Zero-Knowledge Proof of 3 Colorability[a]

```
for i = 1, 2, ..., |E|^2 do
    Peggy chooses a random permutation π of the 3-coloring φ;
    Peggy samples encryption schemes randomly, commits[b] them,
    and sends π(φ(1)), π(φ(2)), ..., π(φ(|V|)) encrypted to Victor;
    Victor chooses at random an edge e ∈ E and sends it to Peggy
    for the coloring of the endpoints of e;
    if e = (u, v) ∈ E then
        Peggy reveals the colors π(φ(u)) and π(φ(v)) and "proves"
        that they correspond to their encryptions;
    else
        Peggy stops;
    end if
```

[a] Goldreich, Micali, and Wigderson (1986).

[b] Contributed by Mr. Ren-Shuo Liu (D98922016) on December 22, 2009.

```
10:     if the “proof” provided in Line 6 is not valid then
11:         Victor rejects and stops;
12:     end if
13:     if π(φ(u)) = π(φ(v)) or π(φ(u)), π(φ(v)) ∉ {1, 2, 3} then
14:         Victor rejects and stops;
15:     end if
16: end for
17: Victor accepts;
```

Analysis

- If the graph is 3-colorable and both Peggy and Victor follow the protocol, then Victor always accepts.
- Suppose the graph is not 3-colorable and Victor follows the protocol.
- Let e be an edge that is *not* colored legally.
- Victor will pick it with probability $1/m$, where $m = |\, E\,|$.
- Then however Peggy plays, Victor will accept with probability $\leq 1 - (1/m)$ per round.

Analysis (concluded)

- So Victor will accept with probability at most

$$\left(1 - m^{-1}\right)^{m^2} \leq e^{-m}.$$

- Thus the protocol is valid.
- This protocol yields no knowledge to Victor as all he gets is a bunch of random pairs.
- The proof that the protocol is zero-knowledge to *any* verifier is intricate.

Comments

- Each $\pi(\phi(i))$ is encrypted by a different cryptosystem in Line 3.[a]
  - Otherwise, all the colors will be revealed in Line 6.
- Each edge e must be picked randomly.[b]
  - Otherwise, Peggy will know Victor's game plan and plot accordingly.

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[a] Contributed by Ms. Yui-Huei Chang (R96922060) on May 22, 2008

[b] Contributed by Mr. Chang-Rong Hung (R96922028) on May 22, 2008

Approximability

All science is dominated by
the idea of approximation.
— Bertrand Russell (1872–1970)

Just because the problem is NP-complete
does not mean that
you should not try to solve it.
— Stephen Cook (2002)

Tackling Intractable Problems

- Many important problems are NP-complete or worse.
- **Heuristics** have been developed to attack them.
- They are **approximation algorithms**.
- How good are the approximations?
  - We are looking for theoretically *guaranteed* bounds, not "empirical" bounds.
- Are there NP problems that cannot be approximated well (assuming NP $\neq$ P)?
- Are there NP problems that cannot be approximated at all (assuming NP $\neq$ P)?

Some Definitions

- Given an **optimization problem**, each problem instance x has a set of **feasible solutions** $F(x)$.
- Each feasible solution $s \in F(x)$ has a cost $c(s) \in \mathbb{Z}^+$.
  - Here, cost refers to the quality of the feasible solution, not the time required to obtain it.
  - It is our **objective function**, e.g., total distance, number of satisfied expressions, or cut size.

Some Definitions (concluded)

- The **optimum cost** is

$$\text{OPT}(x) = \min_{s \in F(x)} c(s)$$

  for a minimization problem.

- It is

$$\text{OPT}(x) = \max_{s \in F(x)} c(s)$$

  for a maximization problem.

