Primality Tests

- PRIMES asks if a number N is a prime.
- The classic algorithm tests if $k \mid N$ for $k = 2, 3, \ldots, \sqrt{N}$.
- But it runs in $\Omega(2^{(\log_2 N)/2})$ steps.

The Fermat Test for Primality

Fermat's "little" theorem (p. 493) suggests the following primality test for any given number N:

Pick a number a randomly from $\{ 1, 2, \ldots, N - 1 \}$;
if $a^{N-1} \not\equiv 1 \bmod N$ **then**
    return "N is composite";
else
    return "N is (probably) a prime";
end if

The Fermat Test for Primality (concluded)

- **Carmichael numbers** are composite numbers that will pass the Fermat test for *all* $a \in \{ 1, 2, \ldots, N-1 \}$.[a]
  - The Fermat test will return "N is a prime" for all Carmichael numbers N.
- Unfortunately, there are infinitely many Carmichael numbers.[b]
- In fact, the number of Carmichael numbers less than N exceeds $N^{2/7}$ for N large enough.
- So the Fermat test is an incorrect algorithm for PRIMES.

[a] Carmichael (1910). Lo (1994) mentions an investment strategy based on such numbers!

[b] Alford, Granville, & Pomerance (1992).

Square Roots Modulo a Prime

- Equation $x^2 \equiv a \bmod p$ has at most two (distinct) roots by Lemma 64 (p. 498).
  - The roots are called **square roots**.
  - Numbers a with square roots *and* $\gcd(a, p) = 1$ are called **quadratic residues**.
    * They are

$$1^2 \bmod p, 2^2 \bmod p, \ldots, (p-1)^2 \bmod p.$$

- We shall show that a number either has two roots or has none, and testing which is the case is trivial.[a]

[a] But no efficient *deterministic* general-purpose square-root-extracting algorithms are known yet.

Euler's Test

Lemma 69 (Euler) *Let p be an odd prime and $a \neq 0 \bmod p$.*

1. *If*
$$a^{(p-1)/2} \equiv 1 \bmod p,$$
then $x^2 \equiv a \bmod p$ has two roots.

2. *If*
$$a^{(p-1)/2} \not\equiv 1 \bmod p,$$
then
$$a^{(p-1)/2} \equiv -1 \bmod p$$
and $x^2 \equiv a \bmod p$ has no roots.

The Proof (continued)

- Let r be a primitive root of p.
- Fermat's "little" theorem says $r^{p-1} \equiv 1 \bmod p$, so

$$r^{(p-1)/2}$$

  is a square root of 1.
- In particular,

$$r^{(p-1)/2} \equiv 1 \text{ or } -1 \bmod p.$$

- But as r is a primitive root, $r^{(p-1)/2} \not\equiv 1 \bmod p$.
- Hence $r^{(p-1)/2} \equiv -1 \bmod p$.

The Proof (continued)

- Let $a = r^k \bmod p$ for some k.
- Suppose $a^{(p-1)/2} \equiv 1 \bmod p$.
- Then

$$1 \equiv a^{(p-1)/2} \equiv r^{k(p-1)/2} \equiv \left[r^{(p-1)/2} \right]^k \equiv (-1)^k \bmod p.$$

- So k must be even.

The Proof (continued)

- Suppose $a = r^{2j} \bmod p$ for some $1 \leq j \leq (p-1)/2$.
- Then

$$a^{(p-1)/2} \equiv r^{j(p-1)} \equiv 1 \bmod p.$$

- The two *distinct* roots of a are

$$r^j, -r^j (\equiv r^{j+(p-1)/2} \bmod p).$$

  - If $r^j \equiv -r^j \bmod p$, then $2r^j \equiv 0 \bmod p$, which implies $r^j \equiv 0 \bmod p$, a contradiction as r is a primitive root.

The Proof (continued)

- As $1 \leq j \leq (p-1)/2$, there are $(p-1)/2$ such a's.
- Each such $a \equiv r^{2j} \bmod p$ has 2 distinct square roots.
- The square roots of all these a's are distinct.
  - The square roots of *different* a's must be different.
- Hence the set of *square roots* is $\{1, 2, \ldots, p-1\}$.
- As a result,

$$a = r^{2j} \bmod p, 1 \leq j \leq (p-1)/2,$$

exhaust all the quadratic residues.

The Proof (concluded)

- Suppose $a = r^{2j+1} \bmod p$ now.
- Then it has no square roots because all the square roots have been taken.
- Finally,

$$a^{(p-1)/2} \equiv \left[r^{(p-1)/2} \right]^{2j+1} \equiv (-1)^{2j+1} \equiv -1 \bmod p.$$

The Legendre Symbol[a] and Quadratic Residuacity Test

- By Lemma 69 (p. 560),

$$a^{(p-1)/2} \bmod p = \pm 1$$

for $a \not\equiv 0 \bmod p$.

- For odd prime p, define the **Legendre symbol** $(a \,|\, p)$ as

$$(a \,|\, p) \triangleq \begin{cases} 0, & \text{if } p \,|\, a, \\ 1, & \text{if } a \text{ is a quadratic residue modulo } p, \\ -1, & \text{if } a \text{ is a } \textbf{quadratic nonresidue} \text{ modulo } p. \end{cases}$$

- It is sometimes pronounced "a over p."

[a] Andrien-Marie Legendre (1752–1833).

The Legendre Symbol and Quadratic Residuacity Test (concluded)

- Euler's test (p. 560) implies

$$a^{(p-1)/2} \equiv (a \,|\, p) \bmod p$$

for any odd prime p and any integer a.

- Note that $(ab \,|\, p) = (a \,|\, p)(b \,|\, p)$.

