Maximum Satisfiability

- Given a set of clauses, MAXSAT seeks the truth assignment that satisfies the most simultaneously.
- MAX2SAT is already NP-complete (p. 352), so MAXSAT is NP-complete.
- Consider the more general k-MAXGSAT for constant k.
  - Let $\Phi = \{ \phi_1, \phi_2, \ldots, \phi_m \}$ be a set of boolean expressions in n variables.
  - Each ϕ_i is a *general* expression involving up to k variables.
  - k-MAXGSAT seeks the truth assignment that satisfies the most expressions simultaneously.

A Probabilistic Interpretation of an Algorithm

- Let ϕ_i involve $k_i \leq k$ variables and be satisfied by s_i of the 2^{k_i} truth assignments.
- A random truth assignment $\in \{ 0, 1 \}^n$ satisfies ϕ_i with probability $p(\phi_i) = s_i / 2^{k_i}$.
  - $p(\phi_i)$ is easy to calculate as k is a constant.
- Hence a random truth assignment satisfies an average of

$$p(\Phi) = \sum_{i=1}^{m} p(\phi_i)$$

expressions ϕ_i.

The Search Procedure

- Clearly
$$p(\Phi) = \frac{p(\Phi[\,x_1 = \texttt{true}\,]) + p(\Phi[\,x_1 = \texttt{false}\,])}{2}.$$
- Select the $t_1 \in \{\,\texttt{true}, \texttt{false}\,\}$ such that $p(\Phi[\,x_1 = t_1\,])$ is the larger one.
- Note that $p(\Phi[\,x_1 = t_1\,]) \geq p(\Phi)$.
- Repeat the procedure with expression $\Phi[\,x_1 = t_1\,]$ until all variables x_i have been given truth values t_i and all ϕ_i are either true or false.

The Search Procedure (continued)

- By our hill-climbing procedure,

$$\begin{aligned} & p(\Phi) \\ \leq \quad & p(\Phi[\,x_1 = t_1\,]) \\ \leq \quad & p(\Phi[\,x_1 = t_1, x_2 = t_2\,]) \\ \leq \quad & \cdots \\ \leq \quad & p(\Phi[\,x_1 = t_1, x_2 = t_2, \ldots, x_n = t_n\,]). \end{aligned}$$

- So at least $p(\Phi)$ expressions are satisfied by truth assignment $(t_1, t_2, \ldots, t_n)$.

The Search Procedure (concluded)

- Note that the algorithm is *deterministic*!
- It is called **the method of conditional expectations**.[a]

[a] Erdős & Selfridge (1973); Spencer (1987).

Approximation Analysis

- The optimum is at most the number of satisfiable ϕ_i—i.e., those with $p(\phi_i) > 0$.
- The ratio of algorithm's output vs. the optimum is[a]

$$\geq \frac{p(\Phi)}{\sum_{p(\phi_i)>0} 1} = \frac{\sum_i p(\phi_i)}{\sum_{p(\phi_i)>0} 1} \geq \min_{p(\phi_i)>0} p(\phi_i).$$

- This is a polynomial-time ϵ-approximation algorithm with $\epsilon = 1 - \min_{p(\phi_i)>0} p(\phi_i)$ by Eq. (20) on p. 732.
- Because $p(\phi_i) \geq 2^{-k}$ for a satisfiable ϕ_i, the heuristic is a polynomial-time ϵ-approximation algorithm with $\epsilon = 1 - 2^{-k}$.

[a] Because $\sum_i a_i / \sum_i b_i \geq \min_i (a_i/b_i)$.

Back to MAXSAT

- In MAXSAT, the ϕ_i's are clauses (like $x \lor y \lor \neg z$).
- Hence $p(\phi_i) \geq 1/2$ (why?).
- The heuristic becomes a polynomial-time ϵ-approximation algorithm with $\epsilon = 1/2$.[a]
- Suppose we set each boolean variable to true with probability $(\sqrt{5} - 1)/2$, the golden ratio.
- Then follow through the method of conditional expectations to **derandomize** it.

---

[a] Johnson (1974).

Back to MAXSAT (concluded)

- We will obtain a $[(3-\sqrt{5})]/2$-approximation algorithm.[a]
  - Note $[(3-\sqrt{5})]/2 \approx 0.382$.
- If the clauses have k *distinct* literals,

$$p(\phi_i) = 1 - 2^{-k}.$$

- The heuristic becomes a polynomial-time ϵ-approximation algorithm with $\epsilon = 2^{-k}$.
  - This is the best possible for $k \geq 3$ unless P = NP.
- All the results hold even if clauses are weighted.

[a] Lieberherr & Specker (1981).

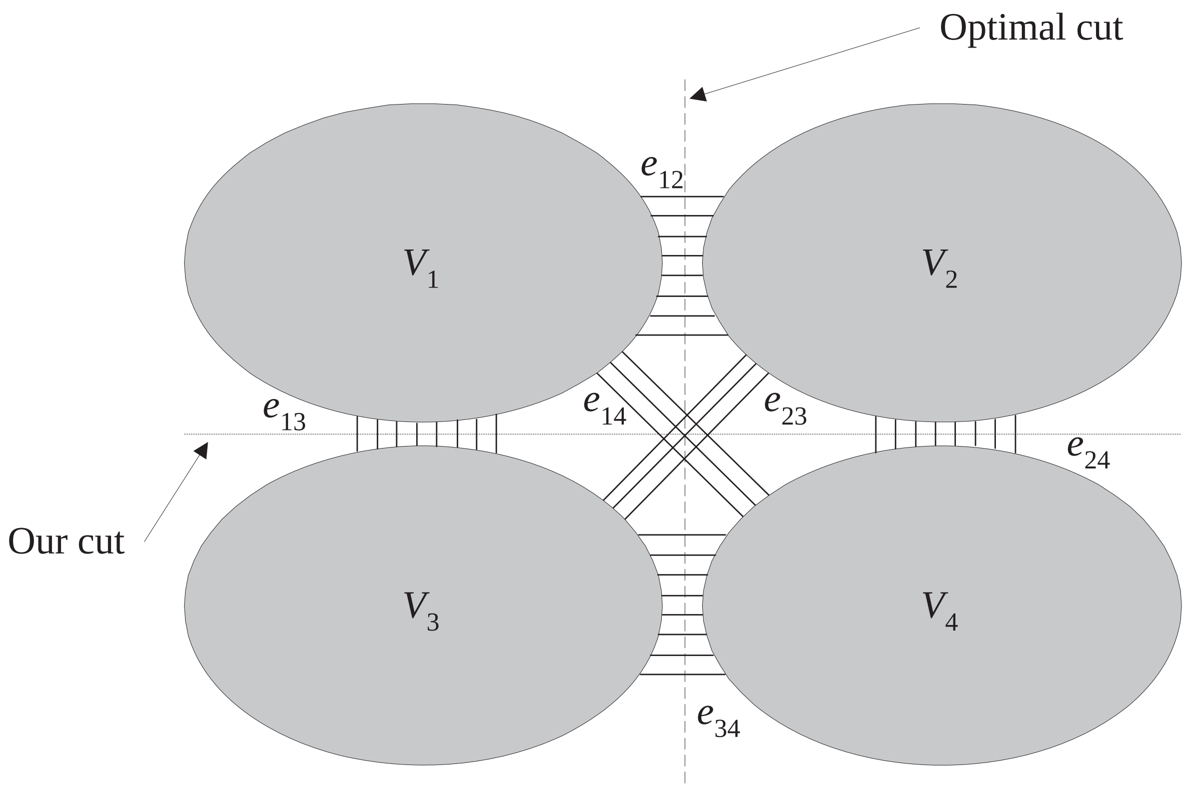
MAX CUT Revisited

- MAX CUT seeks to partition the nodes of graph $G = (V, E)$ into $(S, V - S)$ so that there are as many edges as possible between S and $V - S$.
- It is NP-complete (p. 387).
- **Local search** starts from a feasible solution and performs "local" improvements until none are possible.
- Next we present a local-search algorithm for MAX CUT.