Approximation Algorithms

- Let (polynomial-time) algorithm M on x returns a feasible solution.
- M is an ϵ**-approximation algorithm**, where $\epsilon \geq 0$, if for all x,

$$\frac{|c(M(x)) - \text{OPT}(x)|}{\max(\text{OPT}(x), c(M(x)))} \leq \epsilon.$$

  - For a minimization problem,

$$\frac{c(M(x)) - \min_{s \in F(x)} c(s)}{c(M(x))} \leq \epsilon.$$

  - For a maximization problem,

$$\frac{\max_{s \in F(x)} c(s) - c(M(x))}{\max_{s \in F(x)} c(s)} \leq \epsilon. \tag{17}$$

Lower and Upper Bounds

- For a minimization problem,

$$\min_{s \in F(x)} c(s) \leq c(M(x)) \leq \frac{\min_{s \in F(x)} c(s)}{1-\epsilon}.$$

- For a maximization problem,

$$(1-\epsilon) \times \max_{s \in F(x)} c(s) \leq c(M(x)) \leq \max_{s \in F(x)} c(s). \tag{18}$$

Range Bounds

- ϵ ranges between 0 (best) and 1 (worst).
- For minimization problems, an ϵ-approximation algorithm returns solutions within

$$\left[\text{OPT}, \frac{\text{OPT}}{1-\epsilon}\right].$$

- For maximization problems, an ϵ-approximation algorithm returns solutions within

$$[\,(1-\epsilon) \times \text{OPT}, \text{OPT}\,].$$

Approximation Thresholds

- For each NP-complete optimization problem, we shall be interested in determining the *smallest* ϵ for which there is a polynomial-time ϵ-approximation algorithm.
- But sometimes ϵ has no minimum value.
- The **approximation threshold** is the greatest lower bound of all $\epsilon \geq 0$ such that there is a polynomial-time ϵ-approximation algorithm.
- By a standard theorem in real analysis, such a threshold must exist.[a]

[a]Bauldry (2009).

Approximation Thresholds (concluded)

- The approximation threshold of an optimization problem can be anywhere between 0 (approximation to any desired degree) and 1 (no approximation is possible).
- If P = NP, then all optimization problems *in NP* have an approximation threshold of 0.
- So we assume P $\neq$ NP for the rest of the discussion.

Approximation Ratio

- ϵ-approximation algorithms can also be defined via **approximation ratio**:[a]

$$\frac{c(M(x))}{\text{OPT}(x)}.$$

- For a minimization problem, the approximation ratio is

$$1 \leq \frac{c(M(x))}{\min_{s \in F(x)} c(s)} \leq \frac{1}{1-\epsilon}. \tag{19}$$

- For a maximization problem, the approximation ratio is

$$1 - \epsilon \leq \frac{c(M(x))}{\max_{s \in F(x)} c(s)} \leq 1.$$

[a] Williamson and Shmoys (2011).

NODE COVER

- NODE COVER seeks the smallest $C \subseteq V$ in graph $G = (V, E)$ such that for each edge in E, at least one of its endpoints is in C.
- A heuristic to obtain a good node cover is to iteratively move a node with the *highest degree* to the cover.
- This turns out to produce an approximation ratio of[a]

$$\frac{c(M(x))}{\text{OPT}(x)} = \Theta(\log n).$$

- So it is not an ϵ-approximation algorithm for any constant $\epsilon < 1$ according to Eq. (19).

[a] Chvátal (1979).

A 0.5-Approximation Algorithm[a]

$C := \emptyset$;
while $E \neq \emptyset$ **do**
    Delete an arbitrary edge $\{ u, v \}$ from E;
    Add u and v to C; {Add 2 nodes to C each time.}
    Delete edges incident with u or v from E;
end while
return C;