Gauss's Lemma

Lemma 70 (Gauss) *Let p and q be two distinct odd primes. Then $(q \mid p) = (-1)^m$, where m is the number of residues in $R \stackrel{\Delta}{=} \{\, iq \bmod p : 1 \leq i \leq (p-1)/2 \,\}$ that are greater than $(p-1)/2$.*

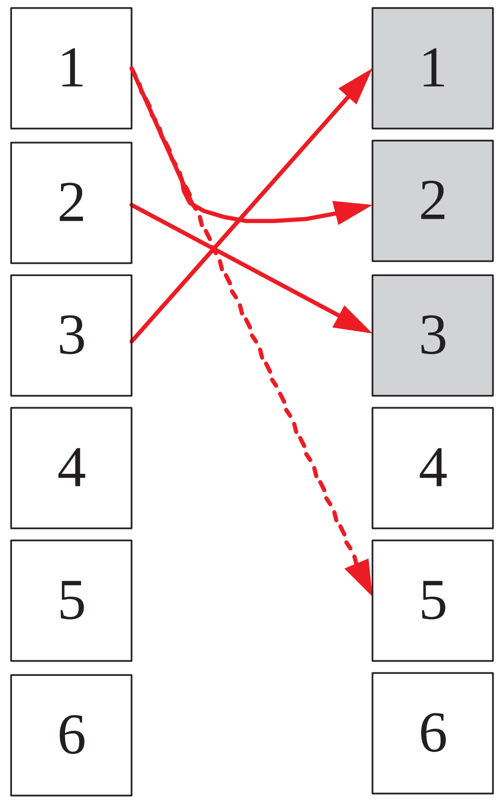
- All residues in R are distinct.
  - If $iq = jq \bmod p$, then $p \mid (j - i)$ or $p \mid q$.
  - But neither is possible.
- No two elements of R add up to p.
  - If $iq + jq \equiv 0 \bmod p$, then $p \mid (i + j)$ or $p \mid q$.
  - But neither is possible.

The Proof (continued)

- Replace each of the m elements $a \in R$ such that $a > (p-1)/2$ by $p - a$.
  - This is equivalent to performing $-a \bmod p$.
- Call the resulting set of residues R'.
- All numbers in R' are at most $(p-1)/2$.
- In fact, $R' = \{ 1, 2, \ldots, (p-1)/2 \}$ (see illustration next page).
  - Otherwise, two elements of R would add up to p,[a] which has been shown to be impossible.

---

[a]Because then $iq \equiv -jq \bmod p$ for some $i \neq j$.



$p = 7$ and $q = 5$.

The Proof (concluded)

- Alternatively, $R' = \{ \pm iq \bmod p : 1 \leq i \leq (p-1)/2 \}$, where exactly m of the elements have the minus sign.
- Take the product of all elements in the two representations of R'.
- So

$$[(p-1)/2]! \equiv (-1)^m q^{(p-1)/2} [(p-1)/2]! \bmod p.$$

- Because $\gcd([(p-1)/2]!, p) = 1$, the above implies

$$1 = (-1)^m q^{(p-1)/2} \bmod p.$$

Legendre's Law of Quadratic Reciprocity[a]

- Let p and q be two distinct odd primes.
- The next result says $(p \,|\, q)$ and $(q \,|\, p)$ are distinct if and only if both p and q are 3 mod 4.

Lemma 71 (Legendre, 1785; Gauss)

$$(p \,|\, q)(q \,|\, p) = (-1)^{\frac{p-1}{2} \frac{q-1}{2}}.$$

---

[a]First stated by Euler in 1751. Legendre (1785) did not give a correct proof. Gauss proved the theorem when he was 19. He gave at least 8 different proofs during his life. The 152nd proof appeared in 1963. A computer-generated formal proof was given in Russinoff (1990). As of 2008, there had been 4 such proofs. Wiedijk (2008), "the Law of Quadratic Reciprocity is the first nontrivial theorem that a student encounters in the mathematics curriculum."

The Proof (continued)

- Sum the elements of R' in the previous proof in mod2.
- On one hand, this is just $\sum_{i=1}^{(p-1)/2} i \bmod 2$.
- On the other hand, the sum equals

$$
\begin{aligned}
& mp + \sum_{i=1}^{(p-1)/2} \left(iq - p \left\lfloor \frac{iq}{p} \right\rfloor \right) \bmod 2 \\
= \quad & mp + \left(q \sum_{i=1}^{(p-1)/2} i - p \sum_{i=1}^{(p-1)/2} \left\lfloor \frac{iq}{p} \right\rfloor \right) \bmod 2.
\end{aligned}
$$

  - m of the $iq \bmod p$ are replaced by $p - iq \bmod p$.
  - But signs are irrelevant under mod2.
  - m is as in Lemma 70 (p. 568).

The Proof (continued)

- Ignore odd multipliers to make the sum equal

$$m + \left(\sum_{i=1}^{(p-1)/2} i - \sum_{i=1}^{(p-1)/2} \left\lfloor \frac{iq}{p} \right\rfloor \right) \bmod 2.$$

- Equate the above with $\sum_{i=1}^{(p-1)/2} i$ modulo 2.
- Now simplify to obtain

$$m \equiv \sum_{i=1}^{(p-1)/2} \left\lfloor \frac{iq}{p} \right\rfloor \bmod 2.$$

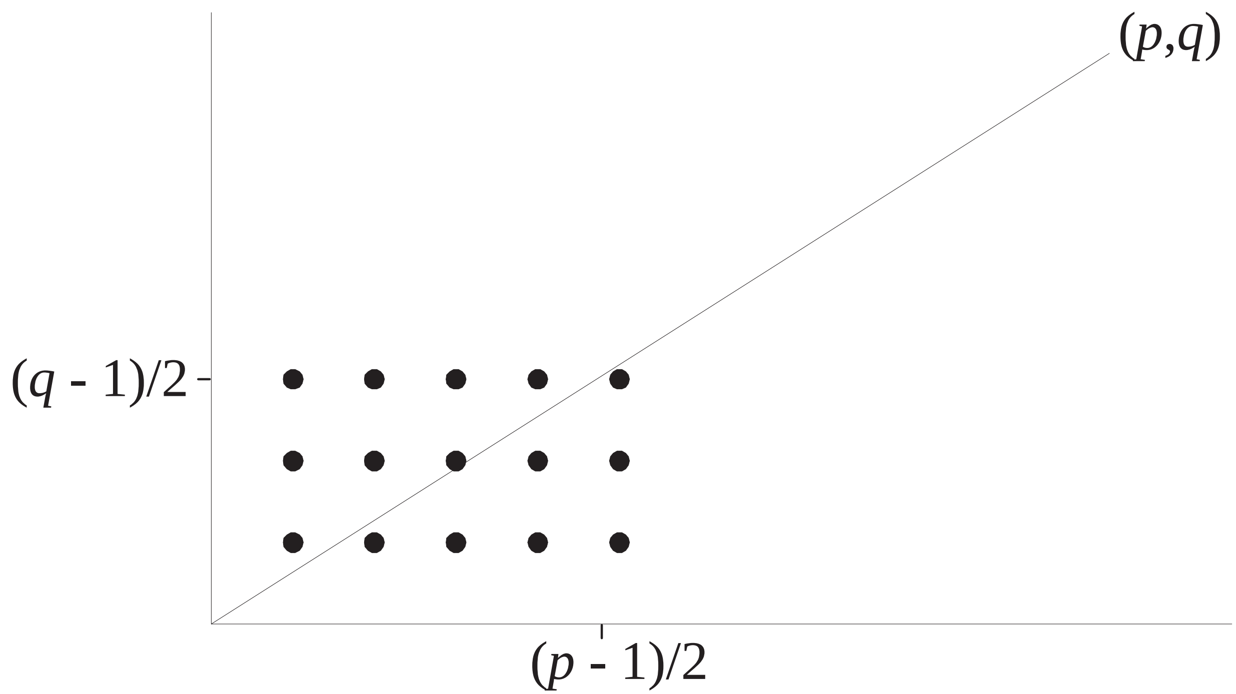
The Proof (continued)

