Reductions and Completeness

It is unworthy of excellent men
to lose hours like slaves in the labor of
computation.
— Gottfried Wilhelm von Leibniz (1646–1716)

I thought perhaps you might be members of
that lowly section of the university
known as the Sheffield Scientific School.
F. Scott Fitzgerald (1920), "May Day"

Degrees of Difficulty

- When is a problem more difficult than another?
- B **reduces to** A if:
  - There is a transformation R which for every problem instance x of B yields a problem instance $R(x)$ of A.[a]
  - The answer to "$R(x) \in$ A?" is the same as the answer to "$x \in$ B?"
  - R is easy to compute.
- We say problem A is at least as hard as[b] problem B if B reduces to A.

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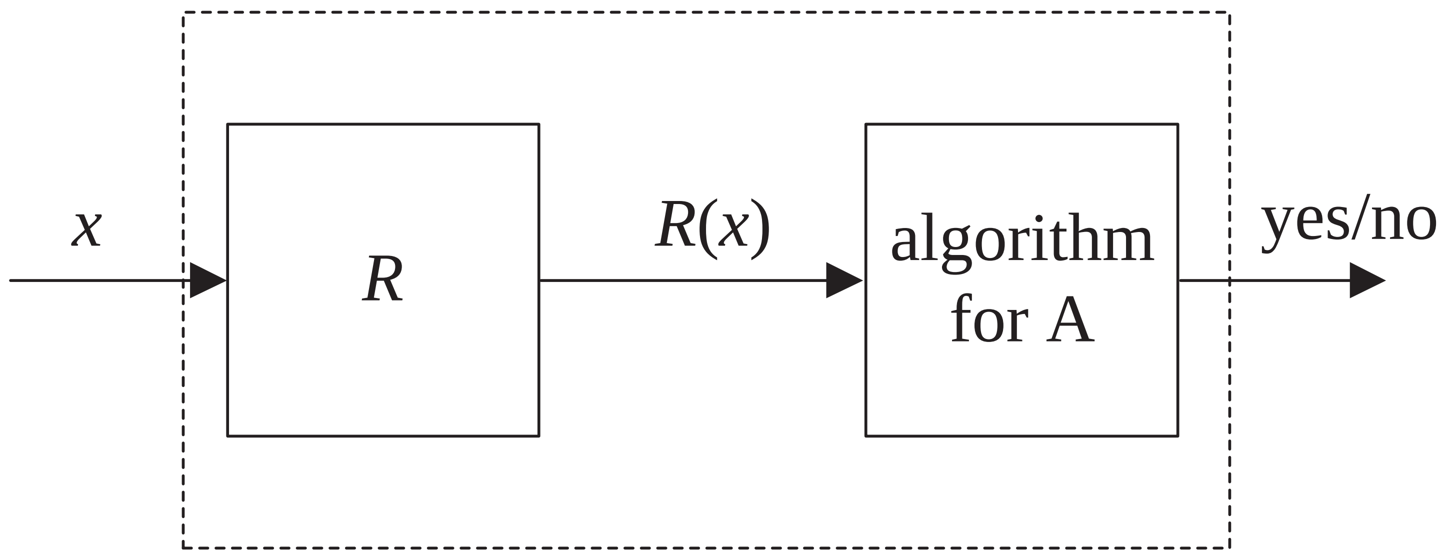
[a]See also p. 141.

[b]Or simply "harder than" for brevity.

Reduction



Solving problem B by calling the algorithm for problem A *once* and *without* further processing its answer.[a]

[a]More general reductions are possible, such as the Turing reduction (1939) and the Cook reduction (1971).

Degrees of Difficulty (concluded)

- This makes intuitive sense: If A is able to solve your problem B after only a little bit of work of R, then A must be at least as hard.
  - If A is easy to solve, it combined with R (which is also easy) would make B easy to solve, too.[a]
  - So if B is hard to solve, A must be hard (if not harder), too!

[a]Thanks to a lively class discussion on October 13, 2009.

Comments[a]

- Suppose B reduces to A via a transformation R.[b]
- The input x is an instance of B.
- The output $R(x)$ is an instance of A.
- $R(x)$ may not span all possible instances of A.[c]
  - Some instances of A may never appear in the range of R.
- But x must be a general instance for B.

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[a]Contributed by Mr. Ming-Feng Tsai (D92922003) on October 29, 2003.

[b]Sometimes, we say "B can be reduced to A."

[c]$R(x)$ may not be onto; Mr. Alexandr Simak (D98922040) on October 13, 2009.

