cycle count is

$$(4 \cdot 1 \cdot 1) \cdot (1) + (1 \cdot 1) \cdot (2 \cdot 3) + (4 \cdot 2 \cdot 1 \cdot 1) = 18.$$

[a] Each edge has weight 1 unless stated otherwise.

Three #P-Complete Counting Problems

Theorem 101 (Valiant, 1979) 0/1 PERMANENT, #BIPARTITE PERFECT MATCHING, *and* #CYCLE COVER *are #P-complete.*

- By Lemma 100 (p. 878), it suffices to prove that #CYCLE COVER is #P-complete.
- #SAT is #P-complete (p. 868).
- #3SAT is #P-complete because it and #SAT are parsimoniously equivalent.
- We shall prove that #3SAT is polynomial-time Turing-reducible to #CYCLE COVER.

The Proof (continued)

- Let ϕ be the given 3SAT formula.
  - It contains n variables and m clauses (hence $3m$ literals).
  - It has $\#\phi$ satisfying truth assignments.
- First we construct a *weighted* directed graph H with cycle count
$$\#H = 4^{3m} \times \#\phi.$$
- Then we construct an unweighted directed graph G.
- We shall make sure $\#H$ (hence $\#\phi$) is polynomial-time Turing-reducible to $\#G$ (G's number of cycle covers).

The Proof: Comments (continued)

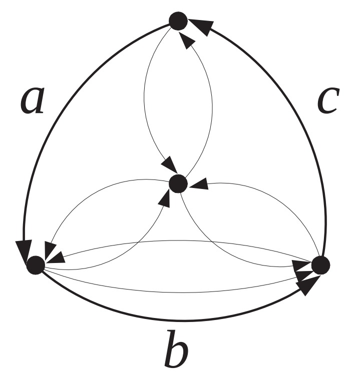
- Our reduction is not expected to be parsimonious.
  - Suppose otherwise and

$$\#\phi = \#G.$$

  - Hence G has a cycle cover if and only if ϕ is satisfiable.
  - But CYCLE COVER $\in P$ (p. 871).
  - Thus 3SAT $\in P$, a most unlikely event!

The Proof: the Clause Gadget (continued)

- Each clause is associated with a **clause gadget**.



- Each edge has weight 1 unless stated otherwise.
- Each bold edge corresponds to one literal in the clause.
  - The assignment of literals to edges is arbitrary.
- Bold edges are schematic only,[a] not *parallel* edges.

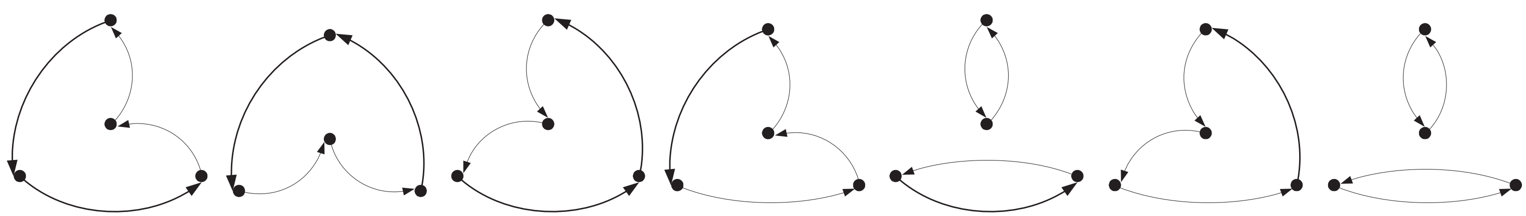
[a]Preview p. 897.

The Proof: the Clause Gadget (continued)

- Following a bold edge makes the literal *false* (0).
- A cycle cover cannot select *all* 3 bold edges.
  - The interior node would be missing.
- Every proper nonempty subset of bold edges corresponds to a unique cycle cover of weight 1 (see next page).