A 0.5-Approximation Algorithm for MAX CUT

$S := \emptyset$;
while $\exists v \in V$ whose switching sides results in a larger cut **do**
    Switch the side of v;
end while
return S;

Analysis



Analysis (continued)

- Partition $V = V_1 \cup V_2 \cup V_3 \cup V_4$, where
  - Our algorithm returns $(V_1 \cup V_2, V_3 \cup V_4)$.
  - The optimum cut is $(V_1 \cup V_3, V_2 \cup V_4)$.
- Let e_{ij} be the number of edges between V_i and V_j.
- Our algorithm returns a cut of size

$$e_{13} + e_{14} + e_{23} + e_{24}.$$

- The optimum cut size is

$$e_{12} + e_{34} + e_{14} + e_{23}.$$

Analysis (continued)

- For each node $v \in V_1$, its edges to $V_3 \cup V_4$ cannot be outnumbered by those to $V_1 \cup V_2$.
  - Otherwise, v would have been moved to $V_3 \cup V_4$ to improve the cut.
- Considering all nodes in V_1 together, we have

$$2e_{11} + e_{12} \leq e_{13} + e_{14}.$$

  - $2e_{11}$, because each edge in V_1 is counted twice.
- The above inequality implies

$$e_{12} \leq e_{13} + e_{14}.$$

Analysis (concluded)

- Similarly,

$$\begin{aligned} e_{12} &\leq e_{23} + e_{24} \\ e_{34} &\leq e_{23} + e_{13} \\ e_{34} &\leq e_{14} + e_{24} \end{aligned}$$

- Add all four inequalities, divide both sides by 2, and add the inequality $e_{14} + e_{23} \leq e_{14} + e_{23} + e_{13} + e_{24}$ to obtain

$$e_{12} + e_{34} + e_{14} + e_{23} \leq 2(e_{13} + e_{14} + e_{23} + e_{24}).$$

- The above says our solution is at least half the optimum.

Remarks

- A 0.12-approximation algorithm exists.[a]
- 0.059-approximation algorithms do not exist unless NP = ZPP.[b]

---

[a] Goemans & Williamson (1995).
[b] Håstad (1997).

Approximability, Unapproximability, and Between

- Some problems have approximation thresholds less than 1.
  - KNAPSACK has a threshold of 0 (p. 782).
  - NODE COVER (p. 738), BIN PACKING, and MAXSAT[a] have a threshold larger than 0.
- The situation is maximally pessimistic for TSP (p. 757) and INDEPENDENT SET,[b] which cannot be approximated
  - Their approximation threshold is 1.

---

[a] Williamson & Shmoys (2011).

[b] See the textbook.

Unapproximability of TSP[a]

Theorem 84 *The approximation threshold of* TSP *is 1 unless* $P = NP$.

- Suppose there is a polynomial-time ϵ-approximation algorithm for TSP for some $\epsilon < 1$.
- We shall construct a polynomial-time algorithm to solve the NP-complete HAMILTONIAN CYCLE.
- Given any graph $G = (V, E)$, construct a TSP with $|\,V\,|$ cities with distances

$$d_{ij} = \begin{cases} 1, & \text{if } [\,i, j\,] \in E, \\ \frac{|\,V\,|}{1-\epsilon}, & \text{otherwise.} \end{cases}$$

[a] Sahni & Gonzales (1976).

The Proof (continued)

- Run the alleged approximation algorithm on this TSP instance.
- Note that if a tour includes edges of length $|V|/(1-\epsilon)$, then the tour costs more than $|V|$.
- Note also that no tour has a cost less than $|V|$.
- Suppose a tour of cost $|V|$ is returned.
  - Then every edge on the tour exists in the *original* graph G.
  - So this tour is a Hamiltonian cycle on G.

The Proof (concluded)

- Suppose a tour that includes an edge of length $|V|/(1-\epsilon)$ is returned.
  - The total length of this tour exceeds $|V|/(1-\epsilon)$.[a]
  - Because the algorithm is ϵ-approximate, the optimum is at least $1-\epsilon$ times the returned tour's length.
  - The optimum tour has a cost exceeding $|V|$.
  - Hence G has no Hamiltonian cycles.

[a] So this reduction is **gap introducing**.

METRIC TSP

- METRIC TSP is similar to TSP.
- But the distances must satisfy the triangular inequality:

$$d_{ij} \leq d_{ik} + d_{kj}$$

for all i, j, k.

- Inductively,

$$d_{ij} \leq d_{ik} + d_{kl} + \cdots + d_{zj}.$$

A 0.5-Approximation Algorithm for METRIC TSP[a]

- It suffices to present an algorithm with the approximation ratio of

$$\frac{c(M(x))}{\text{OPT}(x)} \leq 2$$

(see p. 733).

[a]Choukhmane (1978); Iwainsky, Canuto, Taraszow, & Villa (1986); Kou, Markowsky, & Berman (1981); Plesník (1981).

A 0.5-Approximation Algorithm for METRIC TSP (concluded)

$T :=$ a minimum spanning tree of G;
$T' :=$ duplicate the edges of T plus their cost; {Note: T' is an Eulerian *multigraph*.}
$C :=$ an Euler cycle of T';
Remove repeated nodes of C; {"Shortcutting."}
return C;