Is “Reduction” a Confusing Choice of Word?[a]

- If B reduces to A, doesn’t that intuitively make A smaller and simpler?
- But our definition means just the opposite.
- Our definition says in this case B is a special case of A.[b]
- Hence A is harder.

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[a]Moore and Mertens (2011).
[b]See also p. 144.

Reduction between Languages

- Language L_1 is **reducible to** L_2 if there is a function R computable by a deterministic TM in space $O(\log n)$.
- Furthermore, for all inputs x, $x \in L_1$ if and only if $R(x) \in L_2$.
- R is said to be a (**Karp**) **reduction** from L_1 to L_2.

Reduction between Languages (concluded)

- Note that by Theorem 22 (p. 223), R runs in polynomial time.
  - In most cases, a polynomial-time R suffices for proofs.[a]
- Suppose R is a reduction from L_1 to L_2.
- Then solving "$R(x) \in L_2$?" is an algorithm for solving "$x \in L_1$?"[b]

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[a] In fact, unless stated otherwise, we will only require that the reduction R run in polynomial time.

[b] Of course, it may not be an optimal one.

A Paradox?

- Degree of difficulty is not defined in terms of *absolute* complexity.
- So a language B $\in$ TIME(n^{99}) may be "easier" than a language A $\in$ TIME(n^3).
  - Again, this happens when B reduces to A.
- But isn't this a contradiction if the best algorithm for B requires n^{99} steps?
- That is, how can a problem *requiring* n^{99} steps be reducible to a problem solvable in n^3 steps?

Paradox Resolved

- The so-called contradiction is the result of flawed logic.
- Suppose we solve the problem "$x \in$ B?" via "$R(x) \in$ A?"
- We must consider the time spent by $R(x)$ and its length $|\, R(x) \,|$:
  - Because $R(x)$ (not x) is solved by A.

HAMILTONIAN PATH

- A **Hamiltonian path** of a graph is a path that visits every node of the graph exactly once.
- Suppose graph G has n nodes: $1, 2, \ldots, n$.
- A Hamiltonian path can be expressed as a permutation π of $\{1, 2, \ldots, n\}$ such that
  - $\pi(i) = j$ means the ith position is occupied by node j.
  - $(\pi(i), \pi(i+1)) \in G$ for $i = 1, 2, \ldots, n-1$.

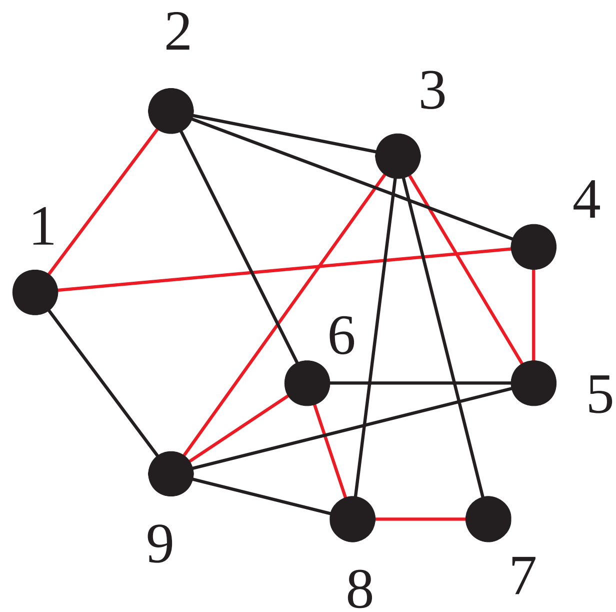
HAMILTONIAN PATH (concluded)

- So
$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}.$$
- HAMILTONIAN PATH asks if a graph has a Hamiltonian path.

Reduction of HAMILTONIAN PATH to SAT

- Given a graph G, we shall construct a CNF $R(G)$ such that $R(G)$ is satisfiable if and only if G has a Hamiltonian path.
- $R(G)$ has n^2 boolean variables x_{ij}, $1 \leq i, j \leq n$.
- x_{ij} means

  the ith position in the Hamiltonian path is occupied by node j.
- Our reduction will produce clauses.



$x_{12} = x_{21} = x_{34} = x_{45} = x_{53} = x_{69} = x_{76} = x_{88} = x_{97} = 1$;
$\pi(1) = 2, \pi(2) = 1, \pi(3) = 4, \pi(4) = 5, \pi(5) = 3, \pi(6) = 9, \pi(7) = 6, \pi(8) = 8, \pi(9) = 7$.