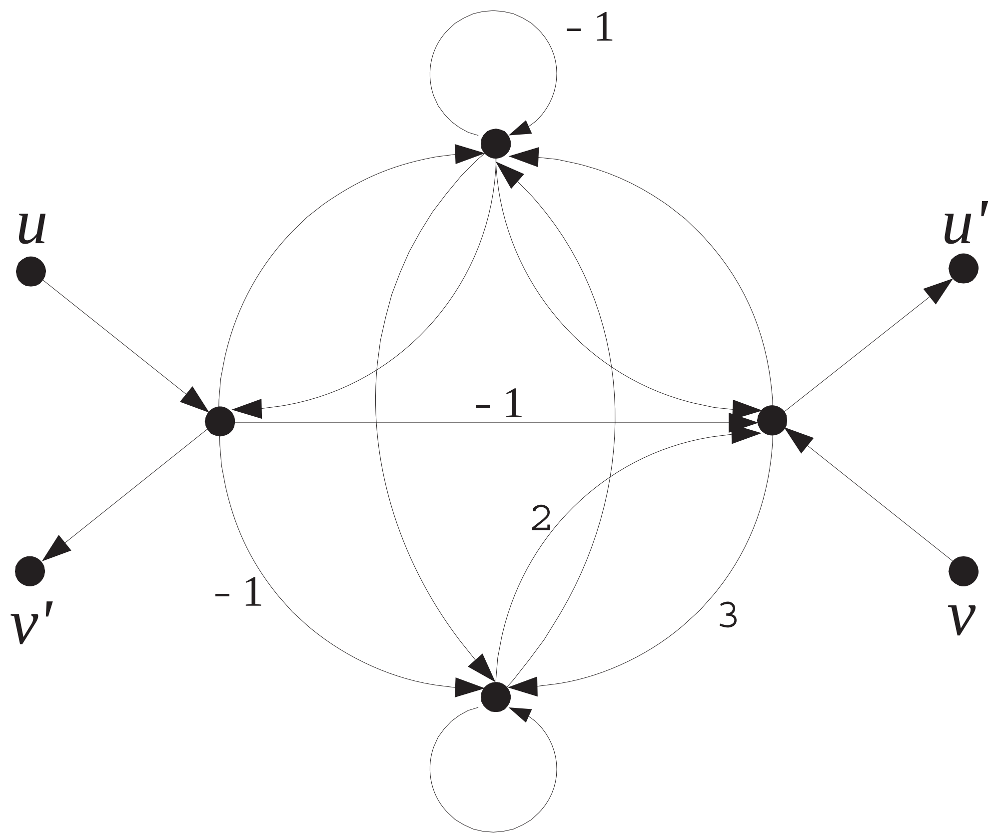
The Proof: the Clause Gadget (continued)

- 7 possible cycle covers, one for each satisfying assignment: (1) $a = 0, b = 0, c = 1$, (2) $a = 0, b = 1, c = 0$, etc.



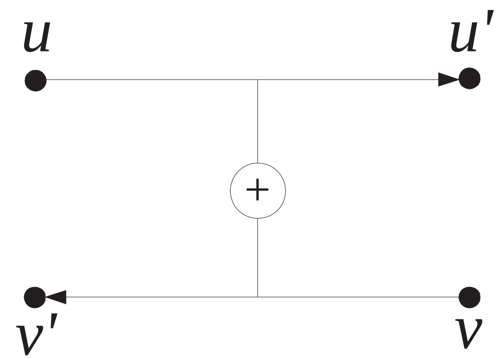
(1) (2) (3) (4) (5) (6) (7)

The Proof: the XOR Gadget (continued)



The Proof: Properties of the XOR Gadget (continued)

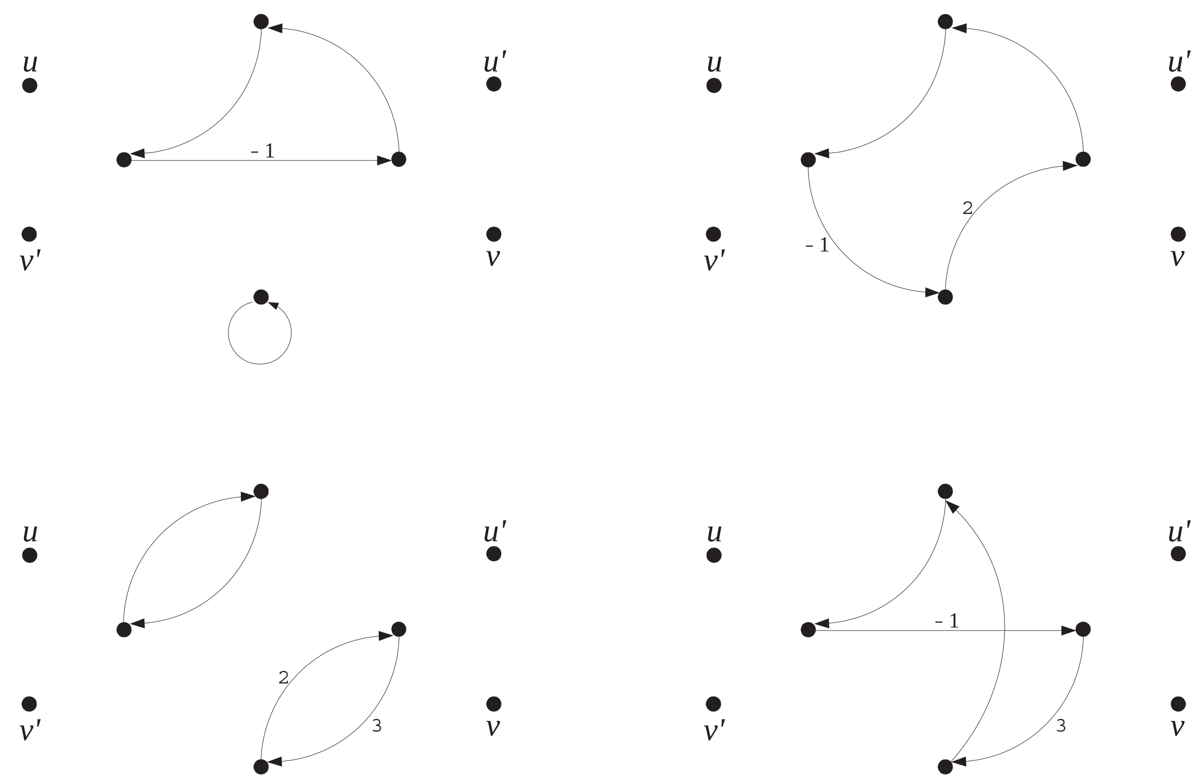
- The XOR gadget schema:



