Problem 1 (20 points) Let \( L = \{ (M, x, 1^k) \mid \text{NTM } M \text{ accepts } x \text{ in less than } k \text{ steps} \} \). Prove that \( L \) is in NP.

Proof: First, we can guess any input \( x \) and run \( M(x) \) to see if \( M \) accepts \( x \) in less than \( k \) steps. It takes \( O(k) \) time. To find \( x \) accepted by \( M \) in less than \( k \) steps, it takes \( O(2^k) \) time to check all possible combinations. So it is clear that \( L \) is in NP.

Problem 2 (20 points) Let \( G(V, E) \) be a directed graph with vertices \( V \) and edges \( E \). BIGCYCLE asks if \( G \) has a cycle of length equal or larger than \( |V|/2 \). Prove that BIGCYCLE is NP-complete. (Need to show that BIGCYCLE is in NP.)

Proof: We first prove that BIGCYCLE is in NP. Given a graph \( G \), one can guess a cycle and accept \( G \) if the length of the cycle is equal or larger than \( |V|/2 \). It can be done in polynomial time.

We proceed to reduce HAMILTONIAN CYCLE to BIGCYCLE. Let \( N \) be an NTM which decides BIGCYCLE. Construct an NTM \( M \) which decides HAMILTONIAN CYCLE as follows:

1: On input \( G(V, E) \) with \( |V| \).
2: Add exactly \( |V| \) isolated vertices to \( G \) to obtain \( G' \).
3: Run \( N(G') \).
4: If \( N \) accepts, accept.
5: Otherwise, reject.

Hence, \( G \in \text{HAMILTONIAN CYCLE} \) if and only if \( G' \in \text{BIGCYCLE} \). \( M \) runs in polynomial time. This completes the proof.

Problem 3 (20 points) Recall that the depth of a gate \( g \) is the length of the longest path in a circuit from \( g \) to an input gate. A circuit is leveled if every input of a gate in depth \( k \) comes from one in depth \( k-1 \). LEVELED CIRCUIT asks if a leveled circuit is satisfiable. Prove that LEVELED CIRCUIT is NP-complete. (No need to show it is in NP.)
Proof: We can obtain a leveled circuit from any circuit $C$ by increasing the number of gates by a polynomial factor, as follows. This holds for the input gates. Inductively, suppose that all gates of depth $k - 1$ have length $k - 1$ for the shortest paths to the input gates. Now consider gates of depth $k$. Pick any gate $g$ with a shorter shortest path to the input gates, say length $l < k$. Insert a series of $k - l \lor$ gates on the edge between $g$ and its predecessor gate on one such path. These $k - l \lor$ gates have their two identical inputs. Note that $k - l = O(|C|)$. So they act as the identity function. The new circuit has size $O(|C|^2)$. Finally, recall that $\text{Circuit SAT}$ is NP-complete by Cook’s Theorem.

Problem 4 (20 points) Prove that $\text{EXACT-}k\text{-COLORING}$, which asks if a graph can be colored where all $k$ colors are used, is NP-complete. (Need to show that it is in NP.)

Proof: The problem is in NP because it is easy to check any coloring uses up all $k$ colors. Given the input graph $G(V, E)$, add a clique with $k$ nodes to obtain $G'$. Then $G$ can be colored using up all $k$ colors if and only if $G'$ can be colored by exactly $k$ colors.

Problem 5 (20 points) Show that $\text{validity}$ is coNP-complete.

Proof: We first construct a TM which verifies the input $x$ and accepts if $x \in \text{validity}$. It takes polynomial time. So $\text{validity} \in \text{coNP}$. Now we proceed to show that $L$ can be reduced to $\text{validity}$ for all $L \in \text{coNP}$. It is known that $\text{SAT}$ is NP-complete. By Proposition 54 (see p. 457 of the slides), $\overline{\text{SAT}}$ is coNP-complete. So it suffices to show that $\overline{\text{SAT}}$ can be reduced to $\text{validity}$. Let $N$ be an NTM which decides $\text{validity}$. Construct an NTM $M$ which decides $\overline{\text{SAT}}$ as follows:

1: On input $x$, let $x' = \overline{x}$.
2: Run $N(x')$
3: If $N$ accepts, halt and accept.
4: Otherwise, halt and reject.

$M$ clearly runs in polynomial time. Hence, $\text{validity}$ is coNP-complete.