A Warmup to Razborov's (1985) Theorem[a]

Lemma 85 (The birthday problem) *The probability of collision, $C(N, q)$, when q balls are thrown randomly into $N \geq q$ bins is at most*

$$\frac{q(q-1)}{2N}.$$

Lemma 86 *If crude circuit $CC(X_1, X_2, \ldots, X_m)$ computes* $\textsc{clique}_{n,k}$*, then $m \geq n^{n^{1/8}/20}$ for n sufficiently large.*

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[a] Arora & Barak (2009).

The Proof (continued)

- Let $k = n^{1/4}$.
- Let $\ell = \sqrt{k}/10$.
- Let $X \subseteq V$.

The Proof (continued)

- Suppose $|X| \leq \ell$.
- A random $f : X \to \{1, 2, \ldots, k-1\}$ has collisions with probability less than 0.01 by Lemma 85 (p. 803).
- Hence f is one-to-one with probability 0.99.
- When f is one-to-one, f is a coloring of X with $k-1$ colors without repeated colors.
- As a result, when f is one-to-one, it generates a clique on X.

The Proof (continued)

- Note that a random negative example is simply a random $g : V \to \{ 1, 2, \ldots, k-1 \}$.
- So our random $f : X \to \{ 1, 2, \ldots, k-1 \}$ is simply a random g restricted to X.
- In summary, the probability that X is not a clique when supplied with a random negative example is at most 0.01.

The Proof (continued)

- Now suppose $|\,X\,| > \ell$.
- Consider the probability that X is a clique when supplied with a random positive example.
- It is the probability that X is part of the clique.
- Hence the desired probability is $\binom{n-\ell}{k-\ell} / \binom{n}{k}$.

The Proof (continued)

- Now,

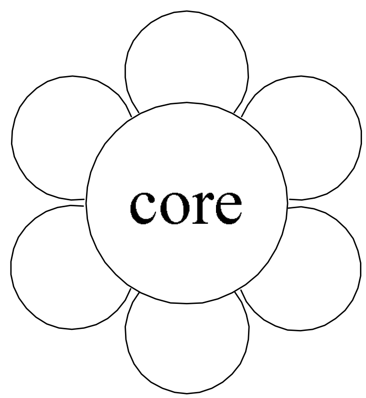
$$\begin{aligned}
\frac{\binom{n-\ell}{k-\ell}}{\binom{n}{k}} &= \frac{k(k-1)\cdots(k-\ell+1)}{n(n-1)\cdots(n-\ell+1)} \\
&\leq \left(\frac{k}{n}\right)^{\ell} \\
&\leq n^{-(3/4)\,\ell} \\
&\leq n^{-\sqrt{k}/20} \\
&= n^{-n^{1/8}/20}.
\end{aligned}$$

The Proof (concluded)

- In summary, the probability that X is a clique when supplied with a random positive example is at most $n^{-n^{1/8}/20}$.
- So we need at least $n^{n^{1/8}/20}$ Xs in the crude circuit.

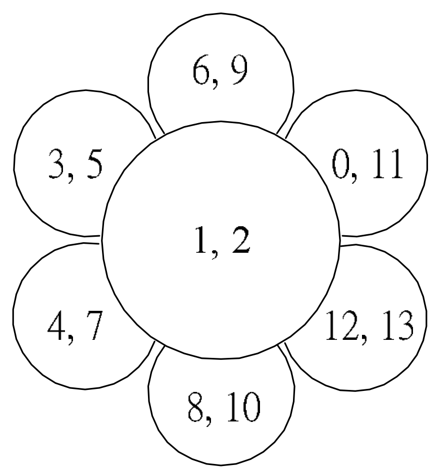
Sunflowers

- Fix $p \in \mathbb{Z}^+$ and $\ell \in \mathbb{Z}^+$.
- A **sunflower** is a family of p sets $\{ P_1, P_2, \ldots, P_p \}$, called **petals**, each of cardinality at most ℓ.
- Furthermore, all pairs of sets in the family must have the same intersection (called the **core** of the sunflower).