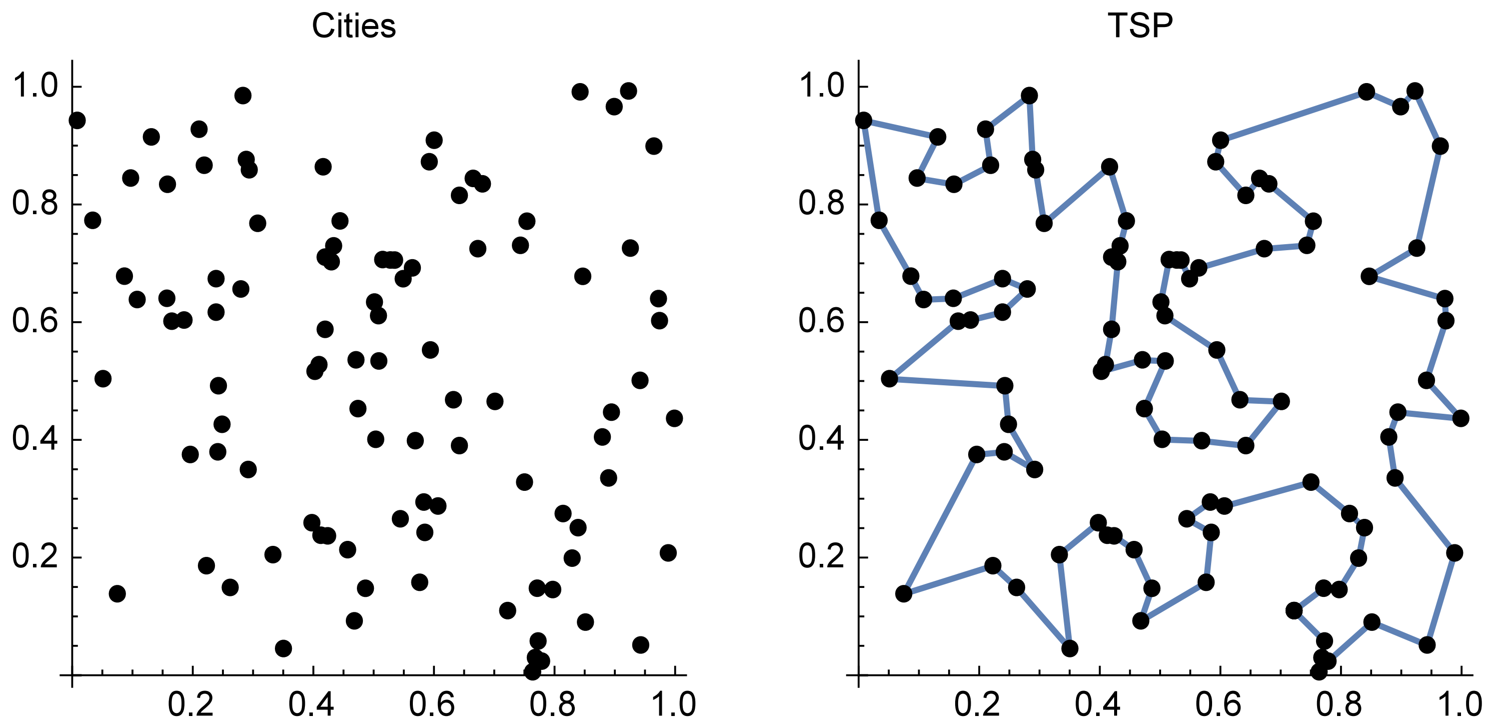
Analysis

- Let C_{opt} be an optimal TSP tour.
- Note first that
$$c(T) \leq c(C_{\text{opt}}). \tag{21}$$
  - C_{opt} is a spanning tree after the removal of one edge.
  - But T is a *minimum* spanning tree.
- Because T' doubles the edges of T,
$$c(T') = 2c(T).$$

Analysis (concluded)

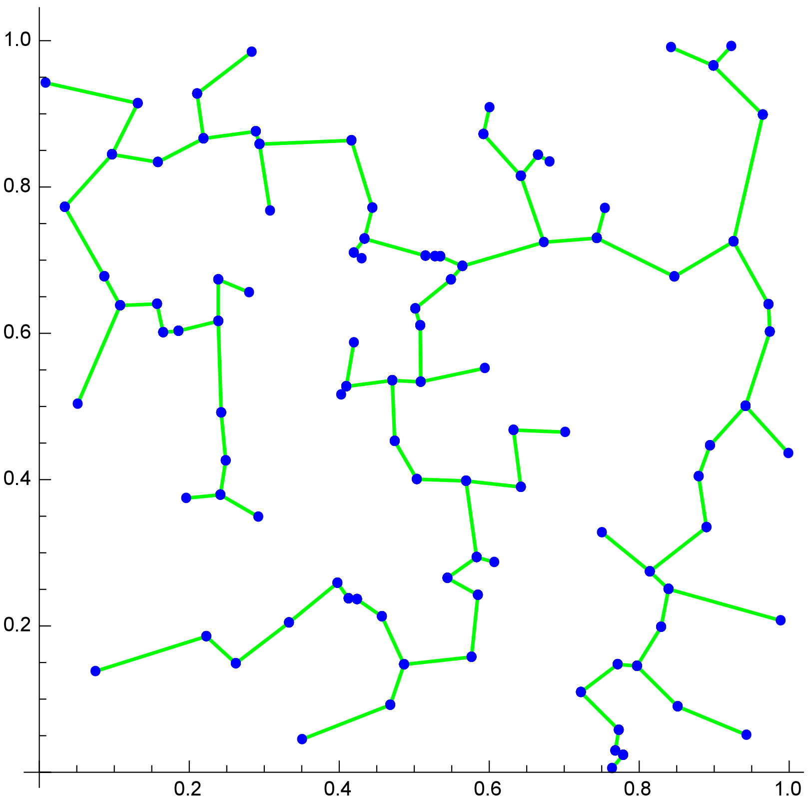
- Because of the triangular inequality, "shortcutting" does not increase the cost.
  - $(1, 2, 3, 2, 1, 4, \ldots) \to (1, 2, 3, 4, \ldots)$, a Hamiltonian cycle.
- Thus
$$c(C) \leq c(T').$$
- Combine all the inequalities to yield
$$c(C) \leq c(T') = 2c(T) \leq 2c(C_{\mathrm{opt}}),$$
as desired.

A 100-Node Example



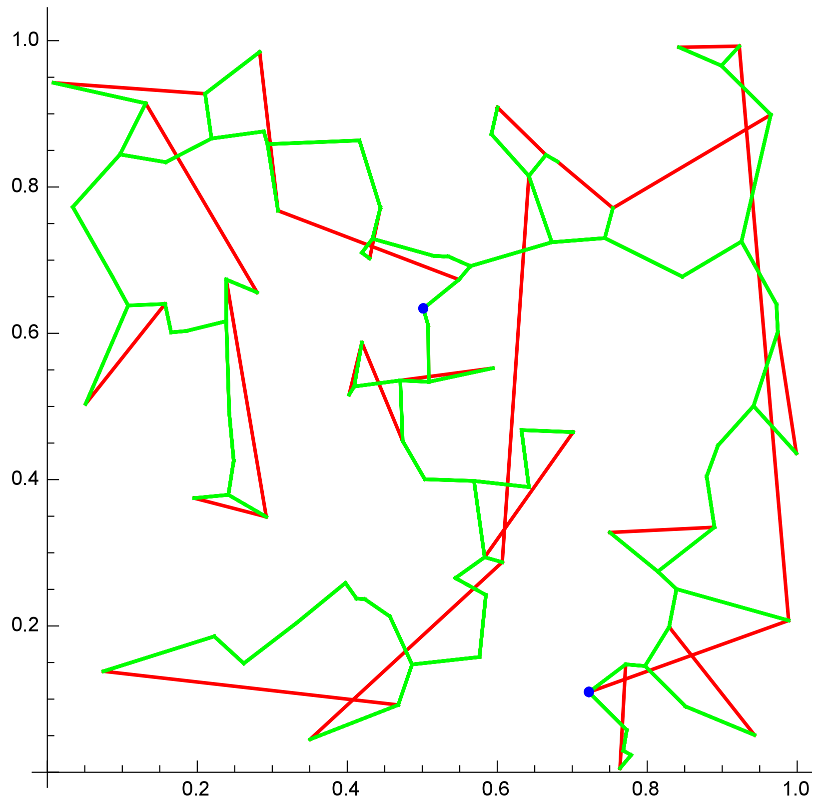
The cost is 7.72877.

A 100-Node Example (continued)



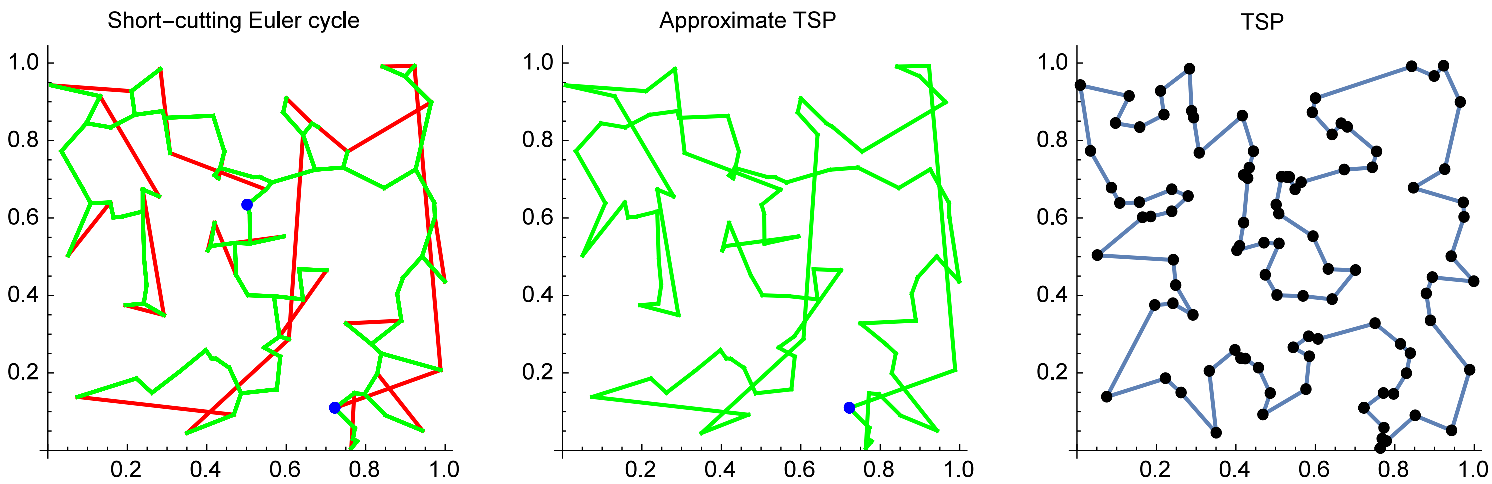
The minimum spanning tree T.