- *At most one* of the 2 schematic edges will be included in a cycle cover.
- There are $3m$ XOR gadgets, one for each literal.
- Only those cycle covers that take exactly one schematic edge in *every* XOR gadget contribute a nonzero weight (see next pages).

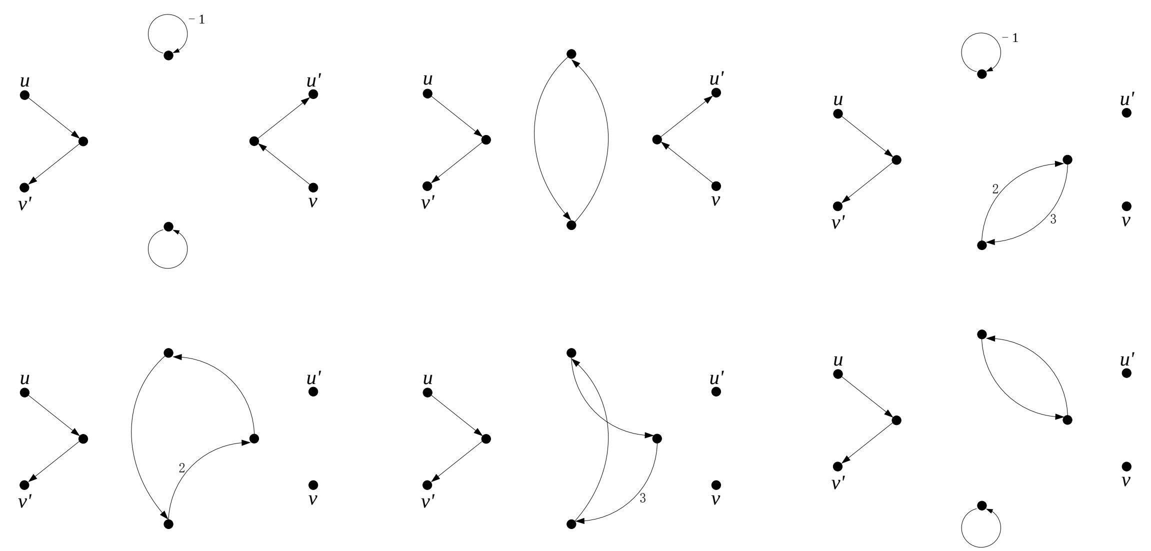
The Proof: Properties of the XOR Gadget (continued)

- Total weight of $-1-2+6-3=0$ for cycle covers not entering or leaving it.



The Proof: Properties of the XOR Gadget (continued)

- Total weight of $-1 + 1 - 6 + 2 + 3 + 1 = 0$ for cycle covers entering at u and leaving at v'.[a]