---

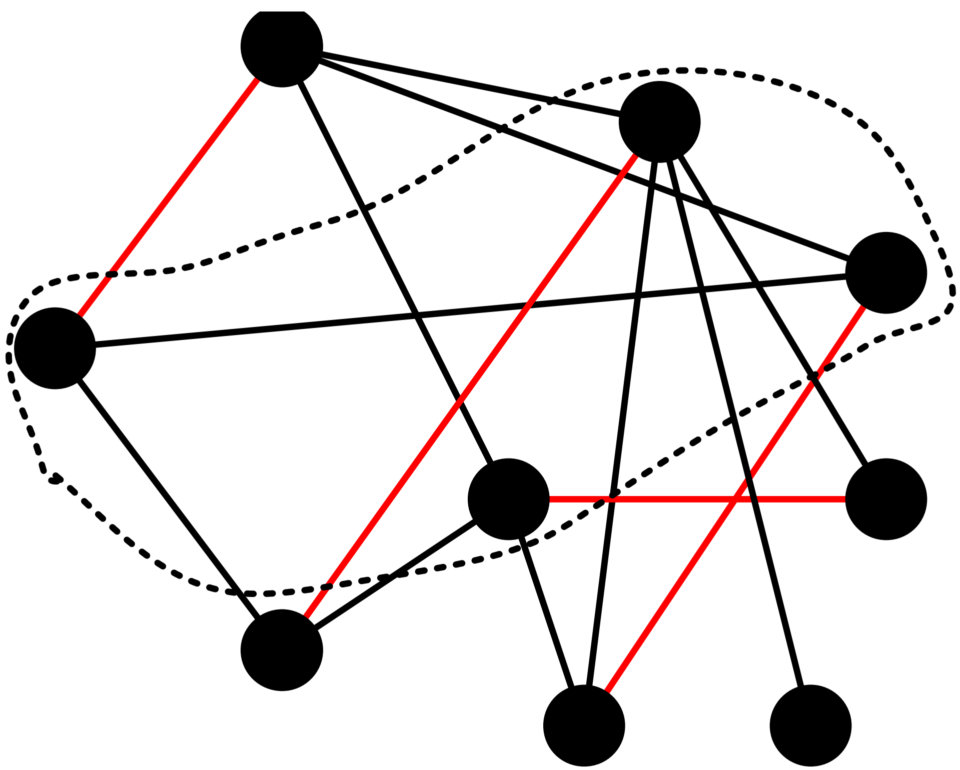
[a] Johnson (1974).

Analysis

- It is easy to see that C is a node cover.
- C contains $|C|/2$ edges.[a]
- No two edges of C share a node.[b]
- *Any* node cover must contain at least one node from each of these edges.
  - If there is an edge in C both of whose ends are outside the cover, then that cover will not be a valid cover.

[a] The edges deleted in Line 3.

[b] In fact, C as a set of edges is a *maximal* matching.



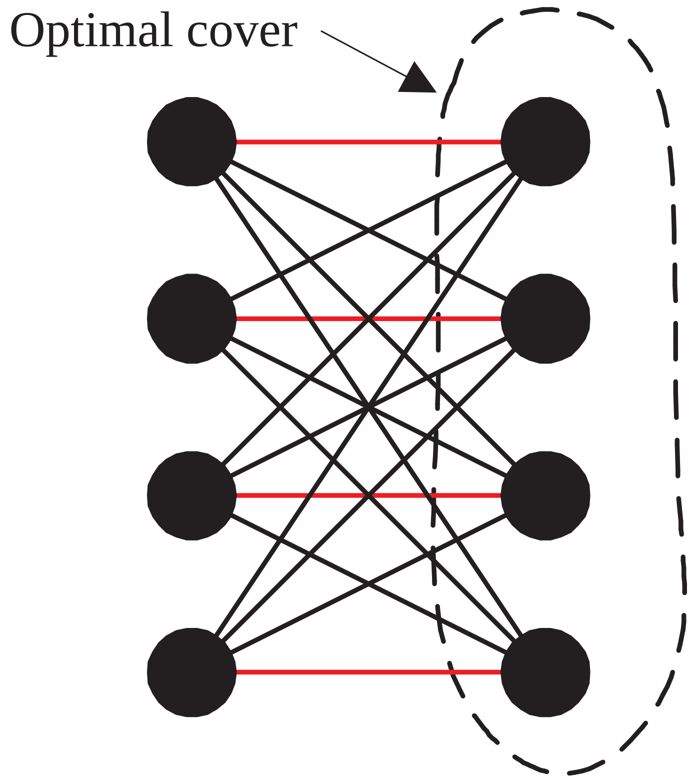
Analysis (concluded)