- $\sum_{i=1}^{(p-1)/2} \lfloor \frac{iq}{p} \rfloor$ is the number of integral points *below* the line
$$y = (q/p)\, x$$
for $1 \leq x \leq (p-1)/2$.
- Gauss's lemma (p. 568) says $(q \,|\, p) = (-1)^m$.
- Repeat the proof with p and q reversed.
- Then $(p \,|\, q) = (-1)^{m'}$, where m' is the number of integral points *above* the line $y = (q/p)\, x$ for $1 \leq y \leq (q-1)/2$.

The Proof (concluded)

- As a result,
$$(p\,|\,q)(q\,|\,p) = (-1)^{m+m'}.$$
- But $m + m'$ is the total number of integral points in the $[1, \frac{p-1}{2}] \times [1, \frac{q-1}{2}]$ rectangle, which is
$$\frac{p-1}{2}\,\frac{q-1}{2}.$$

Eisenstein's Rectangle



Above, $p = 11$, $q = 7$, $m = 7$, $m' = 8$.

The Jacobi Symbol[a]

- The Legendre symbol only works for odd *prime* moduli.
- The **Jacobi symbol** $(a \,|\, m)$ extends it to cases where m is not prime.
  - a is sometimes called the **numerator** and m the **denominator**.
- Trivially, $(1 \,|\, m) = 1$.
- Define $(a \,|\, 1) = 1$.

[a]Carl Jacobi (1804–1851).

The Jacobi Symbol (concluded)

- Let $m = p_1 p_2 \cdots p_k$ be the prime factorization of m.
- When $m > 1$ is odd and $\gcd(a, m) = 1$, then

$$(a \,|\, m) \stackrel{\Delta}{=} \prod_{i=1}^{k} (a \,|\, p_i).$$

  - Note that the Jacobi symbol equals ± 1.
  - It reduces to the Legendre symbol when m is a prime.

Properties of the Jacobi Symbol

The Jacobi symbol has the following properties when it is defined.

1. $(ab \,|\, m) = (a \,|\, m)(b \,|\, m)$.
2. $(a \,|\, m_1 m_2) = (a \,|\, m_1)(a \,|\, m_2)$.
3. If $a \equiv b \bmod m$, then $(a \,|\, m) = (b \,|\, m)$.
4. $(-1 \,|\, m) = (-1)^{(m-1)/2}$ (by Lemma 70 on p. 568).
5. $(2 \,|\, m) = (-1)^{(m^2-1)/8}$.[a]
6. If a and m are both odd, then $(a \,|\, m)(m \,|\, a) = (-1)^{(a-1)(m-1)/4}$.

---

[a]By Lemma 70 (p. 568) and some parity arguments.

Properties of the Jacobi Symbol (concluded)

- Properties 3–6 allow us to calculate the Jacobi symbol *without* factorization.
  - It will also yield the same result as Euler's test[a] when m is an odd prime.
- This situation is similar to the Euclidean algorithm.
- Note also that $(a \mid m) = 1/(a \mid m)$ because $(a \mid m) = \pm 1$.[b]

---

[a] Recall p. 560.

[b] Contributed by Mr. Huang, Kuan-Lin (B96902079, R00922018) on December 6, 2011.

Calculation of $(2200 \,|\, 999)$

$$
\begin{aligned}
(2200 \,|\, 999) &= (202 \,|\, 999) \\
&= (2 \,|\, 999)(101 \,|\, 999) \\
&= (-1)^{(999^2-1)/8}(101 \,|\, 999) \\
&= (-1)^{124750}(101 \,|\, 999) = (101 \,|\, 999) \\
&= (-1)^{(100)(998)/4}(999 \,|\, 101) = (-1)^{24950}(999 \,|\, 101) \\
&= (999 \,|\, 101) = (90 \,|\, 101) = (-1)^{(101^2-1)/8}(45 \,|\, 101) \\
&= (-1)^{1275}(45 \,|\, 101) = -(45 \,|\, 101) \\
&= -(-1)^{(44)(100)/4}(101 \,|\, 45) = -(101 \,|\, 45) = -(11 \,|\, 45) \\
&= -(-1)^{(10)(44)/4}(45 \,|\, 11) = -(45 \,|\, 11) \\
&= -(1 \,|\, 11) = -1.
\end{aligned}
$$

A Result Generalizing Proposition 10.3 in the Textbook

Theorem 72 *The group of set* $\Phi(n)$ *under multiplication* mod n *has a primitive root if and only if* n *is either 1, 2, 4,* p^k*, or* $2p^k$ *for some nonnegative integer* k *and an odd prime* p*.*

This result is essential in the proof of the next lemma.

The Jacobi Symbol and Primality Test[a]

Lemma 73 *If* $(M \mid N) \equiv M^{(N-1)/2} \bmod N$ *for all* $M \in \Phi(N)$*, then* N *is a prime. (Assume* N *is odd.)*

- Assume $N = mp$, where p is an odd prime, $\gcd(m, p) = 1$, and $m > 1$ (not necessarily prime).
- Let $r \in \Phi(p)$ such that $(r \mid p) = -1$.
- The Chinese remainder theorem says that there is an $M \in \Phi(N)$ such that

$$\begin{aligned} M &= r \bmod p, \\ M &= 1 \bmod m. \end{aligned}$$

[a]Mr. Clement Hsiao (`B4506061`, `R88526067`) pointed out that the textbook's proof for Lemma 11.8 is incorrect in January 1999 while he was a senior.