The Clauses of $R(G)$ and Their Intended Meanings

1. Each node j must appear in the path.
   - $x_{1j} \vee x_{2j} \vee \cdots \vee x_{nj}$ for each j.
2. No node j appears twice in the path.
   - $\neg x_{ij} \vee \neg x_{kj} (\equiv \neg(x_{ij} \wedge x_{kj}))$ for all i, j, k with $i \neq k$.
3. Every position i on the path must be occupied.
   - $x_{i1} \vee x_{i2} \vee \cdots \vee x_{in}$ for each i.
4. No two nodes j and k occupy the same position in the path.
   - $\neg x_{ij} \vee \neg x_{ik} (\equiv \neg(x_{ij} \wedge x_{ik}))$ for all i, j, k with $j \neq k$.
5. Nonadjacent nodes i and j cannot be adjacent in the path.
   - $\neg x_{ki} \vee \neg x_{k+1,j} (\equiv \neg(x_{k,i} \wedge x_{k+1,j}))$ for all $(i, j) \notin E$ and $k = 1, 2, \ldots, n-1$.

The Proof

- $R(G)$ contains $O(n^3)$ clauses.
- $R(G)$ can be computed efficiently (simple exercise).
- Suppose $T \models R(G)$.
- From the 1st and 2nd types of clauses, for each node j there is a unique position i such that $T \models x_{ij}$.
- From the 3rd and 4th types of clauses, for each position i there is a unique node j such that $T \models x_{ij}$.
- So there is a permutation π of the nodes such that $\pi(i) = j$ if and only if $T \models x_{ij}$.

The Proof (concluded)

- The 5th type of clauses furthermore guarantee that $(\pi(1), \pi(2), \ldots, \pi(n))$ is a Hamiltonian path.
- Conversely, suppose G has a Hamiltonian path

$$(\pi(1), \pi(2), \ldots, \pi(n)),$$

  where π is a permutation.
- Clearly, the truth assignment

$$T(x_{ij}) = \texttt{true} \text{ if and only if } \pi(i) = j$$

  satisfies all clauses of $R(G)$.

A Comment[a]

- An answer to "Is $R(G)$ satisfiable?" answers the question "Is G Hamiltonian?"
- But a "yes" does not give a Hamiltonian path for G.
  - Providing a witness is not a requirement of reduction.
- A "yes" to "Is $R(G)$ satisfiable?" *plus* a satisfying truth assignment does provide us with a Hamiltonian path for G.

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[a]Contributed by Ms. Amy Liu (`J94922016`) on May 29, 2006.

Reduction of REACHABILITY to CIRCUIT VALUE

- Note that both problems are in P.
- Given a graph $G = (V, E)$, we shall construct a *variable-free* circuit $R(G)$.
- The output of $R(G)$ is true if and only if there is a path from node 1 to node n in G.
- Idea: the Floyd-Warshall algorithm.

The Gates

- The gates are
  - g_{ijk} with $1 \leq i, j \leq n$ and $0 \leq k \leq n$.
  - h_{ijk} with $1 \leq i, j, k \leq n$.
- g_{ijk}: There is a path from node i to node j without passing through a node bigger than k.
- h_{ijk}: There is a path from node i to node j passing through k but not any node bigger than k.
- Input gate $g_{ij0} = \texttt{true}$ if and only if $i = j$ or $(i, j) \in E$.

The Construction

- h_{ijk} is an AND gate with predecessors $g_{i,k,k-1}$ and $g_{k,j,k-1}$, where $k = 1, 2, \ldots, n$.
- g_{ijk} is an OR gate with predecessors $g_{i,j,k-1}$ and $h_{i,j,k}$, where $k = 1, 2, \ldots, n$.
- g_{1nn} is the output gate.
- Interestingly, $R(G)$ uses no $\neg$ gates.
  - It is a **monotone circuit**.

Reduction of CIRCUIT SAT to SAT

- Given a circuit C, we will construct a boolean expression $R(C)$ such that $R(C)$ is satisfiable if and only if C is.
  - $R(C)$ will turn out to be a CNF.
  - $R(C)$ is basically a depth-2 circuit; furthermore, each gate has out-degree 1.
- The variables of $R(C)$ are those of C plus g for each gate g of C.
  - The g's propagate the truth values for the CNF.
- Each gate of C will be turned into equivalent clauses.
- Recall that clauses are $\wedge$ed together by definition.

The Clauses of $R(C)$

g **is a variable gate** x**:** Add clauses $(\neg g \lor x)$ and $(g \lor \neg x)$.