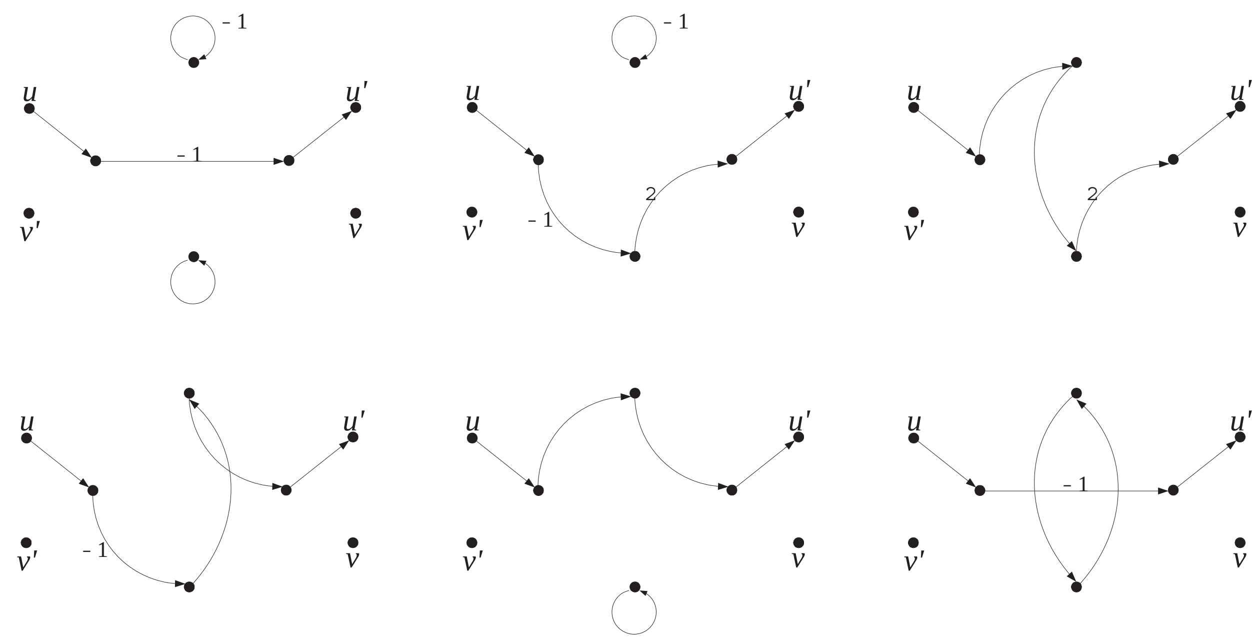


- Same for cycle covers entering at v and leaving at u'.

[a]Corrected by Mr. Yu-Tsung Dai (`B91201046`) and Mr. Che-Wei Chang (`R95922093`) on December 27, 2006.

The Proof: Properties of the XOR Gadget (continued)

- Total weight of $1 + 2 + 2 - 1 + 1 - 1 = 4$ for cycle covers entering at u and leaving at u'.



- Same for cycle covers entering at v and leaving at v'.

The Proof: Summary (continued)

- Cycle covers not entering *all* of the XOR gadgets contribute 0 to the cycle count.
  - Let x denote an XOR gadget not entered for some cycle covers for H.
  - Now, such cycle covers' contribution to the cycle count totals, by p. 889,

$$\begin{aligned}
& \sum_{\text{cycle cover } c \text{ not entering } x} (\text{weight of } c \text{ for } H) \\
= & \sum_{\text{cycle cover } c \text{ not entering } x} (\text{weight of } c \text{ for } H - x) \times (\text{weight of } c \text{ for } x) \\
= & \sum_{\text{cycle cover } c \text{ not entering } x} (\text{weight of } c \text{ for } H - x) \times 0 = 0.
\end{aligned}$$

The Proof: Summary (continued)

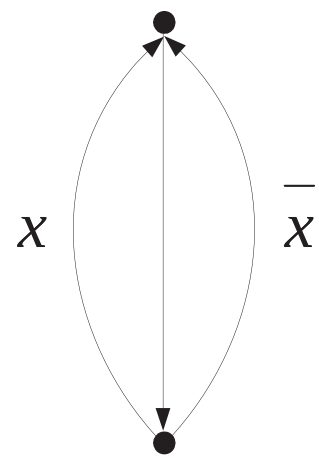
- Cycle covers entering *any* of the XOR gadgets and leaving illegally contribute 0 to the cycle count by p. 890.
- For every XOR gadget entered and exited legally, the total weight of a cycle cover is multiplied by 4.
  - Each such act multiplies the weight by 4 by p. 891.

The Proof: Summary (continued)

- Hereafter we consider only cycle covers which enter every XOR gadget and leaves it legally.
  - Only these cycle covers contribute nonzero weights to the cycle count.
- They are said to **respect** the XOR gadgets.