A 100-Node Example (continued)



"Shortcutting" the repeated nodes on the Euler cycle C.

A 100-Node Example (concluded)



The cost is $10.5718 \leq 2 \times 7.72877 = 15.4576$.

A (1/3)-Approximation Algorithm for METRIC TSP[a]

- It suffices to present an algorithm with the approximation ratio of

$$\frac{c(M(x))}{\text{OPT}(x)} \leq \frac{3}{2}$$

(see p. 733).

- This is the best approximation ratio for METRIC TSP as of 2016!

[a]Christofides (1976).

A (1/3)-Approximation Algorithm for METRIC TSP (concluded)

1: $T :=$ a minimum spanning tree of G;

2: $V' :=$ the set of nodes with an odd degree *in* T; $\{ | V' |$ must be even by a well-known parity result.$\}$

3: $G' :=$ the induced subgraph of G by V'; $\{ G'$ is a complete graph on V'.$\}$

4: $M :=$ a minimum-cost perfect matching of G';

5: $G'' := T \cup M$; $\{ G''$ is an Eulerian *multigraph*.$\}$

6: $C :=$ an Euler cycle of G'';

7: Remove repeated nodes of C; {"Shortcutting."}

8: **return** C;

Analysis

- Let C_{opt} be an optimal TSP tour.
- By Eq. (21) on p. 763,
$$c(T) \leq c(C_{\text{opt}}). \tag{22}$$
- Let C' be C_{opt} on V' by "shortcutting."
  - C_{opt} is a Hamiltonian cycle on V.
  - Replace any path $(v_1, v_2, \ldots, v_k)$ on C_{opt} with (v_1, v_k), where $v_1, v_k \in V'$ but $v_2, \ldots, v_{k-1} \notin V'$.
- So C' is simply the restriction of C_{opt} to V'.

Analysis (continued)

- By the triangular inequality,

$$c(C') \leq c(C_{\text{opt}}).$$

- C' is now a Hamiltonian cycle on V'.
- C' consists of two perfect matchings on G'.[a]
  - The first, third, ... edges constitute one.
  - The second, fourth, ... edges constitute the other.

[a] Note that G' is a complete graph with an even $|V'|$.

Analysis (continued)

- By Eq. (22) on p. 771, the cheaper perfect matching has a cost of

$$\frac{c(C')}{2} \leq \frac{c(C_{\text{opt}})}{2}.$$

- As a result, the minimum-cost one M must satisfy

$$c(M) \leq \frac{c(C')}{2} \leq \frac{c(C_{\text{opt}})}{2}. \tag{23}$$

- Minimum-cost perfect matching can be solved in polynomial time.[a]

[a] Edmonds (1965); Micali & V. Vazirani (1980).

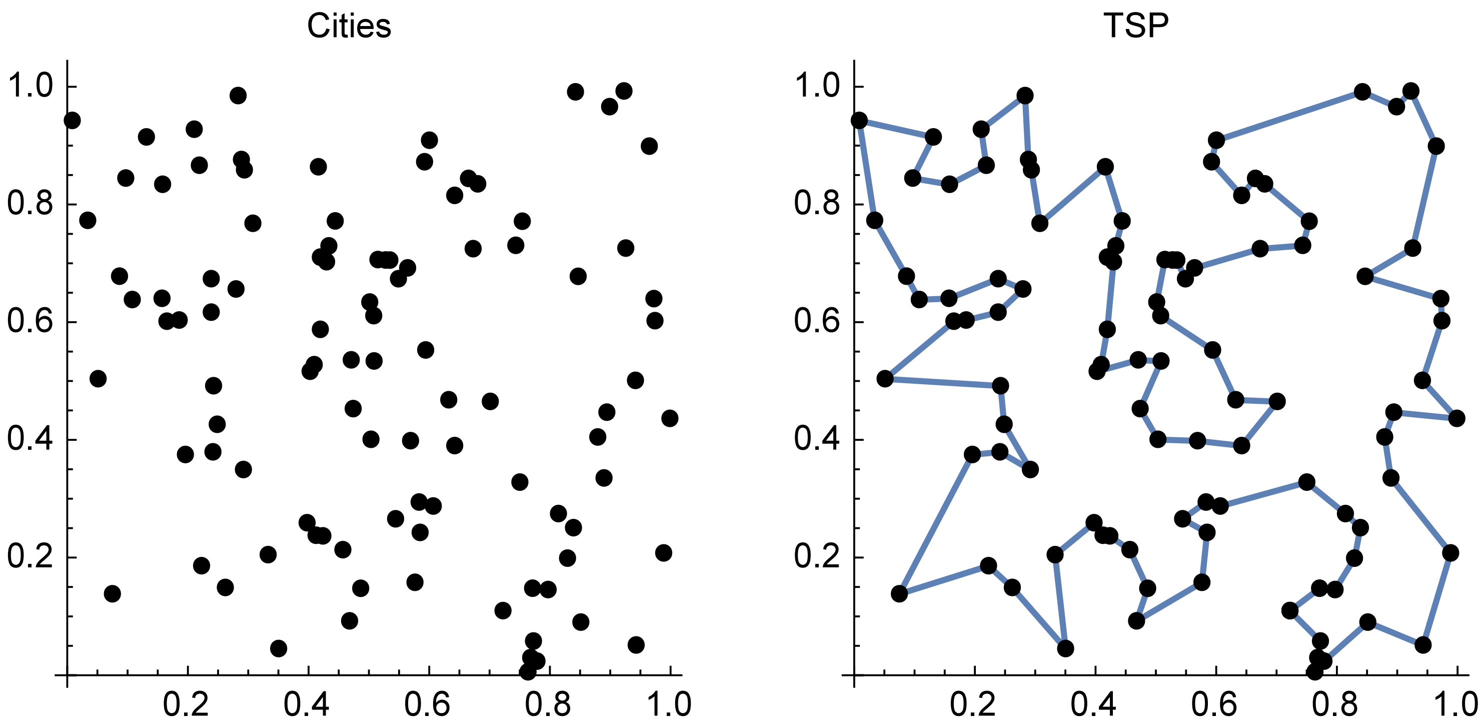
Analysis (concluded)

- By combining the two earlier inequalities, any Euler cycle C has a cost of

$$\begin{aligned} c(C) &\leq c(T) + c(M) && \text{by Line 5 of the algorithm} \\ &\leq c(C_{\text{opt}}) + \frac{c(C_{\text{opt}})}{2} && \text{by inequalities (22) and (23)} \\ &= \frac{3}{2}\, c(C_{\text{opt}}), \end{aligned}$$