The Proof (continued)

- By the hypothesis,

$$M^{(N-1)/2} = (M \mid N) = (M \mid p)(M \mid m) = -1 \bmod N.$$

- Hence

$$M^{(N-1)/2} = -1 \bmod m.$$

- But because $M = 1 \bmod m$,

$$M^{(N-1)/2} = 1 \bmod m,$$

a contradiction.

The Proof (continued)

- Second, assume that $N = p^a$, where p is an odd prime and $a \geq 2$.
- By Theorem 72 (p. 583), there exists a primitive root r modulo p^a.
- From the assumption,

$$M^{N-1} = \left[M^{(N-1)/2} \right]^2 = (M|N)^2 = 1 \bmod N$$

for all $M \in \Phi(N)$.

The Proof (continued)

- As $r \in \Phi(N)$ (prove it), we have
$$r^{N-1} = 1 \bmod N.$$
- As r's exponent modulo $N = p^a$ is $\phi(N) = p^{a-1}(p-1)$,
$$p^{a-1}(p-1) \,|\, (N-1),$$
which implies that $p \,|\, (N-1)$.
- But this is impossible given that $p \,|\, N$.

The Proof (continued)

- Third, assume that $N = mp^a$, where p is an odd prime, $\gcd(m, p) = 1$, $m > 1$ (not necessarily prime), and a is even.
- The proof mimics that of the second case.
- By Theorem 72 (p. 583), there exists a primitive root r modulo p^a.
- From the assumption,

$$M^{N-1} = \left[M^{(N-1)/2} \right]^2 = (M|N)^2 = 1 \bmod N$$

for all $M \in \Phi(N)$.

The Proof (continued)

- In particular,

$$M^{N-1} = 1 \bmod p^a \tag{15}$$

  for all $M \in \Phi(N)$.

- The Chinese remainder theorem says that there is an $M \in \Phi(N)$ such that

$$\begin{aligned} M &= r \bmod p^a, \\ M &= 1 \bmod m. \end{aligned}$$

- Because $M = r \bmod p^a$ and Eq. (15),

$$r^{N-1} = 1 \bmod p^a.$$

The Proof (concluded)

- As r's exponent modulo $N = p^a$ is $\phi(N) = p^{a-1}(p-1)$,

$$p^{a-1}(p-1) \,|\, (N-1),$$

which implies that $p \,|\, (N-1)$.

- But this is impossible given that $p \,|\, N$.

The Number of Witnesses to Compositeness

Theorem 74 (Solovay & Strassen, 1977) *If N is an odd composite, then $(M \mid N) \equiv M^{(N-1)/2} \bmod N$ for at most half of $M \in \Phi(N)$.*

- By Lemma 73 (p. 584) there is at least one $a \in \Phi(N)$ such that $(a \mid N) \not\equiv a^{(N-1)/2} \bmod N$.
- Let $B \stackrel{\Delta}{=} \{ b_1, b_2, \ldots, b_k \} \subseteq \Phi(N)$ be the set of *all* distinct residues such that $(b_i \mid N) \equiv b_i^{(N-1)/2} \bmod N$.
- Let $aB \stackrel{\Delta}{=} \{ ab_i \bmod N : i = 1, 2, \ldots, k \}$.
- Clearly, $aB \subseteq \Phi(N)$, too.

The Proof (concluded)

- $|\, aB \,| = k$.
  - $ab_i \equiv ab_j \bmod N$ implies $N \mid a(b_i - b_j)$, which is impossible because $\gcd(a, N) = 1$ and $N > |\, b_i - b_j \,|$.
- $aB \cap B = \emptyset$ because

  $$(ab_i)^{(N-1)/2} \equiv a^{(N-1)/2} b_i^{(N-1)/2} \not\equiv (a \mid N)(b_i \mid N) \equiv (ab_i \mid N).$$

- Combining the above two results, we know

$$\frac{|\, B \,|}{\phi(N)} \leq \frac{|\, B \,|}{|\, B \cup aB \,|} = 0.5.$$

if N is even but $N \neq 2$ **then**
  return "N is composite";
else if $N = 2$ **then**
  return "N is a prime";
end if
Pick $M \in \{ 2, 3, \ldots, N - 1 \}$ randomly;
if $\gcd(M, N) > 1$ **then**
  return "N is composite";
else
  if $(M \mid N) \equiv M^{(N-1)/2} \bmod N$ **then**
    return "N is (probably) a prime";
  else
    return "N is composite";
  end if
end if

Analysis

- The algorithm certainly runs in polynomial time.
- There are no false positives (for COMPOSITENESS).
  - When the algorithm says the number is composite, it is always correct.

Analysis (concluded)