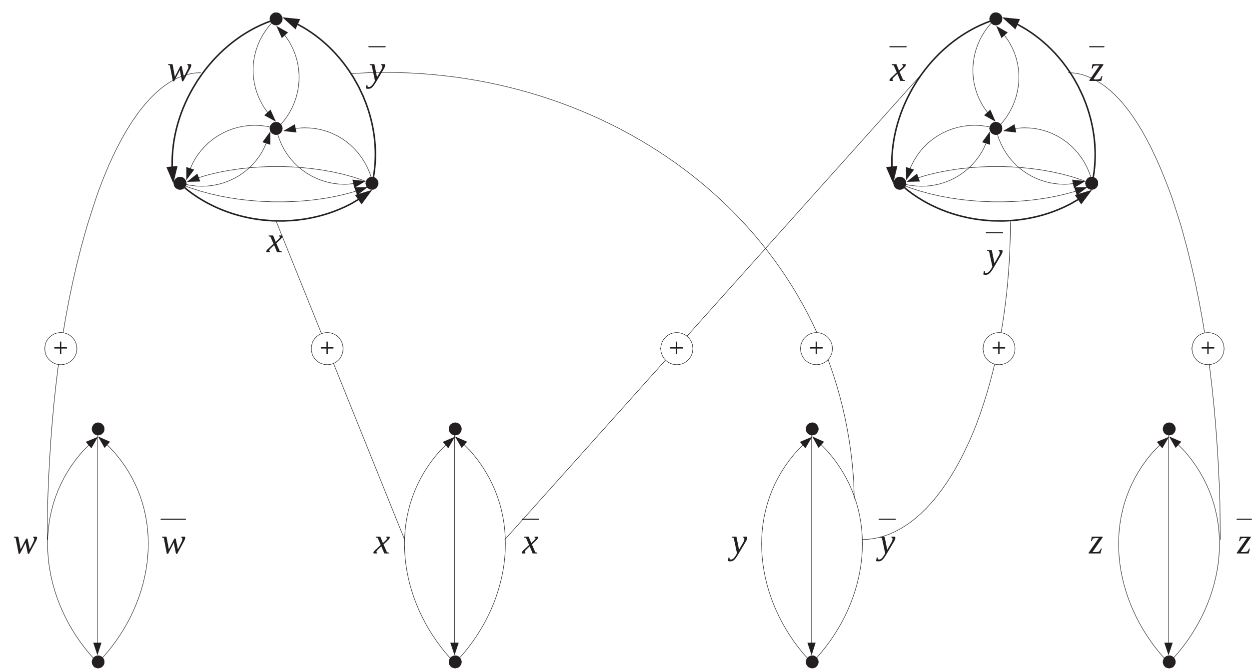
The Proof: the Choice Gadget (continued)

- One choice gadget (a schema) for each *variable*.