- Meaning: $g \Leftrightarrow x$.

g **is a `true` gate:** Add clause (g).

- Meaning: g must be true to make $R(C)$ true.

g **is a `false` gate:** Add clause $(\neg g)$.

- Meaning: g must be false to make $R(C)$ true.

g **is a $\neg$ gate with predecessor gate** h**:** Add clauses $(\neg g \lor \neg h)$ and $(g \lor h)$.

- Meaning: $g \Leftrightarrow \neg h$.

The Clauses of $R(C)$ (concluded)

g **is a** $\vee$ **gate with predecessor gates** h **and** h'**:** Add clauses $(\neg h \vee g)$, $(\neg h' \vee g)$, and $(h \vee h' \vee \neg g)$.

- Meaning: $g \Leftrightarrow (h \vee h')$.

g **is a** $\wedge$ **gate with predecessor gates** h **and** h'**:** Add clauses $(\neg g \vee h)$, $(\neg g \vee h')$, and $(\neg h \vee \neg h' \vee g)$.

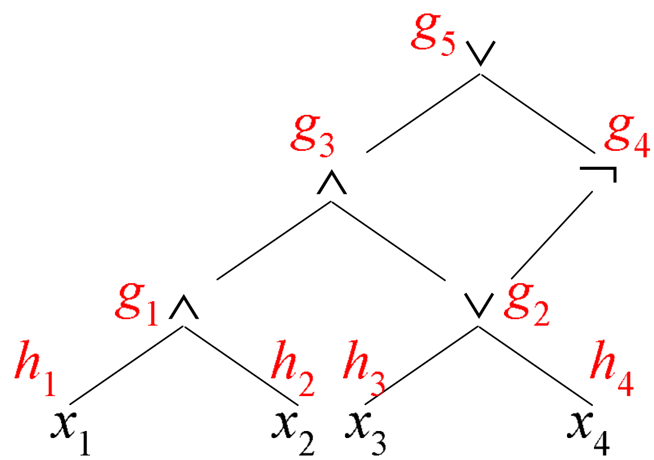
- Meaning: $g \Leftrightarrow (h \wedge h')$.

g **is the output gate:** Add clause (g).

- Meaning: g must be true to make $R(C)$ true.

Note: If gate g feeds gates $h_1, h_2, \ldots$, then variable g appears in the clauses for $h_1, h_2, \ldots$ in $R(C)$.

An Example



$$(h_1 \Leftrightarrow x_1) \wedge (h_2 \Leftrightarrow x_2) \wedge (h_3 \Leftrightarrow x_3) \wedge (h_4 \Leftrightarrow x_4)$$

$$\wedge \quad [\, g_1 \Leftrightarrow (h_1 \wedge h_2) \,] \wedge [\, g_2 \Leftrightarrow (h_3 \vee h_4) \,]$$

$$\wedge \quad [\, g_3 \Leftrightarrow (g_1 \wedge g_2) \,] \wedge (g_4 \Leftrightarrow \neg g_2)$$

$$\wedge \quad [\, g_5 \Leftrightarrow (g_3 \vee g_4) \,] \wedge g_5.$$

An Example (concluded)

- In general, the result is a CNF.
- The CNF has size proportional to the circuit's number of gates.
- The CNF adds new variables to the circuit's original input variables.
- Had we used the idea on p. 193 for the reduction, the resulting formula may have an exponential length because of the copying.[a]

[a]Contributed by Mr. Ching-Hua Yu (D00921025) on October 16, 2012.

Composition of Reductions

Proposition 25 *If* R_{12} *is a reduction from* L_1 *to* L_2 *and* R_{23} *is a reduction from* L_2 *to* L_3*, then the composition* $R_{12} \circ R_{23}$ *is a reduction from* L_1 *to* L_3*.*

- So reducibility is transitive.

Completeness[a]

- As reducibility is transitive, problems can be ordered with respect to their difficulty.
- Is there a *maximal* element (the *hardest* problem)?
- It is not obvious that there should be a maximal element.
  - Many infinite structures (such as integers and real numbers) do not have maximal elements.
- Surprisingly, most of the complexity classes that we have seen so far have maximal elements!

[a]Cook (1971); Levin (1973); Post (1944).

Completeness (concluded)

- Let $\mathcal{C}$ be a complexity class and $L \in \mathcal{C}$.
- L is **$\mathcal{C}$-complete** if every $L' \in \mathcal{C}$ can be reduced to L.
  - Most of the complexity classes we have seen so far have complete problems!
- Complete problems capture the difficulty of a class because they are the hardest problems in the class.[a]

[a]See also p. 155.

