Theory of Computation

Midterm Examination on December 3, 2020

Fall Semester, 2020

Problem 1 (20 points) Let $L = \{ (M, x, 1^k) \mid$ NTM M accepts x in less than k steps$\}$. Prove that L is in NP.

Proof: First, we can guess any input x and run $M(x)$ to see if M accepts x in less than k steps. It takes $O(k)$ time. To find x accepted by M in less than k steps, it takes $O(2^k)$ time to check all of possible combinations. So it is clear that L is in NP.

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Problem 2 (20 points) Let $G(V, E)$ be a directed graph with vertices V and edges E. BIGCYCLE asks if G has a cycle of length equal or larger than $|V|/2$. Prove that BIGCYCLE is NP-complete. (Need to show that BIGCYCLE is in NP.)

Proof: We first prove that BIGCYCLE is in NP. Given a graph G, one can guess a cycle and accept G if the length of the cycle is equal or larger than $|V|/2$. It can be done in polynomial time.

We proceed to reduce HAMILTONIAN CYCLE to BIGCYCLE. Let N be an NTM which decides BIGCYCLE. Construct an NTM M which decides HAMILTONIAN CYCLE as follows:

1: On input $G(V, E)$ with $|V|$.

2: Add exactly $|V|$ isolated vertices to G to obtain G'.

3: Run N(G').

4: If N accepts, accept.

5: Otherwise, reject.

Hence, $G \in$ HAMILTONIAN CYCLE if and only if $G' \in$ BIGCYCLE. M runs in polynomial time. This completes the proof. ■

Problem 3 (20 points) Recall that the **depth** of a gate g is the length of the longest path in a circuit from g to an input gate. A circuit is **leveled** if every input of a gate in depth k comes from one in depth $k-1$. LEVELED CIRCUIT asks if a leveled circuit is satisfiable. Prove that LEVELED CIRCUIT is NP-complete. (No need to show it is in NP.)

Proof: We can obtain a leveled circuit from any circuit C by increasing the number of gates by a polynomial factor, as follows. This holds for the input gates. Inductively, suppose that all gates of depth $k-1$ have length $k-1$ for the shortest paths to the input gates. Now consider gates of depth k. Pick any gate g with a shorter shortest path to the input gates, say length $l < k$. Insert a series of $k - l$ $\vee$ gates on the edge between g and its predecessor gate on one such path. These $k - l$ $\vee$ gates have their two identical inputs. Note that $k - l = O(|\,C\,|)$. So they act as the identity function. The new circuit has size $O(|\,C\,|^2)$. Finally, recall that CIRCUIT SAT is NP-complete by Cook's Theorem. ■

Problem 4 (20 points) Prove that EXACT-k-COLORING, which asks if a graph can be colored where all k colors are used, is NP-complete. (Need to show that it is in NP.)

Proof: The problem is in NP because it is easy to check any coloring uses up all k colors. Given the input graph $G(V, E)$, add a clique with k nodes to obtain G'. Then G can be colored using up all k colors if and only if G' can be colored by exactly k colors. ■

Problem 5 (20 points) Show that VALIDITY is coNP-complete.

Proof: We first construct a TM which verifies the input x and accepts if $x \in$ VALIDITY. It takes polynomial time. So VALIDITY $\in$ coNP. Now we proceed to show that L can be reduced to VALIDITY for all $L \in$ coNP. It is known that SAT is NP-complete. By Proposition 54 (see p. 457 of the slides), $\overline{\text{SAT}}$ is coNP-complete. So it suffices to show that $\overline{\text{SAT}}$ can be reduced to VALIDITY. Let N be an NTM which decides VALIDITY. Construct an NTM M which decides $\overline{\text{SAT}}$ as follows:

1: On input x, let $x' = \neg x$.

2: Run N(x')

3: If N accepts, halt and accept.

4: Otherwise, halt and reject.

M clearly runs in polynomial time. Hence, VALIDITY is coNP-complete. ■