- This means that $\text{OPT}(G) \geq |C|/2$.
- So the approximation ratio

$$\frac{|C|}{\text{OPT}(G)} \leq 2.$$

- So we have a 0.5-approximation algorithm.
- The approximation threshold is therefore ≤ 0.5.

The 0.5 Bound Is Tight for the Algorithm[a]



[a]Contributed by Mr. Jenq-Chung Li (`R92922087`) on December 20, 2003. Recall that König's theorem says the size of a maximum matching equals that of a minimum node cover in a bipartite graph.

Remarks

- The approximation threshold is at least[a]

$$1 - \left(10\sqrt{5} - 21\right)^{-1} \approx 0.2651.$$

- The approximation threshold is 0.5 if one assumes the unique games conjecture.[b]
- This ratio 0.5 is also the lower bound for any "greedy" algorithms.[c]

[a] Dinur and Safra (2002).
[b] Khot and Regev (2008).
[c] Davis and Impagliazzo (2004).

Maximum Satisfiability

- Given a set of clauses, MAXSAT seeks the truth assignment that satisfies the most.
- MAX2SAT is already NP-complete (p. 345), so MAXSAT is NP-complete.
- Consider the more general k-MAXGSAT for constant k.
  - Let $\Phi = \{\phi_1, \phi_2, \ldots, \phi_m\}$ be a set of boolean expressions in n variables.
  - Each ϕ_i is a *general* expression involving k variables.
  - k-MAXGSAT seeks the truth assignment that satisfies the most expressions.

A Probabilistic Interpretation of an Algorithm

- Each ϕ_i involves exactly k variables and is satisfied by s_i of the 2^k truth assignments.
- A random truth assignment $\in \{0,1\}^n$ satisfies ϕ_i with probability $p(\phi_i) = s_i/2^k$.
  - $p(\phi_i)$ is easy to calculate as k is a constant.
- Hence a random truth assignment satisfies an average of

$$p(\Phi) = \sum_{i=1}^{m} p(\phi_i)$$

expressions ϕ_i.

The Search Procedure

- Clearly

$$p(\Phi) = \frac{1}{2}\,\{\, p(\Phi[\, x_1 = \texttt{true}\,]) + p(\Phi[\, x_1 = \texttt{false}\,])\,\}.$$

- Select the $t_1 \in \{\texttt{true}, \texttt{false}\}$ such that $p(\Phi[\, x_1 = t_1\,])$ is the larger one.
- Note that $p(\Phi[\, x_1 = t_1\,]) \geq p(\Phi)$.
- Repeat the procedure with expression $\Phi[\, x_1 = t_1\,]$ until all variables x_i have been given truth values t_i and all ϕ_i are either true or false.

The Search Procedure (continued)

- By our hill-climbing procedure,

$$\begin{aligned} & p(\Phi) \\ \leq \; & p(\Phi[\, x_1 = t_1 \,]) \\ \leq \; & p(\Phi[\, x_1 = t_1, x_2 = t_2 \,]) \\ \leq \; & \cdots \\ \leq \; & p(\Phi[\, x_1 = t_1, x_2 = t_2, \ldots, x_n = t_n \,]). \end{aligned}$$

- So at least $p(\Phi)$ expressions are satisfied by truth assignment $(t_1, t_2, \ldots, t_n)$.

The Search Procedure (concluded)

- Note that the algorithm is *deterministic*!
- It is called **the method of conditional expectations**.[a]

[a] Erdős and Selfridge (1973); Spencer (1987).