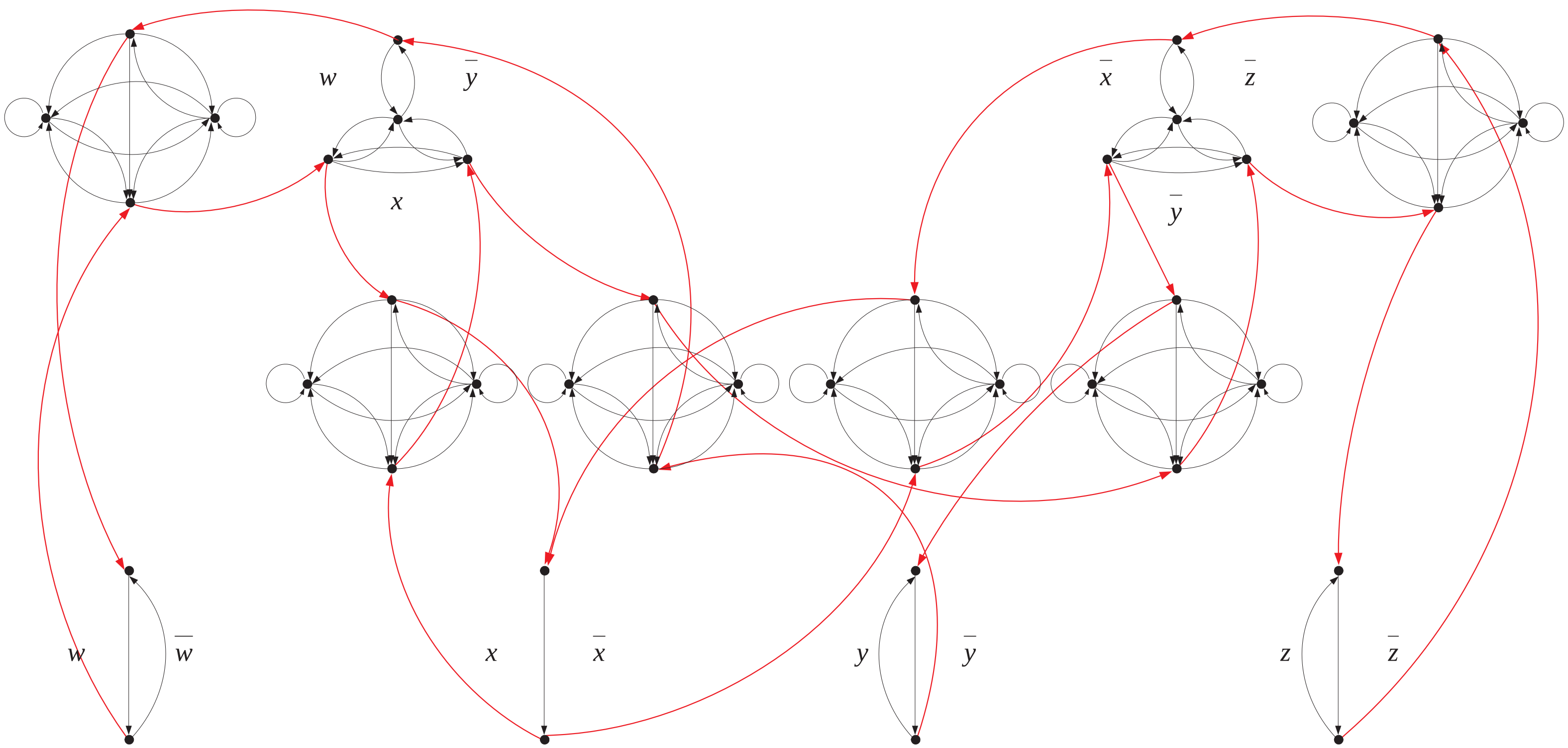


- It gives the truth assignment for the variable.
- Use it with the XOR gadget to enforce consistency.

Schema for $(w \lor x \lor \bar{y}) \land (\bar{x} \lor \bar{y} \lor \bar{z})$



The Full Graph[a] $(w \vee x \vee \bar{y}) \wedge (\bar{x} \vee \bar{y} \vee \bar{z})$

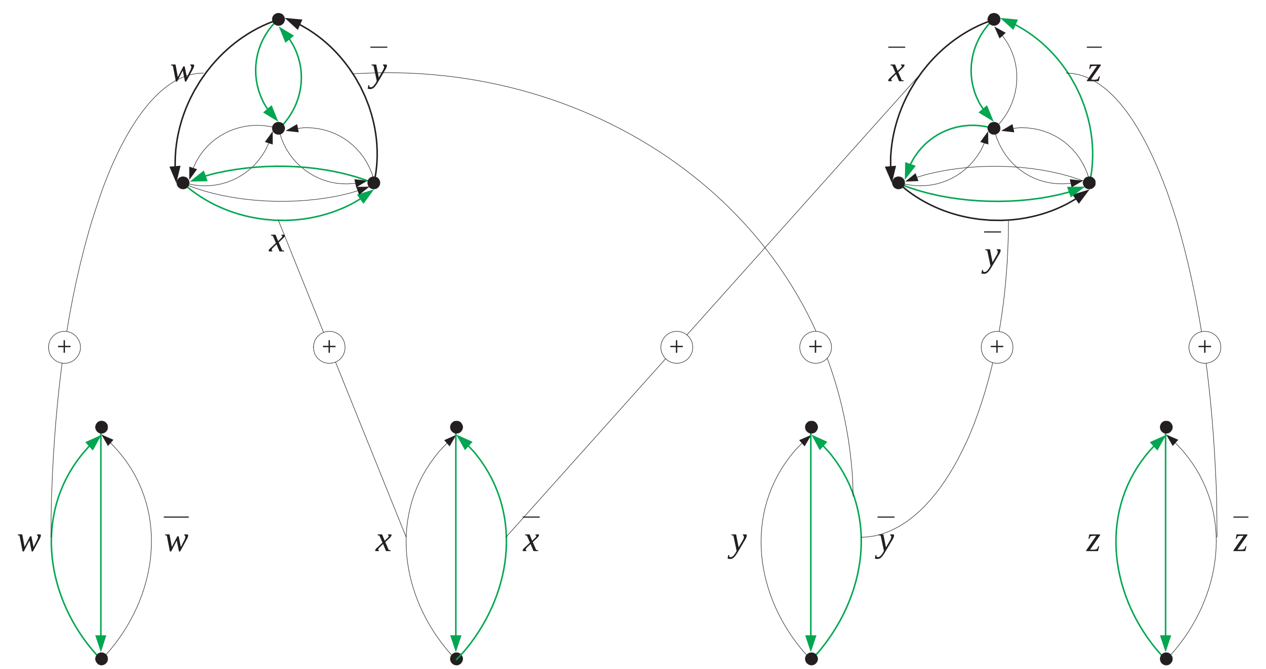


[a]The XOR gadgets (p. 887) are rotated here.

The Proof: a Key Observation (continued)

Each satisfying truth assignment to ϕ corresponds to a schematic cycle cover that respects the XOR gadgets.