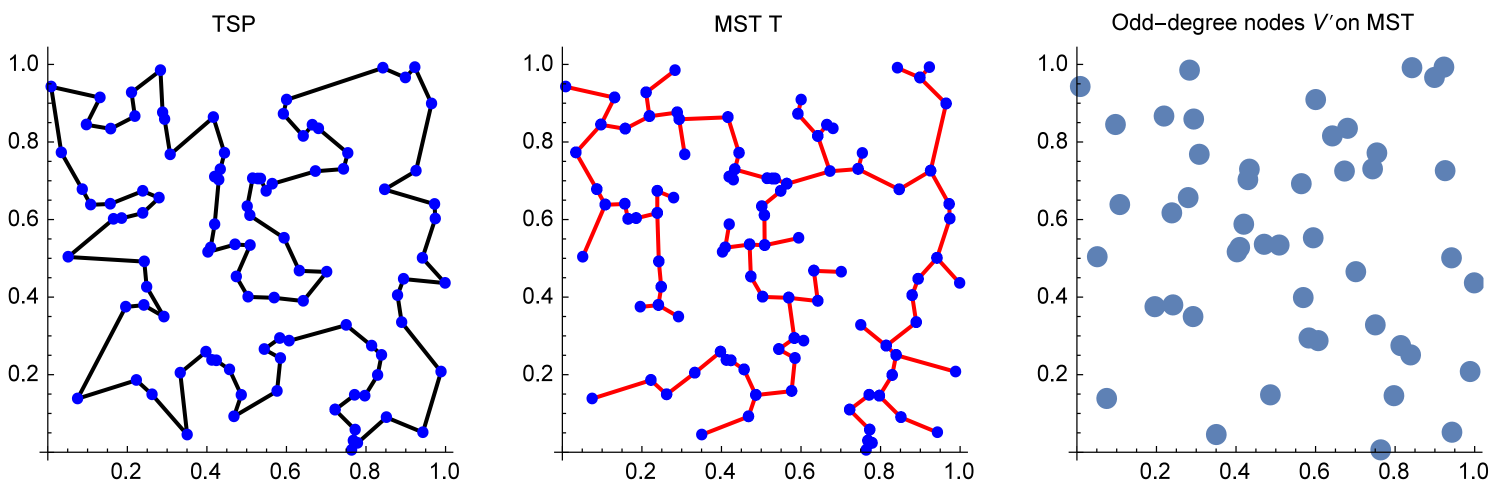
as desired.

A 100-Node Example

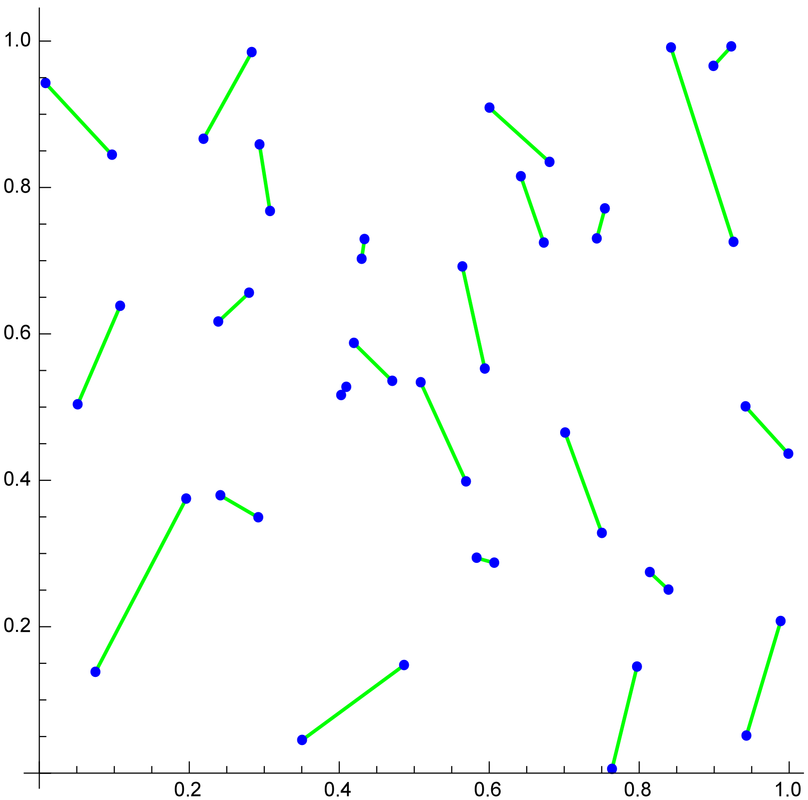


The cost is 7.72877.

A 100-Node Example (continued)

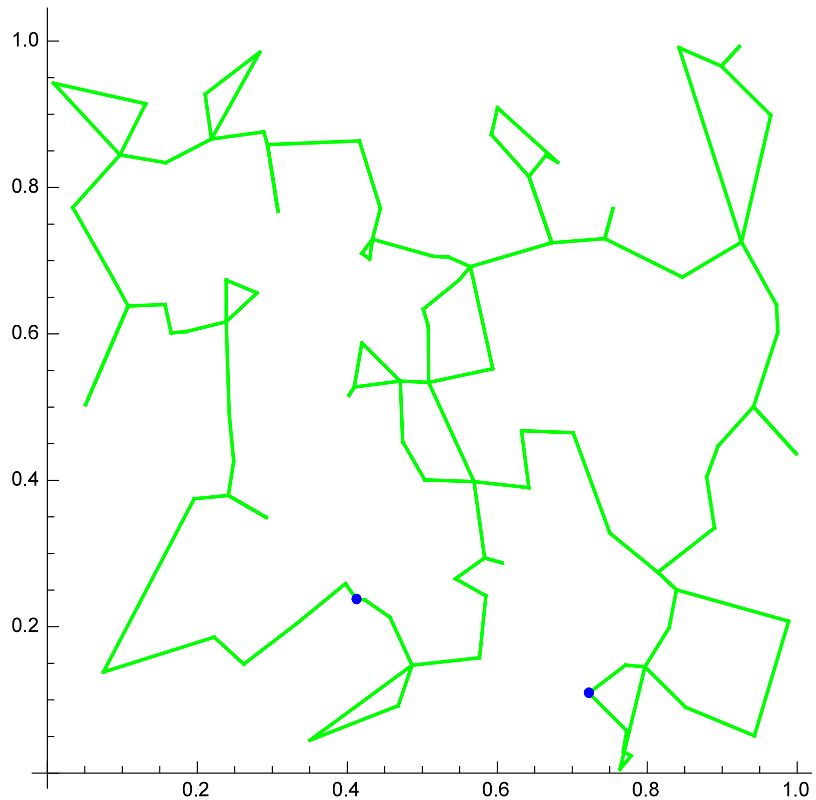


A 100-Node Example (continued)



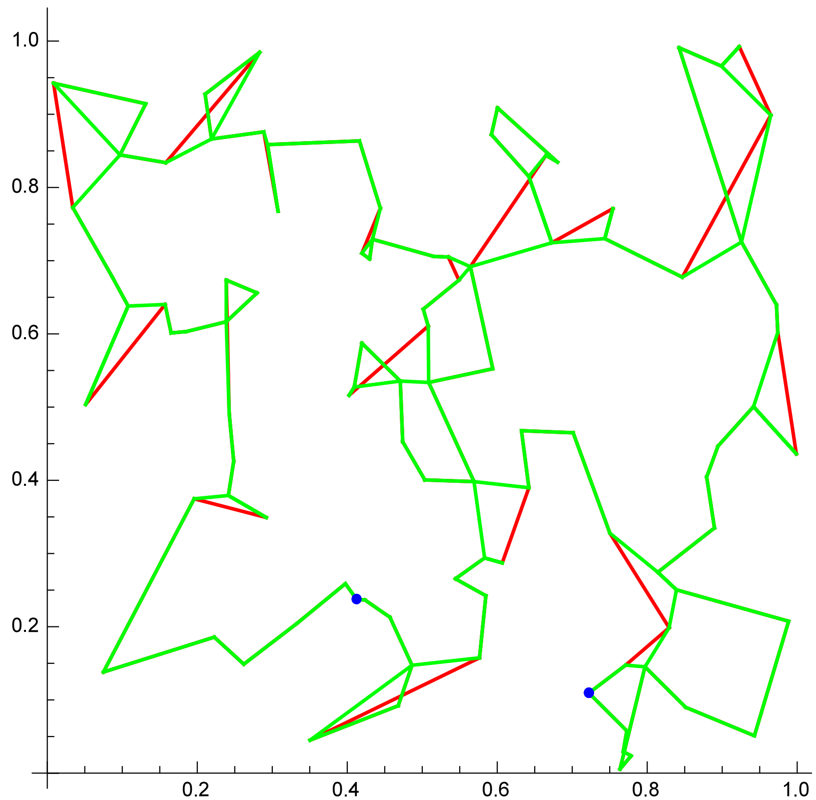
A minimum-cost perfect matching M.