A Sample Sunflower

$$\{\{1,2,3,5\},\{1,2,6,9\},\{0,1,2,11\},$$
$$\{1,2,12,13\},\{1,2,8,10\},\{1,2,4,7\}\}.$$



The Erdős-Rado Lemma

Lemma 87 *Let $\mathcal{Z}$ be a family of more than $M \stackrel{\Delta}{=} (p-1)^{\ell} \ell!$ nonempty sets, each of cardinality ℓ or less. Then $\mathcal{Z}$ must contain a sunflower (with p petals).*

- Induction on ℓ.
- For $\ell = 1$, p different singletons form a sunflower (with an empty core).
- Suppose $\ell > 1$.
- Consider a *maximal* subset $\mathcal{D} \subseteq \mathcal{Z}$ of *disjoint* sets.
  - Every set in $\mathcal{Z} - \mathcal{D}$ intersects some set in $\mathcal{D}$.

The Proof of the Erdős-Rado Lemma (continued)

For example,

$$\begin{aligned}
\mathcal{Z} &= \{\{1,2,3,5\},\{1,3,6,9\},\{0,4,8,11\},\\
&\quad \{4,5,6,7\},\{5,8,9,10\},\{6,7,9,11\}\},\\
\mathcal{D} &= \{\{1,2,3,5\},\{0,4,8,11\}\}.
\end{aligned}$$

The Proof of the Erdős-Rado Lemma (continued)

- Suppose $\mathcal{D}$ contains at least p sets.
  - $\mathcal{D}$ constitutes a sunflower with an empty core.
- Suppose $\mathcal{D}$ contains fewer than p sets.
  - Let C be the union of all sets in $\mathcal{D}$.
  - $|\,C\,| < (p-1)\ell$.
  - C intersects every set in $\mathcal{Z}$ by $\mathcal{D}$'s maximality.
  - There is a $d \in C$ that intersects more than $\frac{M}{(p-1)\ell} = (p-1)^{\ell-1}(\ell-1)!$ sets in $\mathcal{Z}$.
  - Consider $\mathcal{Z}' = \{\, Z - \{\, d \,\} : Z \in \mathcal{Z}, d \in Z \,\}$.

The Proof of the Erdős-Rado Lemma (concluded)

- (continued)
  - $\mathcal{Z}'$ has more than $M' \stackrel{\Delta}{=} (p-1)^{\ell-1}(\ell-1)!$ sets.
  - M' is just M with ℓ replaced with $\ell - 1$.
  - $\mathcal{Z}'$ contains a sunflower by induction, say

$$\{ P_1, P_2, \ldots, P_p \}.$$

  - Now,

$$\{ P_1 \cup \{ d \}, P_2 \cup \{ d \}, \ldots, P_p \cup \{ d \} \}$$

    is a sunflower in $\mathcal{Z}$.

Comments on the Erdős-Rado Lemma

- A family of more than M sets must contain a sunflower.
- **Plucking** a sunflower means replacing the sets in the sunflower by its core.
- By *repeatedly* finding a sunflower and plucking it, we can reduce a family with more than M sets to a family with at most M sets.
- If $\mathcal{Z}$ is a family of sets, the above result is denoted by pluck($\mathcal{Z}$).
- pluck($\mathcal{Z}$) is not unique.[a]

[a] It depends on the sequence of sunflowers one plucks.

An Example of Plucking

- Recall the sunflower on p. 811:

$$\begin{aligned}\mathcal{Z} &= \{\{1,2,3,5\},\{1,2,6,9\},\{0,1,2,11\},\\ &\quad \{1,2,12,13\},\{1,2,8,10\},\{1,2,4,7\}\}\end{aligned}$$

- Then

$$\text{pluck}(\mathcal{Z}) = \{\{1,2\}\}.$$

Razborov's Theorem

Theorem 88 (Razborov, 1985) *There is a constant c such that for large enough n, all monotone circuits for* $\text{CLIQUE}_{n,k}$ *with* $k = n^{1/4}$ *have size at least* $n^{cn^{1/8}}$.

- We shall approximate any monotone circuit for $\text{CLIQUE}_{n,k}$ by a restricted kind of crude circuit.
- The approximation will proceed in steps: one step for each gate of the monotone circuit.
- Each step introduces few errors (false positives and false negatives).
- Yet, the final crude circuit has exponentially many errors.