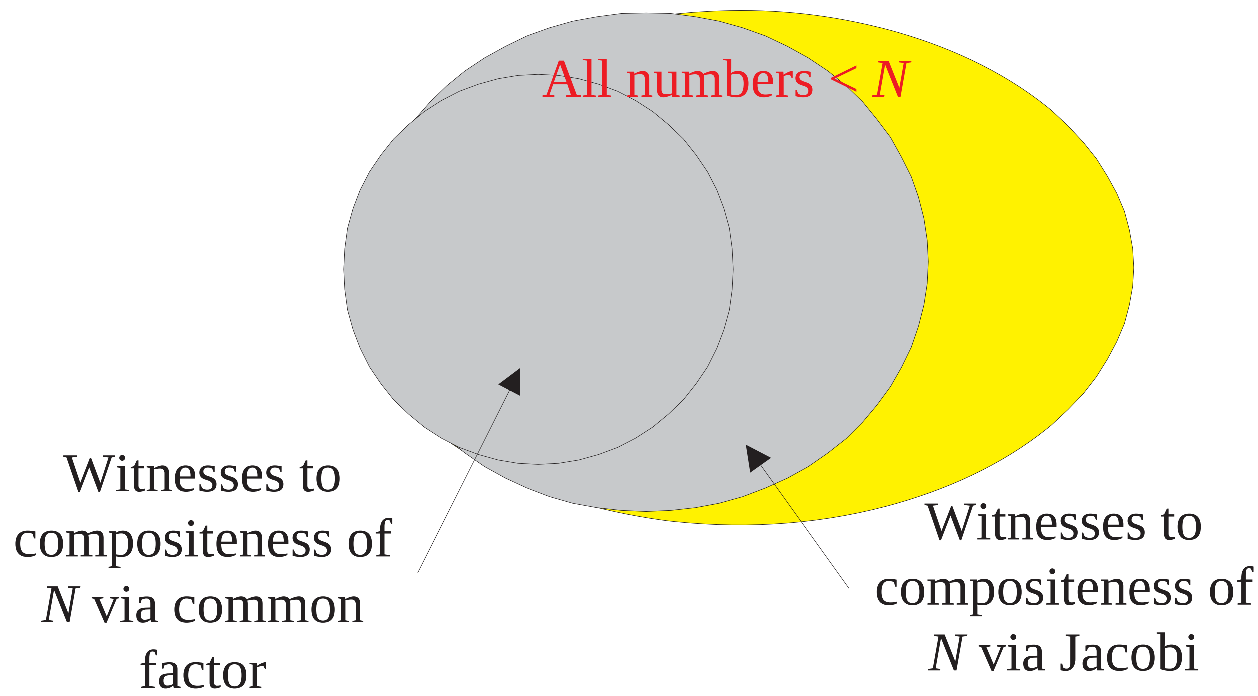
- The probability of a false negative (again, for COMPOSITENESS) is at most one half.
  - Suppose the input is composite.
  - By Theorem 74 (p. 591),

$$\text{prob}[\,\text{algorithm answers "no"} \mid N \text{ is composite}\,] \leq 0.5.$$

  - Note that we are not referring to the probability that N is composite when the algorithm says "no."
- So it is a Monte Carlo algorithm for COMPOSITENESS[a] by the definition on p. 539.

[a] Not PRIMES.

The Improved Density Attack for COMPOSITENESS



Randomized Complexity Classes; RP

- Let N be a polynomial-time precise NTM that runs in time $p(n)$ and has 2 nondeterministic choices at each step.
- N is a **polynomial Monte Carlo Turing machine** for a language L if the following conditions hold:
  - If $x \in L$, then at least half of the $2^{p(n)}$ computation paths of N on x halt with "yes" where $n = |\, x \,|$.
  - If $x \notin L$, then all computation paths halt with "no."
- The class of all languages with polynomial Monte Carlo TMs is denoted **RP** (**randomized polynomial time**).[a]

[a] Adleman & Manders (1977).

Comments on RP

- In analogy to Proposition 41 (p. 335), a "yes" instance of an RP problem has many certificates (witnesses).
- There are no false positives.
- If we associate nondeterministic steps with flipping fair coins, then we can phrase RP in the language of probability.
  - If $x \in L$, then $N(x)$ halts with "yes" with probability at least 0.5.
  - If $x \notin L$, then $N(x)$ halts with "no."

Comments on RP (concluded)

- The probability of false negatives is ≤ 0.5.
- But *any* constant ϵ between 0 and 1 can replace 0.5.
  - Repeat the algorithm

$$k \triangleq \left\lceil -\frac{1}{\log_2 \epsilon} \right\rceil$$

    times.
  - Answer "no" only if all the runs answer "no."
  - The probability of false negatives becomes $\epsilon^k \leq 0.5$.

Where RP Fits

- $\text{P} \subseteq \text{RP} \subseteq \text{NP}$.
  - A deterministic TM is like a Monte Carlo TM except that all the coin flips are ignored.
  - A Monte Carlo TM is an NTM with more demands on the number of accepting paths.
- COMPOSITENESS $\in$ RP;[a] PRIMES $\in$ coRP; PRIMES $\in$ RP.[b]
  - In fact, PRIMES $\in$ P.[c]
- $\text{RP} \cup \text{coRP}$ is an alternative "plausible" notion of efficient computation.

---

[a] Rabin (1976); Solovay & Strassen (1977).
[b] Adleman & Huang (1987).
[c] Agrawal, Kayal, & Saxena (2002).

ZPP[a] (Zero Probabilistic Polynomial)

- The class **ZPP** is defined as RP $\cap$ coRP.
- A language in ZPP has *two* Monte Carlo algorithms, one with no false positives (RP) and the other with no false negatives (coRP).
- If we repeatedly run both Monte Carlo algorithms, *eventually* one definite answer will come (unlike RP).
  - A *positive* answer from the one without false positives.
  - A *negative* answer from the one without false negatives.

---

[a] Gill (1977).

The ZPP Algorithm (**Las Vegas**)

{Suppose $L \in$ ZPP.}
{N_1 has no false positives, and N_2 has no false negatives.}
while `true` **do**
    if $N_1(x) =$ "yes" **then**
        return "yes";
    end if
    if $N_2(x) =$ "no" **then**
        return "no";
    end if
end while

ZPP (concluded)

- The *expected* running time for the correct answer to emerge is polynomial.
  - The probability that a run of the 2 algorithms does not generate a definite answer is 0.5 (why?).
  - Let $p(n)$ be the running time of each run of the while-loop.
  - The expected running time for a definite answer is

$$\sum_{i=1}^{\infty} 0.5^i i p(n) = 2p(n).$$

- Essentially, ZPP is the class of problems that can be solved, without errors, in expected polynomial time.

Large Deviations

- Suppose you have a *biased* coin.
- One side has probability $0.5 + \epsilon$ to appear and the other $0.5 - \epsilon$, for some $0 < \epsilon < 0.5$.
- But you do not know which is which.
- How to decide which side is the more likely side—with high confidence?
- Answer: Flip the coin many times and pick the side that appeared the most times.
- Question: Can you quantify your confidence?

The (Improved) Chernoff Bound[a]

Theorem 75 (Chernoff, 1952) *Suppose* $x_1, x_2, \ldots, x_n$ *are independent random variables taking the values 1 and 0 with probabilities* p *and* $1 - p$*, respectively. Let* $X = \sum_{i=1}^{n} x_i$*. Then for all* $0 \leq \theta \leq 1$,

$$\text{prob}[\, X \geq (1 + \theta)\, pn \,] \leq e^{-\theta^2 pn/3}.$$

- The probability that the deviate of a **binomial random variable** from its expected value $E[\, X\,] = E\left[\, \sum_{i=1}^{n} x_i \,\right] = pn$ decreases exponentially with the deviation.