A 100-Node Example (continued)



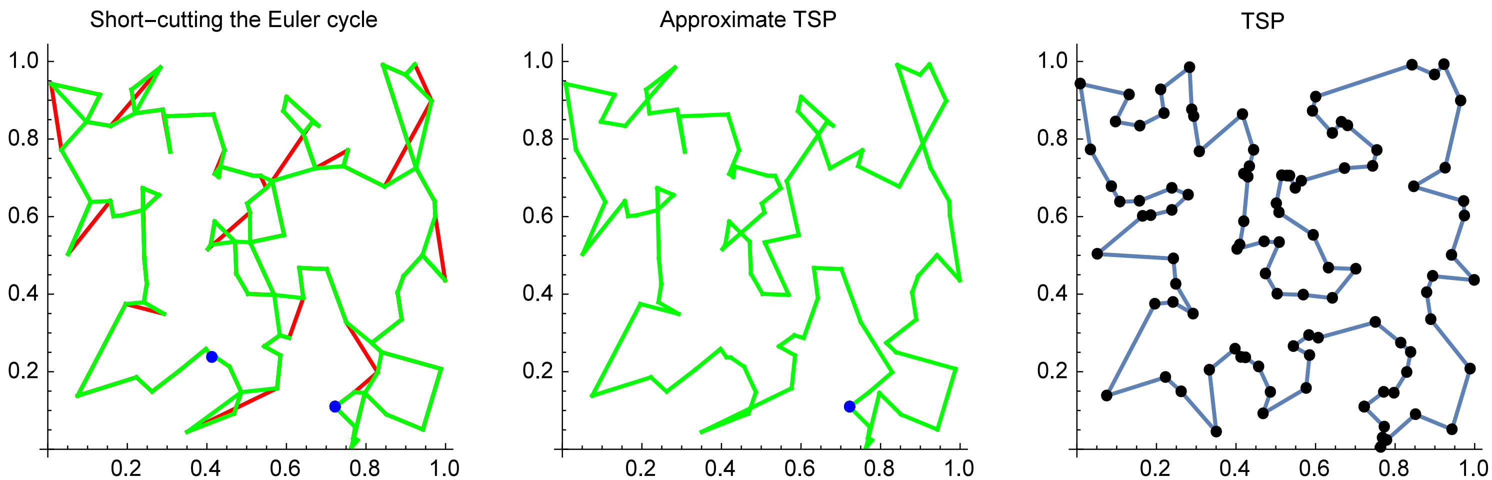
An Euler cycle C of $G'' = T \cup M$.

A 100-Node Example (continued)



"Shortcutting" the repeated nodes on the Euler cycle C.

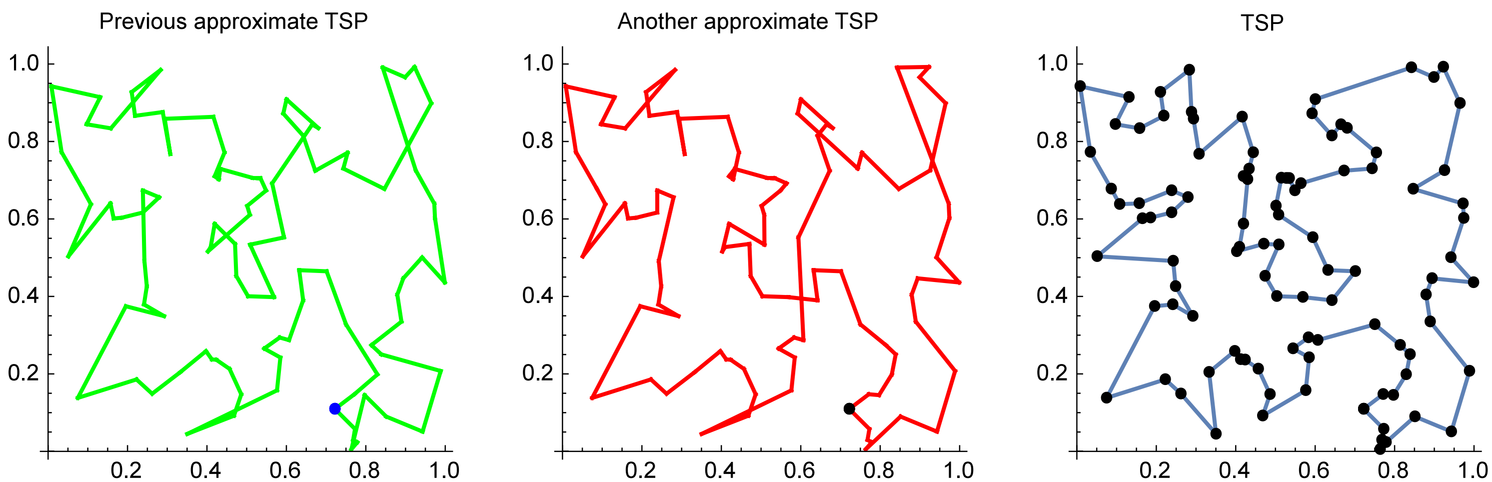
A 100-Node Example (continued)



The cost is $8.74583 \leq (3/2) \times 7.72877 = 11.5932.$[a]

[a] In comparison, the earlier 0.5-approximation algorithm gave a cost of 10.5718 on p. 768.

A 100-Node Example (concluded)



If a different Euler cycle were generated on p. 778, the cost could be different, such as 8.54902 (above), 8.85674, 8.53410, 9.20841, and 8.87152.[a]

[a]Contributed by Mr. Yu-Chuan Liu (`B00507010`, `R04922040`) on July 15, 2017.

KNAPSACK Has an Approximation Threshold of Zero[a]

Theorem 85 *For any ϵ, there is a polynomial-time ϵ-approximation algorithm for* KNAPSACK.

- We have n weights $w_1, w_2, \ldots, w_n \in \mathbb{Z}^+$, a weight limit W, and n values $v_1, v_2, \ldots, v_n \in \mathbb{Z}^+$.[b]
- We must find an $I \subseteq \{1, 2, \ldots, n\}$ such that $\sum_{i \in I} w_i \leq W$ and $\sum_{i \in I} v_i$ is the largest possible.

---

[a] Ibarra & Kim (1975). This algorithm can be used to derive good approximation algorithms for some NP-complete scheduling problems (Bansal & Sviridenko, 2006).

[b] If the values are fractional, the result is slightly messier, but the main conclusion remains correct. Contributed by Mr. Jr-Ben Tian (`B89902011`, `R93922045`) on December 29, 2004.

The Proof (continued)

- Let
$$V = \max\{ v_1, v_2, \ldots, v_n \}.$$
- Clearly, $\sum_{i \in I} v_i \leq nV$.
- Let $0 \leq i \leq n$ and $0 \leq v \leq nV$.
- $W(i, v)$ is the minimum weight attainable by selecting only from the *first* i items and with a total value of v.
  - It is an $(n + 1) \times (nV + 1)$ table.