The Proof

- Fix $k = n^{1/4}$.
- Fix $\ell = n^{1/8}$.
- Note that[a]
$$2\binom{\ell}{2} \leq k - 1.$$
- p will be fixed later to be $n^{1/8} \log n$.
- Fix $M = (p-1)^{\ell} \ell!$.
  - Recall the Erdős-Rado lemma (p. 812).

[a]Corrected by Mr. Moustapha Bande (`D98922042`) on January 5, 2010.

The Proof (continued)

- Each crude circuit used in the approximation process is of the form $\mathrm{CC}(X_1, X_2, \ldots, X_m)$, where:
  - $X_i \subseteq V$.
  - $|X_i| \leq \ell$.
  - $m \leq M$.
- It answers true if any X_i is a clique.
- We shall show how to approximate any monotone circuit for $\text{CLIQUE}_{n,k}$ by such a crude circuit, inductively.
- The induction basis is straightforward:
  - Input gate g_{ij} is the crude circuit $\mathrm{CC}(\{i, j\})$.

The Proof (continued)

- A monotone circuit is the OR or AND of two subcircuits.
- We will build approximators of the overall circuit from the approximators of the two subcircuits.
  - Start with two crude circuits $\text{CC}(\mathcal{X})$ and $\text{CC}(\mathcal{Y})$.
  - $\mathcal{X}$ and $\mathcal{Y}$ are two families of at most M sets of nodes, each set containing at most ℓ nodes.
  - We will construct the approximate OR and the approximate AND of these subcircuits.
  - Then show both approximations introduce few errors.

The Proof: OR

- $\text{CC}(\mathcal{X} \cup \mathcal{Y})$ is *equivalent to* the OR of $\text{CC}(\mathcal{X})$ and $\text{CC}(\mathcal{Y})$.
  - Trivially, a node set $\mathcal{C} \in \mathcal{X} \cup \mathcal{Y}$ is a clique if and only if $\mathcal{C} \in \mathcal{X}$ is a clique or $\mathcal{C} \in \mathcal{Y}$ is a clique.
- Violations in using $\text{CC}(\mathcal{X} \cup \mathcal{Y})$ occur when $|\,\mathcal{X} \cup \mathcal{Y}\,| > M$.
- Such violations are eliminated by using

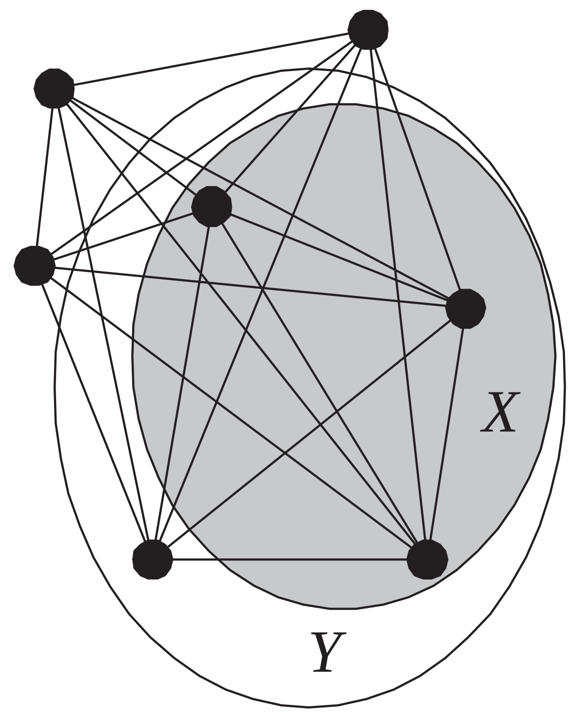
$$\text{CC}(\text{pluck}(\mathcal{X} \cup \mathcal{Y}))$$

as the approximate OR of $\text{CC}(\mathcal{X})$ and $\text{CC}(\mathcal{Y})$.

The Proof: OR

- If $\text{CC}(\mathcal{Z})$ is true, then $\text{CC}(\text{pluck}(\mathcal{Z}))$ must be true.
  - The quick reason: If Y is a clique, then a subset of Y must also be a clique.
  - Let $Y \in \mathcal{Z}$ be a clique.
  - There must exist an $X \in \text{pluck}(\mathcal{Z})$ such that $X \subseteq Y$.
  - This X is also a clique.