Hardness

- Let $\mathcal{C}$ be a complexity class.
- L is **$\mathcal{C}$-hard** if every $L' \in \mathcal{C}$ can be reduced to L.
- It is not required that $L \in \mathcal{C}$.
- If L is $\mathcal{C}$-hard, then by definition, every $\mathcal{C}$-complete problem can be reduced to L.[a]

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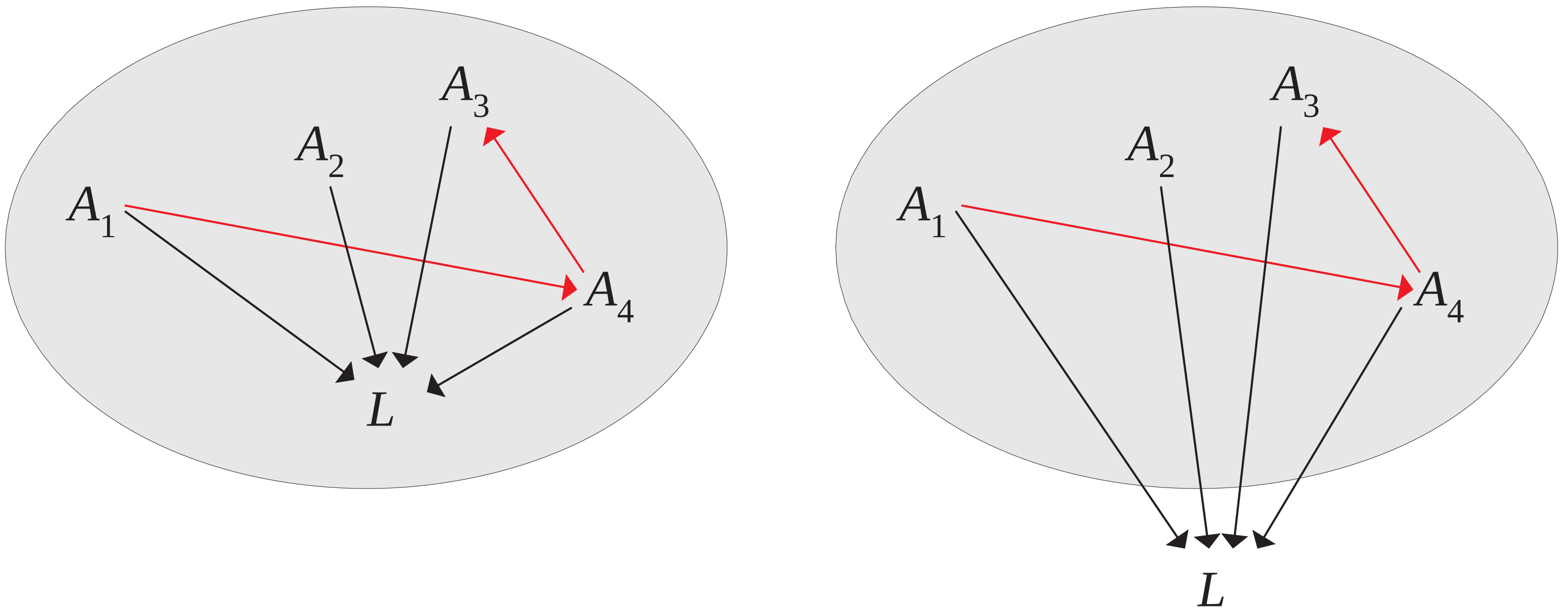
[a] Contributed by Mr. Ming-Feng Tsai (`D92922003`) on October 15, 2003.

Illustration of Completeness and Hardness



Closedness under Reductions

- A class $\mathcal{C}$ is **closed under reductions** if whenever L is reducible to L' and $L' \in \mathcal{C}$, then $L \in \mathcal{C}$.
- It is easy to show that P, NP, coNP, L, NL, PSPACE, and EXP are all closed under reductions.
- E is not closed under reductions.[a]

[a] Balcázar, Díaz, and Gabarró (1988).