The Proof (continued)

- Set $W(0, v) = \infty$ for $v \in \{1, 2, \ldots, nV\}$ and $W(i, 0) = 0$ for $i = 0, 1, \ldots, n$.[a]
- Then, for $0 \leq i < n$ and $1 \leq v \leq nV$,[b]

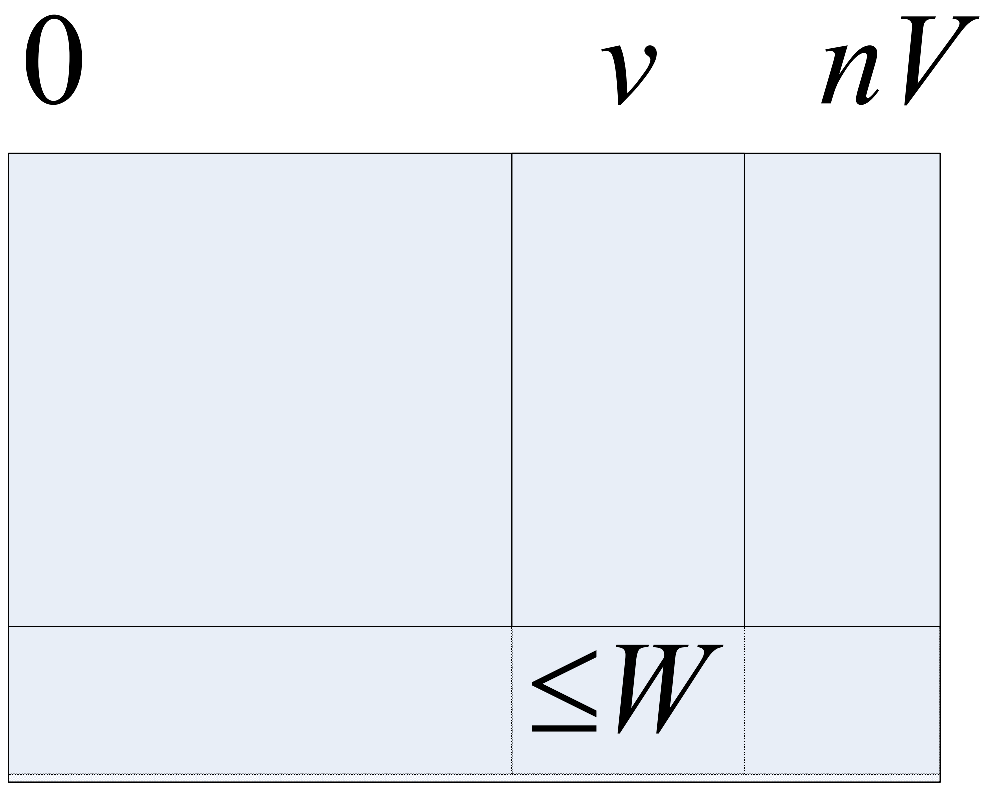
$$W(i+1, v) = \begin{cases} \min\{W(i, v), W(i, v - v_{i+1}) + w_{i+1}\}, & \text{if } v \geq v_{i+1}, \\ W(i, v), & \text{otherwise.} \end{cases}$$

- Finally, pick the largest v such that $W(n, v) \leq W$.[c]

[a]Contributed by Mr. Ren-Shuo Liu (`D98922016`) and Mr. Yen-Wei Wu (`D98922013`) on December 28, 2009.

[b]The textbook's formula has an error here.

[c]Lawler (1979).


0
v
nV
≤W

The Proof (continued)

With 6 items, values $(4, 3, 3, 3, 2, 3)$, weights $(3, 3, 1, 3, 2, 1)$, and $W = 12$, the maximum total value 16 is achieved with $I = \{1, 2, 3, 4, 6\}$; I's weight is 11.

0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
0	∞	∞	∞	3	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
0	∞	∞	3	3	∞	∞	6	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
0	∞	∞	1	3	∞	4	4	∞	∞	7	∞	∞	∞	∞	∞	∞	∞	∞
0	∞	∞	1	3	∞	4	4	∞	7	7	∞	∞	10	∞	∞	∞	∞	∞
0	∞	2	1	3	3	4	4	6	6	7	9	9	10	∞	12	∞	∞	∞
0	∞	2	1	3	3	2	4	4	5	5	7	7	8	10	10	11	∞	13

The Proof (continued)

- The running time $O(n^2V)$ is not polynomial.
- Call the problem instance

$$x = (w_1, \ldots, w_n, W, v_1, \ldots, v_n).$$

- Additional idea: Limit the number of precision bits.
- Define

$$v_i' = \left\lfloor \frac{v_i}{2^b} \right\rfloor.$$

- Note that

$$v_i \geq 2^b v_i' > v_i - 2^b.$$

The Proof (continued)

- Call the approximate instance

$$x' = (w_1, \ldots, w_n, W, v'_1, \ldots, v'_n).$$

- Solving x' takes time $O(n^2 V/2^b)$.
  - Use $v'_i = \lfloor v_i/2^b \rfloor$ and $V' = \max(v'_1, v'_2, \ldots, v'_n)$ in the dynamic programming.
  - It is now an $(n+1) \times (nV+1)/2^b$ table.
- The selection I' is optimal for x'.
- But I' may not be optimal for x, although it still satisfies the weight budget W.

The Proof (continued)

With the same parameters as p. 786 and $b = 1$: Values are $(2, 1, 1, 1, 1, 1)$ and the optimal selection $I' = \{1, 2, 3, 5, 6\}$ for x' has a *smaller* maximum value $4 + 3 + 3 + 2 + 3 = 15$ for x than I's 16; its weight is $10 < W = 12$.[a]

0	∞	∞	∞	∞	∞	∞	∞
0	∞	3	∞	∞	∞	∞	∞
0	3	3	6	∞	∞	∞	∞
0	1	3	4	7	∞	∞	∞
0	1	3	4	7	10	∞	∞
0	1	3	4	6	9	12	∞
0	1	2	4	5	7	10	13

[a] The *original* optimal $I = \{1, 2, 3, 4, 6\}$ on p. 786 has the same value 6 and but higher weight 11 for x'.

The Proof (continued)

- The value of I' for x is close to that of the optimal I:

$$\begin{aligned}
\sum_{i \in I'} v_i &\geq \sum_{i \in I'} 2^b v_i' = 2^b \sum_{i \in I'} v_i' \\
&\geq 2^b \sum_{i \in I} v_i' = \sum_{i \in I} 2^b v_i' \\
&\geq \sum_{i \in I} (v_i - 2^b) \\
&\geq \left(\sum_{i \in I} v_i \right) - n2^b.
\end{aligned}$$

The Proof (continued)

- In summary,

$$\sum_{i \in I'} v_i \geq \left(\sum_{i \in I} v_i \right) - n2^b.$$

- Without loss of generality, assume $w_i \leq W$ for all i.
  - Otherwise, item i is redundant and can be removed early on.
- V is a lower bound on OPT.
  - Picking one single item with value V is a legitimate choice.

The Proof (concluded)

- The relative error from the optimum is:

$$\frac{\sum_{i\in I} v_i - \sum_{i\in I'} v_i}{\sum_{i\in I} v_i} \leq \frac{\sum_{i\in I} v_i - \sum_{i\in I'} v_i}{V} \leq \frac{n2^b}{V}.$$

- Suppose we pick $b = \lfloor \log_2 \frac{\epsilon V}{n} \rfloor$.
- The algorithm becomes ϵ-approximate.[a]
- The running time is then $O(n^2V/2^b) = O(n^3/\epsilon)$, a polynomial in n and $1/\epsilon$.[b]

---

[a] See Eq. (17) on p. 727.

[b] It hence depends on the *value* of $1/\epsilon$. Thanks to a lively class discussion on December 20, 2006. If we fix ϵ and let the problem size increase, then the complexity is cubic. Contributed by Mr. Ren-Shan Luoh (`D97922014`) on December 23, 2008.

Comments

- INDEPENDENT SET and NODE COVER are reducible to each other (Corollary 46, p. 378).
- NODE COVER has an approximation threshold at most 0.5 (p. 740).
- But INDEPENDENT SET is unapproximable (see the textbook).
- INDEPENDENT SET limited to graphs with degree $\leq k$ is called k-DEGREE INDEPENDENT SET.
- k-DEGREE INDEPENDENT SET is approximable (see the textbook).