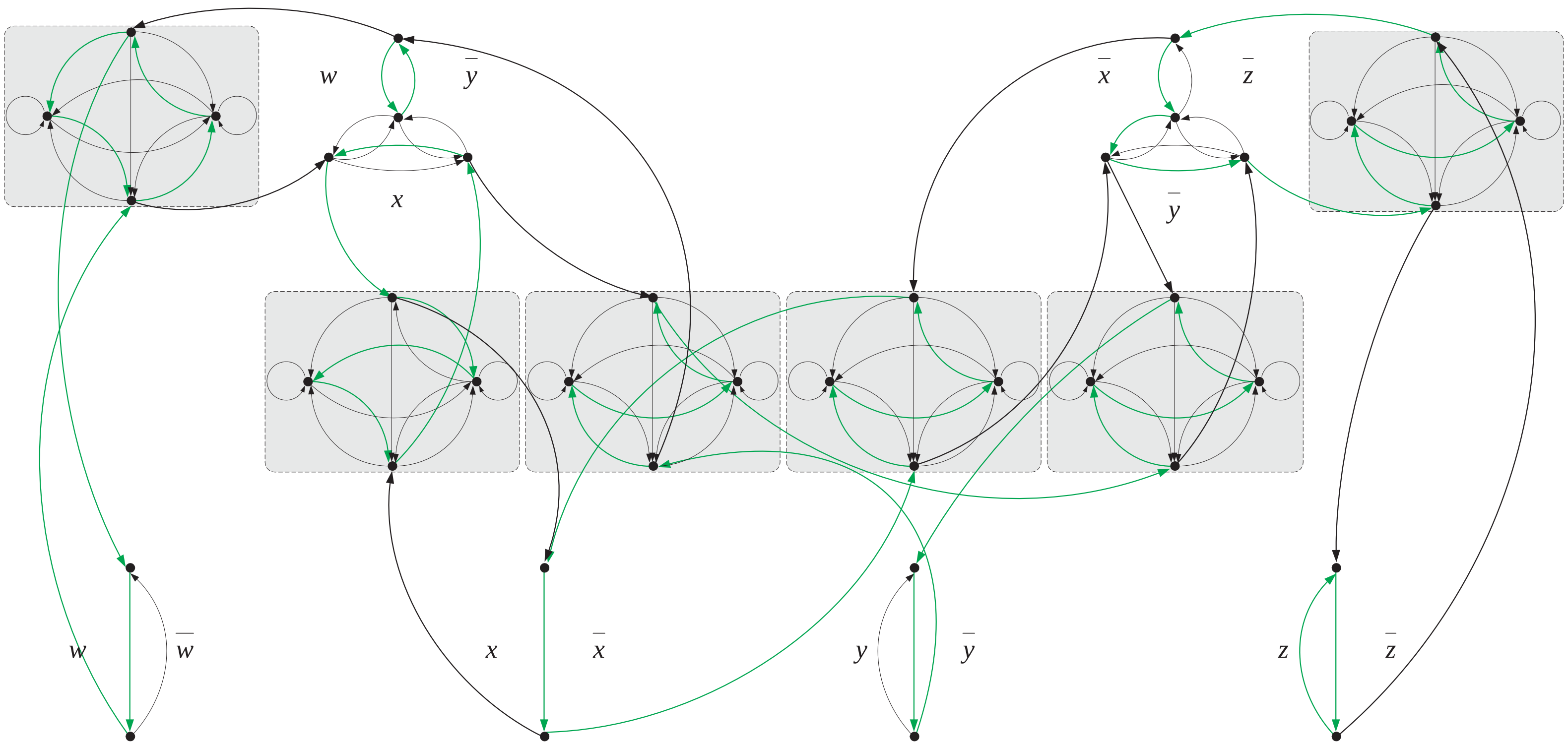
$w = 1, x = 0, y = 0, z = 1 \Leftrightarrow$ One Cycle Cover



The Proof: a Key Corollary (continued)

- Recall that there are $3m$ XOR gadgets.
- Each satisfying truth assignment to ϕ contributes 4^{3m} to the cycle count $\#H$.
- Hence
$$\#H = 4^{3m} \times \#\phi, \tag{28}$$
as desired.

"$w = 1, x = 0, y = 0, z = 1$" Adds 4^6 to Cycle Count



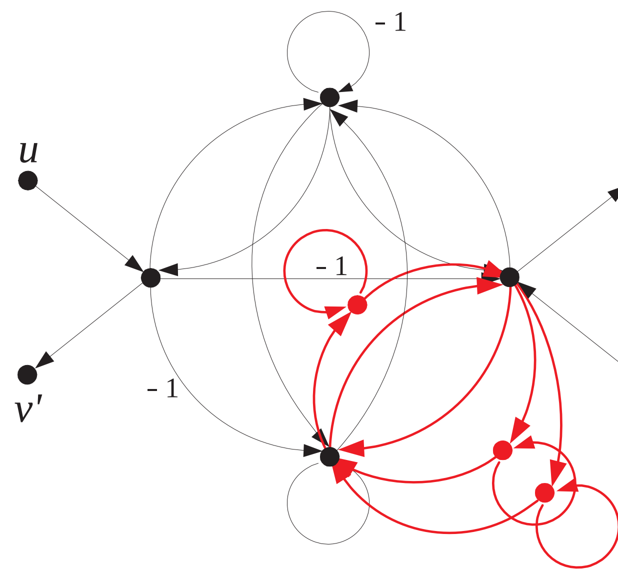
The Proof (continued)