Complete Problems and Complexity Classes

Proposition 26 *Let $\mathcal{C}'$ and $\mathcal{C}$ be two complexity classes such that $\mathcal{C}' \subseteq \mathcal{C}$. Assume $\mathcal{C}'$ is closed under reductions and L is $\mathcal{C}$-complete. Then $\mathcal{C} = \mathcal{C}'$ if and only if $L \in \mathcal{C}'$.*

- Suppose $L \in \mathcal{C}'$ first.
- Every language $A \in \mathcal{C}$ reduces to $L \in \mathcal{C}'$.
- Because $\mathcal{C}'$ is closed under reductions, $A \in \mathcal{C}'$.
- Hence $\mathcal{C} \subseteq \mathcal{C}'$.
- As $\mathcal{C}' \subseteq \mathcal{C}$, we conclude that $\mathcal{C} = \mathcal{C}'$.

The Proof (concluded)

- On the other hand, suppose $\mathcal{C} = \mathcal{C}'$.
- As L is $\mathcal{C}$-complete, $L \in \mathcal{C}$.
- Thus, trivially, $L \in \mathcal{C}'$.

Two Important Corollaries

Proposition 26 implies the following.

Corollary 27 *$P = NP$ if and only if an NP-complete problem in P.*

Corollary 28 *$L = P$ if and only if a P-complete problem is in L.*

Complete Problems and Complexity Classes, Again

Proposition 29 *Let $\mathcal{C}'$ and $\mathcal{C}$ be two complexity classes closed under reductions. If L is complete for both $\mathcal{C}$ and $\mathcal{C}'$, then $\mathcal{C} = \mathcal{C}'$.*

- All languages $A \in \mathcal{C}$ reduce to $L \in \mathcal{C}$ and $L \in \mathcal{C}'$.
- Since $\mathcal{C}'$ is closed under reductions, $A \in \mathcal{C}'$.
- Hence $\mathcal{C} \subseteq \mathcal{C}'$.
- The proof for $\mathcal{C}' \subseteq \mathcal{C}$ is symmetric.

Table of Computation

- Let $M = (K, \Sigma, \delta, s)$ be a single-string polynomial-time deterministic TM deciding L.
- Its computation on input x can be thought of as a $|x|^k \times |x|^k$ table, where $|x|^k$ is the time bound.
  - It is essentially a sequence of configurations.
- Rows correspond to time steps 0 to $|x|^k - 1$.
- Columns are positions in the string of M.
- The (i, j)th table entry represents the contents of position j of the string *after* i steps of computation.

Some Conventions To Simplify the Table

- M halts after at most $|x|^k - 2$ steps.
- Assume a large enough k to make it true for $|x| \geq 2$.
- Pad the table with $\sqcup$s so that each row has length $|x|^k$.
  - The computation will never reach the right end of the table for lack of time.
- If the cursor scans the jth position at time i when M is at state q and the symbol is σ, then the (i, j)th entry is a *new* symbol σ_q.

Some Conventions To Simplify the Table (continued)

- If q is "yes" or "no," simply use "yes" or "no" instead of σ_q.
- Modify M so that the cursor starts not at $\triangleright$ but at the first symbol of the input.
- The cursor never visits the leftmost $\triangleright$ by telescoping two moves of M each time the cursor is about to move to the leftmost $\triangleright$.
- So the first symbol in every row is a $\triangleright$ and not a $\triangleright_q$.

Some Conventions To Simplify the Table (concluded)

- Suppose M has halted before its time bound of $|x|^k$, so that "yes" or "no" appears at a row before the last.
- Then all subsequent rows will be identical to that row.
- M accepts x if and only if the $(|x|^k - 1, j)$th entry is "yes" for some position j.

Comments

- Each row is essentially a configuration.
- If the input $x = 010001$, then the first row is

$$\overbrace{\triangleright 0_{\mathrm{s}} 10001 \sqcup \sqcup \cdots \sqcup}^{|\, x\,|^k}$$

- A typical row looks like

$$\overbrace{\triangleright 10100_{\mathrm{q}} 01110100 \sqcup \sqcup \cdots \sqcup}^{|\, x\,|^k}$$

Comments (concluded)

- The last rows must look like

$$\rhd \overbrace{\cdots \text{"yes"} \cdots \sqcup}^{|x|^k} \quad \text{or} \quad \rhd \overbrace{\cdots \text{"no"} \cdots \sqcup}^{|x|^k}$$

- Three out of the table's 4 borders are known:

$$\begin{array}{cccccccc} \rhd & a & b & c & d & e & f & \sqcup \\ \rhd & & & & & & & \sqcup \\ \rhd & & & & & & & \sqcup \\ \rhd & & & & & & & \sqcup \\ \rhd & & & & & & & \sqcup \\ & & & & \vdots & & & \end{array}$$