On P vs. NP

If 50 million people believe a foolish thing,
it's still a foolish thing.
— George Bernard Shaw (1856–1950)

Exponential Circuit Complexity for NP-Complete Problems

- We shall prove exponential lower bounds for NP-complete problems using *monotone* circuits.
  - Monotone circuits are circuits without $\neg$ gates.[a]
- Note that this result does *not* settle the P vs. NP problem.

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[a]Recall p. 316.

The Power of Monotone Circuits

- Monotone circuits can only compute monotone boolean functions.
- They are powerful enough to solve a P-complete problem: MONOTONE CIRCUIT VALUE (p. 317).
- There are NP-complete problems that are not monotone; they cannot be computed by monotone circuits at all.
- There are NP-complete problems that are monotone; they can be computed by monotone circuits.
  - HAMILTONIAN PATH and CLIQUE.

$\text{CLIQUE}_{n,k}$

- $\text{CLIQUE}_{n,k}$ is the boolean function deciding whether a graph $G = (V, E)$ with n nodes has a clique of size k.
- The input gates are the $\binom{n}{2}$ entries of the adjacency matrix of G.
  - Gate g_{ij} is set to true if the associated undirected edge $\{ i, j \}$ exists.
- $\text{CLIQUE}_{n,k}$ is a monotone function.
- Thus it can be computed by a monotone circuit.
- This does not rule out that *non*monotone circuits for $\text{CLIQUE}_{n,k}$ may use *fewer* gates.

Crude Circuits

- One possible circuit for CLIQUE$_{n,k}$ does the following.
  1. For each $S \subseteq V$ with $|S| = k$, there is a circuit with $O(k^2)$ $\wedge$-gates testing whether S forms a clique.
  2. We then take an OR of the outcomes of all the $\binom{n}{k}$ subsets $S_1, S_2, \ldots, S_{\binom{n}{k}}$.
- This is a monotone circuit with $O(k^2 \binom{n}{k})$ gates, which is exponentially large unless k or $n - k$ is a constant.
- A **crude circuit** $\mathrm{CC}(X_1, X_2, \ldots, X_m)$ tests if there is an $X_i \subseteq V$ that forms a clique.
  - The above-mentioned circuit is $\mathrm{CC}(S_1, S_2, \ldots, S_{\binom{n}{k}})$.

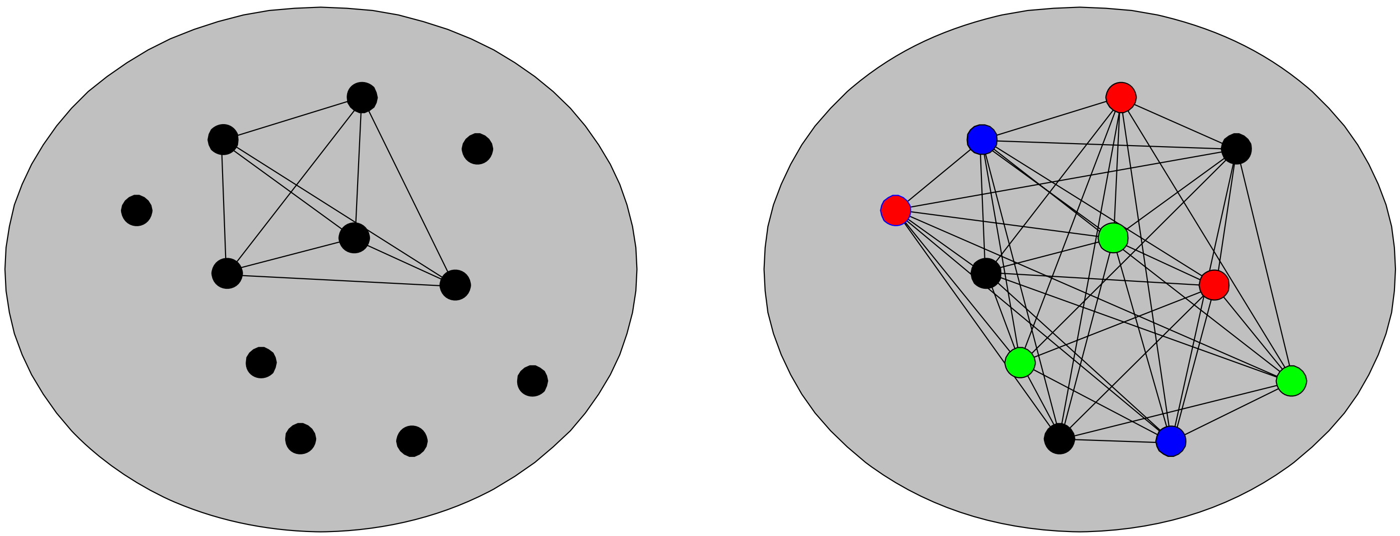
The Proof: Positive Examples

- Analysis will be applied to only the following **positive examples** and **negative examples** as input graphs.
- A positive example is a graph that has $\binom{k}{2}$ edges connecting k nodes in all possible ways.
- There are $\binom{n}{k}$ such graphs.
- They all should elicit a true output from $\textsc{clique}_{n,k}$.

The Proof: Negative Examples

- Color the nodes with $k-1$ different colors and join by an edge any two nodes that are colored differently.
- There are $(k-1)^n$ such graphs.
- They all should elicit a false output from $\text{CLIQUE}_{n,k}$.
  - Each set of k nodes must have 2 identically colored nodes; hence there is no edge between them.

Positive and Negative Examples with $k = 5$



A positive example

A negative example

A Warmup to Razborov's (1985) Theorem[a]

Lemma 86 (The birthday problem) *The probability of collision, $C(N, q)$, when q balls are thrown randomly into $N \geq q$ bins is at most*

$$\frac{q(q-1)}{2N}.$$

Lemma 87 *If crude circuit $CC(X_1, X_2, \ldots, X_m)$ computes $\text{CLIQUE}_{n,k}$, then $m \geq n^{n^{1/8}/20}$ for n sufficiently large.*

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[a] Arora & Barak (2009).

The Proof (continued)

- Let $k = n^{1/4}$.
- Let $\ell = \sqrt{k}/10$.
- Let $X \subseteq V$.

The Proof (continued)

- Suppose $|X| \leq \ell$.
- A random $f : X \rightarrow \{1, 2, \ldots, k-1\}$ has collisions with probability less than 0.01 by Lemma 86 (p. 803).
- Hence f is one-to-one with probability 0.99.
- When f is one-to-one, f is a coloring of X with $k-1$ colors without repeated colors.
- As a result, when f is one-to-one, it generates a clique on X.

The Proof (continued)

- Note that a random negative example is simply a random $g : V \rightarrow \{ 1, 2, \ldots, k-1 \}$.
- So our random $f : X \rightarrow \{ 1, 2, \ldots, k-1 \}$ is simply a random g restricted to X.
- In summary, the probability that X is not a clique when supplied with a random negative example is at most 0.01.

The Proof (continued)

- Now suppose $|\, X \,| > \ell$.
- Consider the probability that X is a clique when supplied with a random positive example.
- It is the probability that X is part of the clique.
- Hence the desired probability is

$$\frac{\binom{n-\ell}{k-\ell}}{\binom{n}{k}}.$$

The Proof (continued)

- Now,

$$\begin{aligned}\frac{\binom{n-\ell}{k-\ell}}{\binom{n}{k}} &= \frac{k(k-1)\cdots(k-\ell+1)}{n(n-1)\cdots(n-\ell+1)} \\ &\leq \left(\frac{k}{n}\right)^{\ell} \\ &\leq n^{-(3/4)\,\ell} \\ &\leq n^{-\sqrt{k}/20} \\ &= n^{-n^{1/8}/20}.\end{aligned}$$

The Proof (concluded)

- In summary, the probability that X is a clique when supplied with a random positive example is at most
$$n^{-n^{1/8}/20}.$$
- So we need at least
$$n^{n^{1/8}/20}$$
Xs in the crude circuit.