Approximation Analysis

- The optimum is at most the number of satisfiable ϕ_i—i.e., those with $p(\phi_i) > 0$.
- Hence the ratio of algorithm's output vs. the optimum is[a]

$$\geq \frac{p(\Phi)}{\sum_{p(\phi_i)>0} 1} = \frac{\sum_i p(\phi_i)}{\sum_{p(\phi_i)>0} 1} \geq \min_{p(\phi_i)>0} p(\phi_i).$$

- So this is a polynomial-time ϵ-approximation algorithm with $\epsilon = 1 - \min_{p(\phi_i)>0} p(\phi_i)$.
- Because $p(\phi_i) \geq 2^{-k}$, the heuristic is a polynomial-time ϵ-approximation algorithm with $\epsilon = 1 - 2^{-k}$.

[a]Recall that $(\sum_i a_i)/(\sum_i b_i) \geq \min_i a_i/b_i$.

Back to MAXSAT

- In MAXSAT, the ϕ_i's are clauses (like $x \vee y \vee \neg z$).
- Hence $p(\phi_i) \geq 1/2$, which happens when ϕ_i contains a single literal.
- And the heuristic becomes a polynomial-time ϵ-approximation algorithm with $\epsilon = 1/2$.[a]
  - Suppose we set each boolean variable to true with probability $(\sqrt{5} - 1)/2$, the golden ratio.
  - Then follow through the method of conditional expectations to derandomize it.
  - We will obtain a $[\,(3 - \sqrt{5}\,)\,]/2$-approximation algorithm, where $[\,(3 - \sqrt{5}\,)\,]/2 \approx 0.382$.[b]

[a] Johnson (1974).
[b] Lieberherr and Specker (1981).

Back to MAXSAT (concluded)

- If the clauses have k *distinct* literals,

$$p(\phi_i) = 1 - 2^{-k}.$$

- And the heuristic becomes a polynomial-time ϵ-approximation algorithm with $\epsilon = 2^{-k}$.
  - This is the best possible for $k \geq 3$ unless P = NP.

MAX CUT Revisited

- MAX CUT seeks to partition the nodes of graph $G = (V, E)$ into $(S, V - S)$ so that there are as many edges as possible between S and $V - S$.
- It is NP-complete.[a]
- **Local search** starts from a feasible solution and performs "local" improvements until none are possible.
- Next we present a local-search algorithm for MAX CUT.

[a]Recall p. 375.

A 0.5-Approximation Algorithm for MAX CUT

$S := \emptyset$;
while $\exists v \in V$ whose switching sides results in a larger cut **do**
    Switch the side of v;
end while
return S;

- A 0.12-approximation algorithm exists.[a]
- 0.059-approximation algorithms do not exist unless NP = ZPP.