The Proof: OR (continued)



The Proof: OR (concluded)

- $\mathrm{CC}(\mathrm{pluck}(\mathcal{X} \cup \mathcal{Y}))$ *introduces* a **false positive** if a negative example makes both $\mathrm{CC}(\mathcal{X})$ and $\mathrm{CC}(\mathcal{Y})$ return false but makes $\mathrm{CC}(\mathrm{pluck}(\mathcal{X} \cup \mathcal{Y}))$ return true.

- $\mathrm{CC}(\mathrm{pluck}(\mathcal{X} \cup \mathcal{Y}))$ *introduces* a **false negative** if a positive example makes either $\mathrm{CC}(\mathcal{X})$ or $\mathrm{CC}(\mathcal{Y})$ return true but makes $\mathrm{CC}(\mathrm{pluck}(\mathcal{X} \cup \mathcal{Y}))$ return false.

- We next count the number of false positives and false negatives introduced[a] by $\mathrm{CC}(\mathrm{pluck}(\mathcal{X} \cup \mathcal{Y}))$.

- Let us work on false negatives first.

[a]Compared with $\mathrm{CC}(\mathcal{X} \cup \mathcal{Y})$ of course.

The Number of False Negatives

Lemma 89 CC(pluck($\mathcal{X} \cup \mathcal{Y}$)) *introduces no false negatives.*

- Each plucking replaces sets in a crude circuit by their common subset.
- This makes the test for cliqueness less stringent (p. 823).[a]

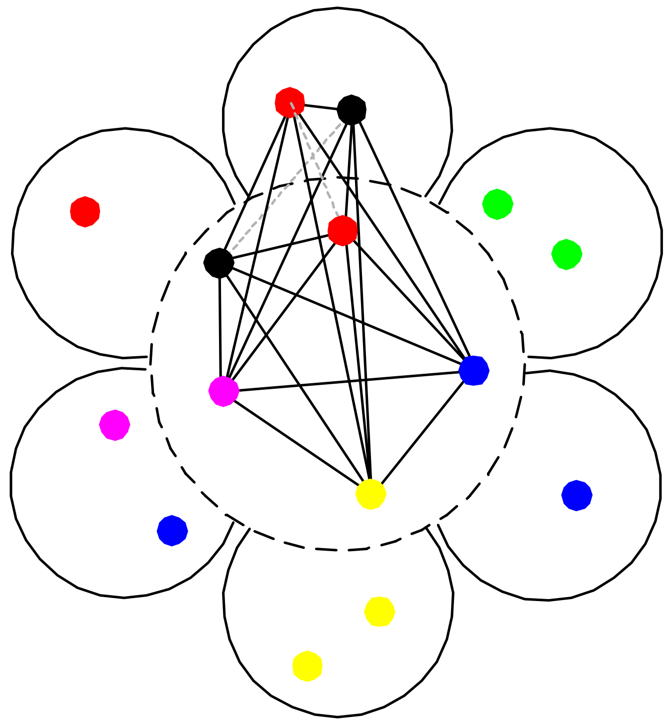
[a]Recall that CC(pluck($\mathcal{X} \cup \mathcal{Y}$)) introduces a false negative if a positive example makes either CC($\mathcal{X}$) or CC($\mathcal{Y}$) return true but makes CC(pluck($\mathcal{X} \cup \mathcal{Y}$)) return false.

The Number of False Positives

Lemma 90 $\mathrm{CC}(\mathrm{pluck}(\mathcal{X} \cup \mathcal{Y}))$ *introduces at most* $\frac{2M}{p-1}\, 2^{-p}(k-1)^n$ *false positives.*

- Each plucking operation replaces the sunflower $\{ Z_1, Z_2, \ldots, Z_p \}$ with its common core Z.
- A false positive is *necessarily* a coloring such that:
  - There is a pair of identically colored nodes in each petal Z_i (and so $\mathrm{CC}(Z_1, Z_2, \ldots, Z_p)$ returns false).
  - But the core contains distinctly colored nodes.
  - This implies at least one node from each identical-color pair was plucked away.