- We are almost done.
- The weighted directed graph H needs to be *efficiently* replaced by an unweighted graph G.
- Furthermore, knowing $\#G$ should enable us to calculate $\#H$ *efficiently*.
  - This done, $\#\phi$ will have been Turing-reducible to $\#G$.[a]
- We proceed to construct this graph G.

[a] By way of $\#H$ of course.

The Proof: Construction of G (continued)

- Replace edges of the XOR gadget (p. 887) with weights 2 and 3 without creating parallel edges:



- The cycle count $\#H$ remains *unchanged*.

The Proof: Construction of G (continued)

- We move on to edges with weight -1.
- First, we count the number of nodes, M.
- Each clause gadget (p. 884) contains 4 nodes, and there are m of them (one per clause).
- Each choice gadget (p. 895)contains 2 nodes, and there are $n \leq 3m$ of them (one per variable).
- Each revised XOR gadget (p. 903) contains 7 nodes, and there are $3m$ of them (one per literal).
- So

$$M \leq 4m + 6m + 21m = 31m.$$

The Proof: Construction of G (continued)

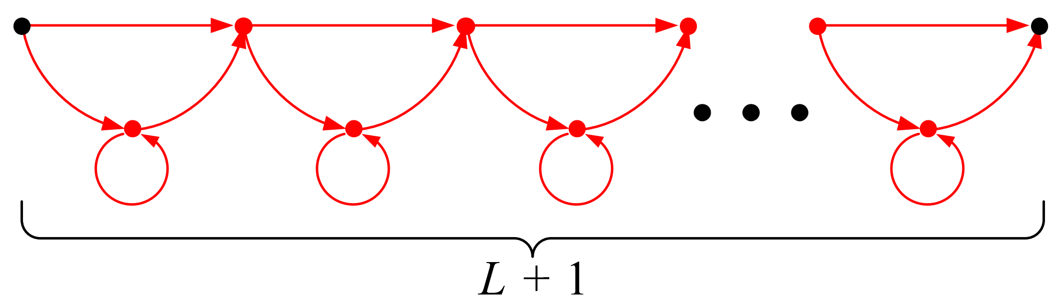
- $\# H \leq 2^L$ for some $L = O(m \log m)$.
  - The maximum absolute value of the edge weight is 1.
  - Hence each term in the permanent is at most 1.
  - There are $M! \leq (31m)!$ terms.
  - Hence

$$
\begin{aligned}
\# H &\leq \sqrt{2\pi(31m)} \left(\frac{31m}{e}\right)^{31m} e^{\frac{1}{12 \times (31m)}} \\
&= 2^{O(m \log m)} \qquad (29)
\end{aligned}
$$

by a refined Stirling's formula.

The Proof: Construction of G (continued)

- Replace each edge with weight -1 with the following:



- Each increases the number of cycle covers 2^{L+1}-fold.
- The desired unweighted G has been obtained.

The Proof (continued)

- $\#G$ equals $\#H$ after replacing each appearance -1 in $\#H$ with 2^{L+1}:

$$\begin{aligned}
\#H &= \cdots + \overbrace{1 \cdot 1 \cdots (-1) \cdots 1}^{\text{a cycle cover (n terms)}} + \cdots\,, \\
\#G &= \cdots + \overbrace{1 \cdot 1 \cdots 2^{L+1} \cdots 1}^{\text{a cycle cover (n terms)}} + \cdots\,.
\end{aligned}$$

- Let $\#G = \sum_{i=0}^{n} a_i \times (2^{L+1})^i$, where $0 \leq a_i < 2^{L+1}$.
- Recall that $\#H \leq 2^L$ (p. 905).
- So each a_i counts the number of cycle covers with i edges of weight -1 as there is no "overflow" in $\#G$.

The Proof (concluded)

- We conclude that

$$\#H = a_0 - a_1 + a_2 - \cdots + (-1)^n a_n,$$

  indeed easily computable from $\#G$.

- We know $\#H = 4^{3m} \times \#\phi$ from Eq. (28) on p. 900.

- So

$$\#\phi = \frac{a_0 - a_1 + a_2 - \cdots + (-1)^n a_n}{4^{3m}}.$$

  - Equivalently,

$$\#\phi = \frac{\#G \bmod (2^{L+1} + 1)}{4^{3m}}.$$

Finis