Theory of Computation

Midterm Examination on October 29, 2020

Fall Semester, 2020

Problem 1 (20 points) Answer the following questions.

(1) Suppose that L is decidable and $L' = L - \{ y \}$, where $y \in L$. Prove that L' is decidable.

(2) Suppose L_2 is decidable and $L_1 \subseteq L_2$. Suppose to determine if $x \in L_1$, we reply "no" if $x \notin L_2$; otherwise, invoke (1) repeatedly over elements y of L_2 and return "no" if a "no" is returned and "yes" otherwise. In other words, it relies on the valid statement: $x \in L_1$ if and only if x in L_2 but $x \notin L_2 - L_1$. Is this proof sketch correct? Why?

Proof:

(1) The following TM decides L':

if $x = y$ **then**
    "no";
else
    $M(x)$;
end if

(2) No, it may never end because L_2 may be of infinite size.

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Problem 2 (20 points) Answer the following questions.

(1) Write down the property of being a recursive language.

(2) Use Rice's theorem to prove that this property is undecidable.

Proof:

(1) The property of being a *recursive* languages is

$$\{ L : \text{a TM } M \text{ decides } L \}.$$

(2) Because $R \subseteq RE$, the above property is a subset of RE. We also know that the halting problem is recursively enumerable but not recursive. Finally, we also know that recursive languages exist (such as primality, etc.). Hence the said property is nontrivial. So Rice's theorem applies.

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Problem 3 (25 points) Answer the following questions.

(1) (10 points) Consider the following boolean functions

$$f : \{ \texttt{true}, \texttt{false} \}^n \to \{ \texttt{true}, \texttt{false} \}^{\sqrt{n}}.$$

How many such functions are there? For simplicity, assume that $\sqrt{x}$ is an integer. Same below. (Use parentheses to avoid ambiguity.)

(2) (15 points) What is a good lower bound for any circuit that computes such functions (assuming $n \geq 2$ or any convenient constant)?

Proof:

(1) $2^{(2^n \sqrt{n})}$.

(2) From slide p. 213, it suffices to show that $((n + 5) \times m^2)^m < 2^{(2^n \sqrt{n})}$ when $m = 2^{n-1}/\sqrt{n + 5}$:

$$\begin{aligned} m \log_2((n + 5) \times m^2) &= \left(\frac{n - 1}{\sqrt{n + 5}} \right) \times 2^n \\ &< \sqrt{n} \times 2^n \end{aligned}$$

for $n \geq 2$.

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Problem 4 (20 points) Answer the following questions with proof.

(1) Show that either $L \neq P$ or $P \neq PSPACE$.

(2) Show that $NPSPACE \subseteq EXP$.

Proof:

(1) Suppose $L = P$ and $P = PSPACE$ instead. Then $L = PSPACE$. However, we know these two classes are different by the space hierarchy theorem, a contradiction.

(2) By Savitch's theorem, $NPSPACE = PSPACE$. We also know $PSPACE \subseteq EXP$ from slide p. 241.

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Problem 5 (15 points) A class C is closed under linear-time reductions if whenever L is reducible to L' and $L' \in C$, then $L \in C$. Prove that E is closed under linear-time reductions.

Proof: Suppose that L is reducible to L' by a linear reduction and $L' \in$ E. We proceed to prove that $L \in$ E. Let R be a linear-time reduction from L to $L' \in$ TIME(2^{kn}) for some positive integer k. By definition, $x \in L$ if and only if $R(x) \in L'$. But $|\,R(x)\,| = O(|\,x\,|)$ as R runs in linear time. ■