Proof of Lemma 90 (continued)



Proof of Lemma 90 (continued)

- We now count the number of such colorings.
- Color nodes in V at random with $k-1$ colors.
- Let $R(X)$ denote the event that there are repeated colors in set X.

Proof of Lemma 90 (continued)

- Now

$$
\begin{aligned}
& \text{prob}[\, R(Z_1) \wedge \cdots \wedge R(Z_p) \wedge \neg R(Z) \,] & (23) \\
\leq \;& \text{prob}[\, R(Z_1) \wedge \cdots \wedge R(Z_p) \,|\, \neg R(Z) \,] \\
= \;& \prod_{i=1}^{p} \text{prob}[\, R(Z_i) \,|\, \neg R(Z) \,] \\
\leq \;& \prod_{i=1}^{p} \text{prob}[\, R(Z_i) \,]. & (24)
\end{aligned}
$$

  - First equality holds because $R(Z_i)$ are independent given $\neg R(Z)$ as Z contains their *only common* nodes.
  - Last inequality holds as the likelihood of repetitions in Z_i decreases given no repetitions in its subset Z.

Proof of Lemma 90 (continued)

- Consider two nodes in Z_i.
- The probability that they have identical color is
$$\frac{1}{k-1}.$$
- Now
$$\text{prob}[\, R(Z_i)\,] \leq \frac{\binom{|Z_i|}{2}}{k-1} \leq \frac{\binom{\ell}{2}}{k-1} \leq \frac{1}{2}.$$
- So the probability[a] that a random coloring is a *new* false positive is at most 2^{-p} by inequality (24) on p. 830.

[a] Proportion, i.e.

Proof of Lemma 90 (continued)

- As there are $(k-1)^n$ different colorings, each plucking introduces at most $2^{-p}(k-1)^n$ false positives.
- Recall that $|\,\mathcal{X} \cup \mathcal{Y}\,| \leq 2M$.
- When the procedure pluck($\mathcal{X} \cup \mathcal{Y}$) ends, the set system contains $\leq M$ sets.

Proof of Lemma 90 (concluded)

- Each plucking reduces the number of sets by $p - 1$.
- Hence at most $2M/(p-1)$ pluckings occur in pluck$(\mathcal{X} \cup \mathcal{Y})$.
- At most

$$\frac{2M}{p-1}\, 2^{-p}(k-1)^n$$

false positives are introduced.[a]

[a]Note that the numbers of errors are added not multiplied. Recall that we count how many *new* errors are introduced by each approximation step. Contributed by Mr. Ren-Shuo Liu (D98922016) on January 5, 2010.

The Proof: AND

- The approximate AND of crude circuits $\mathrm{CC}(\mathcal{X})$ and $\mathrm{CC}(\mathcal{Y})$ is

$$\mathrm{CC}(\mathrm{pluck}(\{\, X_i \cup Y_j : X_i \in \mathcal{X}, Y_j \in \mathcal{Y}, |\, X_i \cup Y_j \,| \leq \ell \,\})).$$

- We now count the number of errors this approximate AND introduces on the positive and negative examples.

The Proof: AND (concluded)

- The approximate AND *introduces* a **false positive** if a negative example makes either $\mathrm{CC}(\mathcal{X})$ or $\mathrm{CC}(\mathcal{Y})$ return false but makes the approximate AND return true.
- The approximate AND *introduces* a **false negative** if a positive example makes both $\mathrm{CC}(\mathcal{X})$ and $\mathrm{CC}(\mathcal{Y})$ return true but makes the approximate AND return false.
- We now bound the number of false positives and false negatives introduced[a] by the approximate AND.

[a] Compared with $\mathrm{CC}(\{ X_i \cup Y_j : X_i \in \mathcal{X}, Y_j \in \mathcal{Y} \})$.