[a]Herman Chernoff (1923–). This bound is asymptotically optimal. The original bound is $e^{-2\theta^2 p^2 n}$ (McDiarmid, 1998).

The Proof

- Let t be any positive real number.
- Then

$$\text{prob}[\, X \geq (1+\theta)\, pn \,] = \text{prob}[\, e^{tX} \geq e^{t(1+\theta)\, pn} \,].$$

- Markov's inequality (p. 542) generalized to real-valued random variables says that

$$\text{prob}\left[\, e^{tX} \geq kE[\, e^{tX} \,] \,\right] \leq 1/k.$$

- With $k = e^{t(1+\theta)\, pn} / E[\, e^{tX} \,]$, we have[a]

$$\text{prob}[\, X \geq (1+\theta)\, pn \,] \leq e^{-t(1+\theta)\, pn} E[\, e^{tX} \,].$$

[a] Note that X does not appear in k. Contributed by Mr. Ao Sun (`R05922147`) on December 20, 2016.

The Proof (continued)

- Because $X = \sum_{i=1}^{n} x_i$ and x_i's are independent,

$$E[\, e^{tX} \,] = (E[\, e^{tx_1} \,])^n = [\, 1 + p(e^t - 1) \,]^n.$$

- Substituting, we obtain

$$\begin{aligned} \text{prob}[\, X \geq (1+\theta)\, pn \,] &\leq e^{-t(1+\theta)\, pn} [\, 1 + p(e^t - 1) \,]^n \\ &\leq e^{-t(1+\theta)\, pn} e^{pn(e^t - 1)} \end{aligned}$$

as $(1 + a)^n \leq e^{an}$ for all $a > 0$.

The Proof (concluded)

- With the choice of $t = \ln(1+\theta)$, the above becomes

$$\text{prob}[\, X \geq (1+\theta)\, pn\,] \leq e^{pn[\,\theta-(1+\theta)\ln(1+\theta)\,]}.$$

- The exponent expands to[a]

$$-\frac{\theta^2}{2} + \frac{\theta^3}{6} - \frac{\theta^4}{12} + \cdots$$

for $0 \leq \theta \leq 1$.

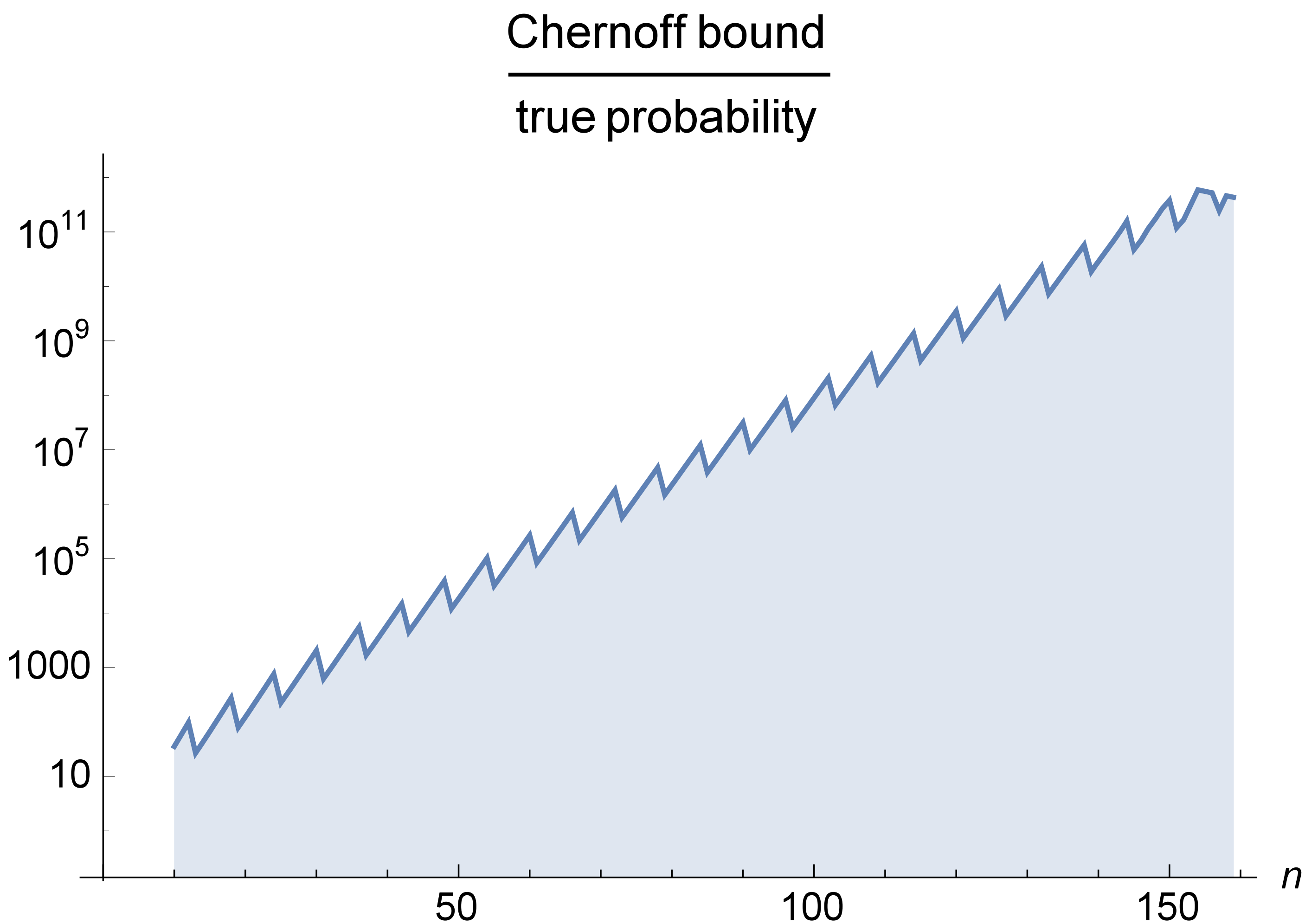
- But it is less than

$$-\frac{\theta^2}{2} + \frac{\theta^3}{6} \leq \theta^2 \left(-\frac{1}{2} + \frac{\theta}{6} \right) \leq \theta^2 \left(-\frac{1}{2} + \frac{1}{6} \right) = -\frac{\theta^2}{3}.$$

---

[a]Or McDiarmid (1998): $x-(1+x)\ln(1+x) \leq -3x^2/(6+2x)$ for all $x \geq 0$.