The Number of False Positives

Lemma 91 *The approximate* AND *introduces at most* $M^2 2^{-p}(k-1)^n$ *false positives.*

- We prove this claim in stages.
- $\mathrm{CC}(\{ X_i \cup Y_j : X_i \in \mathcal{X}, Y_j \in \mathcal{Y} \})$ introduces no false positives.
  - If $X_i \cup Y_j$ is a clique, both X_i and Y_j must be cliques, making both $\mathrm{CC}(\mathcal{X})$ and $\mathrm{CC}(\mathcal{Y})$ return true.
- $\mathrm{CC}(\{ X_i \cup Y_j : X_i \in \mathcal{X}, Y_j \in \mathcal{Y}, | X_i \cup Y_j | \leq \ell \})$ introduces no additional false positives because we are testing only a subset of sets for cliqueness.

Proof of Lemma 91 (concluded)

- $|\{ X_i \cup Y_j : X_i \in \mathcal{X}, Y_j \in \mathcal{Y}, | X_i \cup Y_j | \leq \ell \}| \leq M^2$.
- Each plucking reduces the number of sets by $p - 1$.
- So pluck($\{ X_i \cup Y_j : X_i \in \mathcal{X}, Y_j \in \mathcal{Y}, | X_i \cup Y_j | \leq \ell \}$) involves $\leq M^2/(p-1)$ pluckings.
- Each plucking introduces at most $2^{-p}(k-1)^n$ false positives by the proof of Lemma 90 (p. 827).
- The desired upper bound is

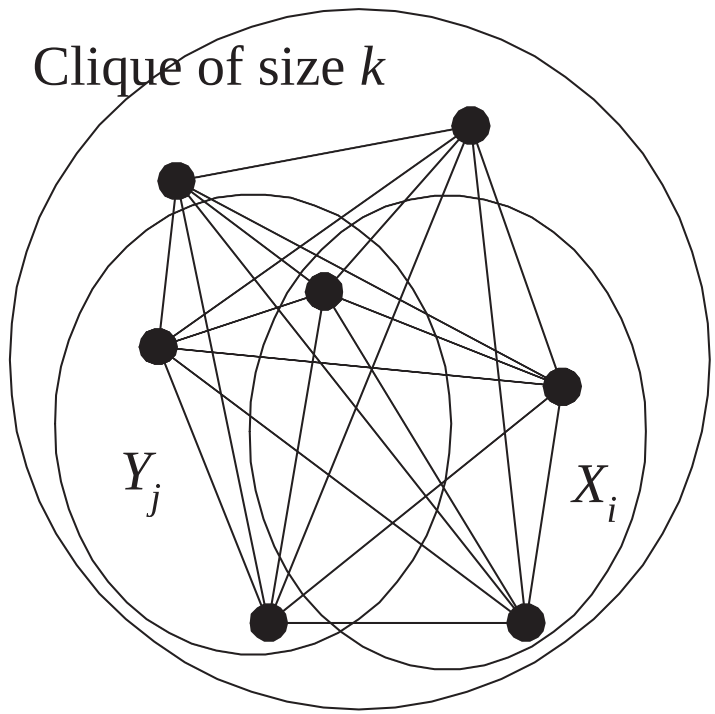
$$[\, M^2/(p-1)\,]\, 2^{-p}(k-1)^n \leq M^2 2^{-p}(k-1)^n.$$

The Number of False Negatives

Lemma 92 *The approximate* AND *introduces at most* $M^2\binom{n-\ell-1}{k-\ell-1}$ *false negatives.*

- We again prove this claim in stages.
- $\mathrm{CC}(\{\, X_i \cup Y_j : X_i \in \mathcal{X}, Y_j \in \mathcal{Y} \,\})$ introduces no false negatives.
  - Suppose *both* $\mathrm{CC}(\mathcal{X})$ and $\mathrm{CC}(\mathcal{Y})$ accept a positive example with a clique $\mathcal{C}$ of size k.
  - This clique $\mathcal{C}$ must contain an $X_i \in \mathcal{X}$ and a $Y_j \in \mathcal{Y}$.
    - This is why both $\mathrm{CC}(\mathcal{X})$ and $\mathrm{CC}(\mathcal{Y})$ return true.
  - As this clique $\mathcal{C}$ also contains $X_i \cup Y_j$, the new circuit returns true.