How Good Is the Bound?



Other Variations of the Chernoff Bound

The following can be proved similarly (prove it).

Theorem 76 *Given the same terms as Theorem 75 (p. 605),*

$$\text{prob}[\, X \leq (1-\theta)\, pn \,] \leq e^{-\theta^2 pn/2}.$$

The following slightly looser inequalities achieve symmetry.

Theorem 77 (Karp, Luby, & Madras, 1989) *Given the same terms as Theorem 75 (p. 605) except with* $0 \leq \theta \leq 2$*,*

$$\begin{aligned} \text{prob}[\, X \geq (1+\theta)\, pn \,] &\leq e^{-\theta^2 pn/4}, \\ \text{prob}[\, X \leq (1-\theta)\, pn \,] &\leq e^{-\theta^2 pn/4}. \end{aligned}$$

Power of the Majority Rule

The next result follows from Theorem 76 (p. 610).

Corollary 78 *If* $p = (1/2) + \epsilon$ *for some* $0 \leq \epsilon \leq 1/2$, *then*

$$\text{prob}\left[\sum_{i=1}^{n} x_i \leq n/2\right] \leq e^{-\epsilon^2 n/2}.$$

- The textbook's corollary to Lemma 11.9 seems too loose, at $e^{-\epsilon^2 n/6}$.[a]
- Our original problem (p. 604) hence demands, e.g., $n \approx 1.4k/\epsilon^2$ independent coin flips to guarantee making an error with probability $\leq 2^{-k}$ with the majority rule.

---

[a]See Dubhashi & Panconesi (2012) for many Chernoff-type bounds.

BPP[a] (Bounded Probabilistic Polynomial)

- The class **BPP** contains all languages L for which there is a precise polynomial-time NTM N such that:
  - If $x \in L$, then at least $3/4$ of the computation paths of N on x lead to "yes."
  - If $x \notin L$, then at least $3/4$ of the computation paths of N on x lead to "no."
- So N accepts or rejects by a *clear* majority.

[a] Gill (1977).

Magic 3/4?

- The number 3/4 bounds the probability (ratio) of a right answer away from 1/2.
- Any constant *strictly* between 1/2 and 1 can be used without affecting the class BPP.
- In fact, as with RP,

$$\frac{1}{2} + \frac{1}{q(n)}$$

for any polynomial $q(n)$ can replace 3/4.

- The next algorithm shows why.

The Majority Vote Algorithm

Suppose L is decided by N by majority $(1/2) + \epsilon$.

for $i = 1, 2, \ldots, 2k + 1$ **do**
    Run N on input x;
end for
if "yes" is the majority answer **then**
    "yes";
else
    "no";
end if

Analysis

- By Corollary 78 (p. 611), the probability of a false answer is at most $e^{-\epsilon^2 k}$.
- By taking $k = \lceil 2/\epsilon^2 \rceil$, the error probability is at most 1/4.
- Even if ϵ is any inverse polynomial, k remains a polynomial in n.
- The running time remains polynomial: $2k+1$ times N's running time.

Aspects of BPP

- BPP is the most comprehensive yet plausible notion of efficient computation.
  - If a problem is in BPP, we take it to mean that the problem can be solved efficiently.
  - In this aspect, BPP has effectively replaced P.
- $(\text{RP} \cup \text{coRP}) \subseteq (\text{NP} \cup \text{coNP})$.
- $(\text{RP} \cup \text{coRP}) \subseteq \text{BPP}$.
- Whether $\text{BPP} \subseteq (\text{NP} \cup \text{coNP})$ is unknown.
- But it is unlikely that $\text{NP} \subseteq \text{BPP}$.[a]

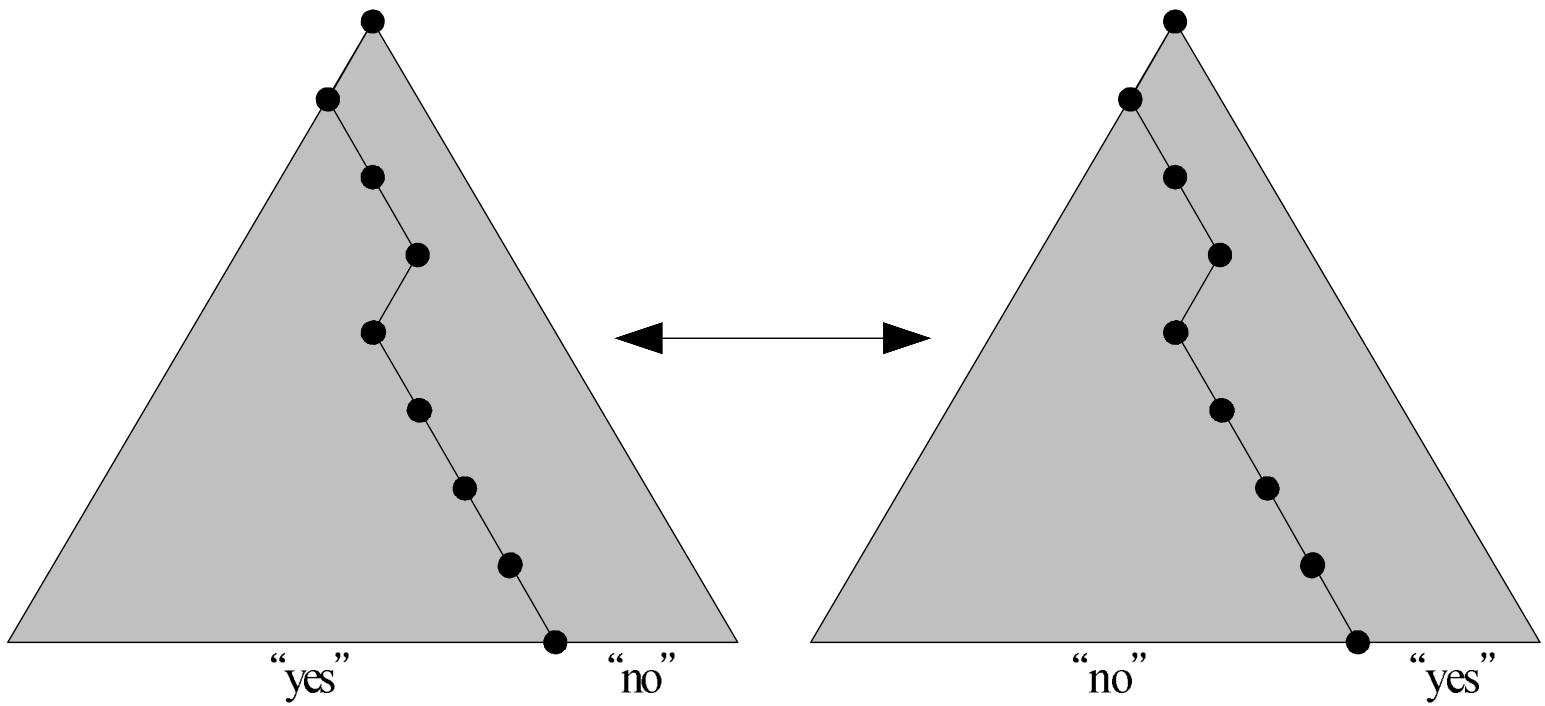
[a] See p. 628.