Proof of Lemma 92 (continued)



Proof of Lemma 92 (continued)

- $\mathrm{CC}(\{ X_i \cup Y_j : X_i \in \mathcal{X}, Y_j \in \mathcal{Y}, |\, X_i \cup Y_j \,| \leq \ell \})$ introduces $\leq M^2 \binom{n-\ell-1}{k-\ell-1}$ false negatives.
  - Deletion of set $Z \stackrel{\Delta}{=} X_i \cup Y_j$ larger than ℓ introduces false negatives *only if* Z is part of a clique.
  - There are $\binom{n-|\,Z\,|}{k-|\,Z\,|}$ such cliques.
    * It is the number of positive examples whose clique contains Z.
  - $\binom{n-|\,Z\,|}{k-|\,Z\,|} \leq \binom{n-\ell-1}{k-\ell-1}$ as $|\,Z\,| > \ell$.
  - There are at most M^2 such Zs.

Proof of Lemma 92 (concluded)

- Plucking introduces no false negatives.
  - Recall that if $\text{CC}(\mathcal{Z})$ is true, then $\text{CC}(\text{pluck}(\mathcal{Z}))$ must be true (p. 823).

Two Summarizing Lemmas

From Lemmas 90 (p. 827) and 91 (p. 836), we have:

Lemma 93 *Each approximation step introduces at most* $M^2 2^{-p}(k-1)^n$ *false positives.*

From Lemmas 89 (p. 826) and 92 (p. 838), we have:

Lemma 94 *Each approximation step introduces at most* $M^2\binom{n-\ell-1}{k-\ell-1}$ *false negatives.*

The Proof (continued)

- The above two lemmas show that each approximation step introduces "few" false positives and false negatives.
- We next show that the resulting crude circuit has "a lot" of false positives or false negatives.

The Final Crude Circuit

Lemma 95 *Every final crude circuit is:*

1. *Identically false—thus wrong on all positive examples.*
2. *Or outputs true on at least half of the negative examples.*

- Suppose it is not identically false.
- By construction, it accepts at least those graphs that have a clique on some set X of nodes, with $|X| \leq \ell$, which at $n^{1/8}$ is less than $k = n^{1/4}$.

Proof of Lemma 95 (concluded)

- The proof of Lemma 90 (p. 827ff) shows that at least half of the colorings assign different colors to nodes in X.
- So at least half of the negative examples have a clique in X and are accepted.

The Proof (continued)

- Recall the constants on p. 819:

$$
\begin{aligned}
k &\stackrel{\Delta}{=} n^{1/4}, \\
\ell &\stackrel{\Delta}{=} n^{1/8}, \\
p &\stackrel{\Delta}{=} n^{1/8} \log n, \\
M &\stackrel{\Delta}{=} (p-1)^{\ell} \ell! < n^{(1/3)n^{1/8}} \quad \text{for large } n.
\end{aligned}
$$

The Proof (continued)

- Suppose the final crude circuit is identically false.
  - By Lemma 94 (p. 842), each approximation step introduces at most $M^2\binom{n-\ell-1}{k-\ell-1}$ false negatives.
  - There are $\binom{n}{k}$ positive examples.
  - The original monotone circuit for $\text{CLIQUE}_{n,k}$ has at least

$$\frac{\binom{n}{k}}{M^2\binom{n-\ell-1}{k-\ell-1}} \geq \frac{1}{M^2}\left(\frac{n-\ell}{k}\right)^{\ell} \geq n^{(1/12)n^{1/8}}$$

gates for large n.

The Proof (concluded)

- Suppose the final crude circuit is not identically false.
  - Lemma 95 (p. 844) says that there are at least $(k-1)^n/2$ false positives.
  - By Lemma 93 (p. 842), each approximation step introduces at most $M^2 2^{-p}(k-1)^n$ false positives
  - The original monotone circuit for $\text{CLIQUE}_{n,k}$ has at least
$$\frac{(k-1)^n/2}{M^2 2^{-p}(k-1)^n} = \frac{2^{p-1}}{M^2} \geq n^{(1/3)n^{1/8}}$$
    gates.

Alexander Razborov (1963–)

$P \neq NP$ Proved?

- Razborov's theorem says that there is a monotone language in NP that has no polynomial monotone circuits.
- If we can prove that all monotone languages in P have polynomial monotone circuits, then $P \neq NP$.
- But Razborov proved in 1985 that some monotone languages in P have no polynomial monotone circuits!

Finis