coBPP

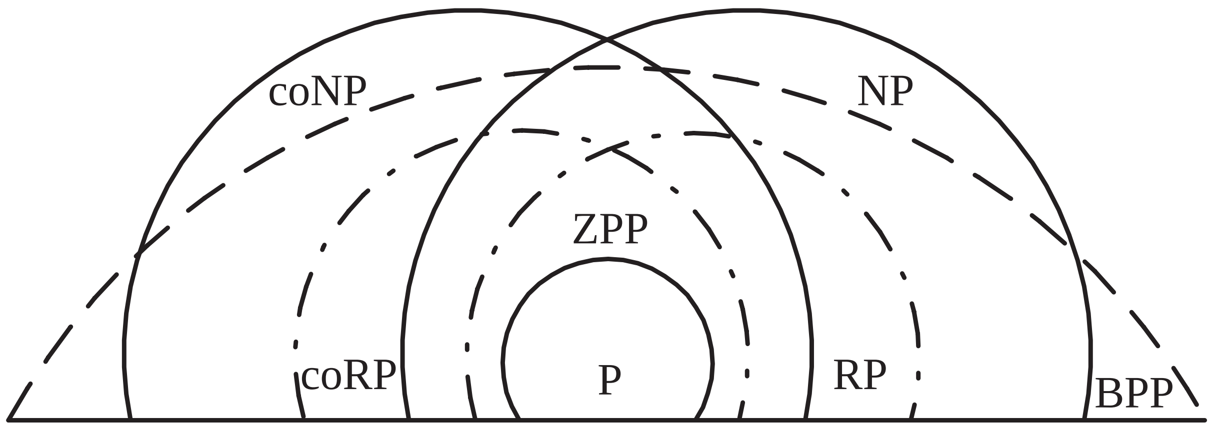
- The definition of BPP is symmetric: acceptance by clear majority and rejection by clear majority.
- An algorithm for $L \in$ BPP becomes one for $\bar{L}$ by reversing the answer.
- So $\bar{L} \in$ BPP and BPP $\subseteq$ coBPP.
- Similarly coBPP $\subseteq$ BPP.
- Hence BPP $=$ coBPP.
- This approach does not work for RP.[a]

[a] It did not work for NP either.

BPP and coBPP



"The Good, the Bad, and the Ugly"



Circuit Complexity

- Circuit complexity is based on boolean circuits instead of Turing machines.
- A boolean circuit with n inputs computes a boolean function of n variables.
- Now, identify `true`/1 with "yes" and `false`/0 with "no."
- Then a boolean circuit with n inputs accepts certain strings in $\{0, 1\}^n$.
- To relate circuits with an arbitrary language, we need one circuit for each possible input length n.

Formal Definitions

- The **size** of a circuit is the number of *gates* in it.
- A **family of circuits** is an infinite sequence $\mathcal{C} = (C_0, C_1, \ldots)$ of boolean circuits, where C_n has n boolean inputs.
- For input $x \in \{ 0, 1 \}^*$, $C_{|x|}$ outputs 1 if and only if $x \in L$.
- In other words,

$$C_n \text{ accepts } L \cap \{ 0, 1 \}^n.$$

Formal Definitions (concluded)

- $L \subseteq \{0,1\}^*$ has **polynomial circuits** if there is a family of circuits $\mathcal{C}$ such that:
  - The size of C_n is at most $p(n)$ for some fixed polynomial p.
  - C_n accepts $L \cap \{0,1\}^n$.

Exponential Circuits Suffice for All Languages

- Theorem 16 (p. 212) implies that there are languages that cannot be solved by circuits of size $2^n/(2n)$.
- But surprisingly, circuits of size 2^{n+2} can solve *all* problems, decidable or otherwise!

Exponential Circuits Suffice for All Languages (continued)

Proposition 79 *All decision problems (decidable or otherwise) can be solved by a circuit of size* 2^{n+2} *and depth* $2n$.

- We will show that for any language $L \subseteq \{0,1\}^*$, $L \cap \{0,1\}^n$ can be decided by a circuit of size 2^{n+2}.
- Define boolean function $f : \{0,1\}^n \to \{0,1\}$, where

$$f(x_1 x_2 \cdots x_n) = \begin{cases} 1, & x_1 x_2 \cdots x_n \in L, \\ 0, & x_1 x_2 \cdots x_n \notin L. \end{cases}$$

The Proof (concluded)

- Clearly, any circuit that implements f decides $L \cap \{0, 1\}^n$.
- Now,

$$f(x_1 x_2 \cdots x_n) = (x_1 \wedge f(1x_2 \cdots x_n)) \vee (\neg x_1 \wedge f(0x_2 \cdots x_n)).$$

- The circuit size $s(n)$ for $f(x_1 x_2 \cdots x_n)$ hence satisfies

$$s(n) = 4 + 2s(n-1)$$

  with $s(1) = 1$.
- Solve it to obtain $s(n) = 5 \times 2^{n-1} - 4 \leq 2^{n+2}$.
- The longest path consists of an alternating sequence of $\vee$s and $\wedge$s.