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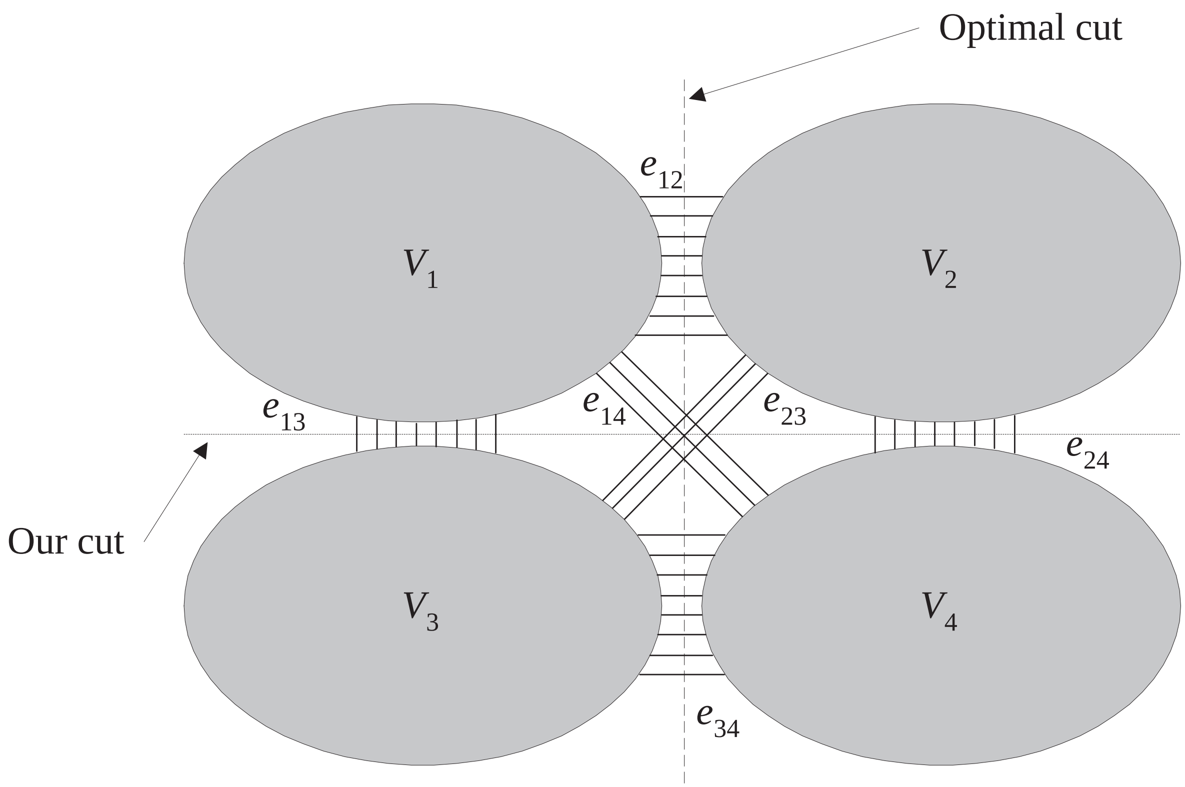
[a] Goemans and Williamson (1995).

Analysis



Analysis (continued)

- Partition $V = V_1 \cup V_2 \cup V_3 \cup V_4$, where
  - Our algorithm returns $(V_1 \cup V_2, V_3 \cup V_4)$.
  - The optimum cut is $(V_1 \cup V_3, V_2 \cup V_4)$.
- Let e_{ij} be the number of edges between V_i and V_j.
- Our algorithm returns a cut of size

$$e_{13} + e_{14} + e_{23} + e_{24}.$$

- The optimum cut size is

$$e_{12} + e_{34} + e_{14} + e_{23}.$$

Analysis (continued)

- For each node $v \in V_1$, its edges to $V_1 \cup V_2$ are outnumbered by those to $V_3 \cup V_4$.
  - Otherwise, v would have been moved to $V_3 \cup V_4$ to improve the cut.
- Considering all nodes in V_1 together, we have

$$2e_{11} + e_{12} \leq e_{13} + e_{14}.$$

  - It is $2e_{11}$ is because each edge in V_1 is counted twice.
- The above inequality implies

$$e_{12} \leq e_{13} + e_{14}.$$

Analysis (concluded)

- Similarly,

$$\begin{aligned} e_{12} &\leq e_{23} + e_{24} \\ e_{34} &\leq e_{23} + e_{13} \\ e_{34} &\leq e_{14} + e_{24} \end{aligned}$$

- Add all four inequalities, divide both sides by 2, and add the inequality $e_{14} + e_{23} \leq e_{14} + e_{23} + e_{13} + e_{24}$ to obtain

$$e_{12} + e_{34} + e_{14} + e_{23} \leq 2(e_{13} + e_{14} + e_{23} + e_{24}).$$

- The above says our solution is at least half the optimum.

Approximability, Unapproximability, and Between

- KNAPSACK, NODE COVER, MAXSAT, and MAX CUT have approximation thresholds less than 1.
  - KNAPSACK has a threshold of 0 (p. 736).
  - But NODE COVER (p. 714) and MAXSAT have a threshold larger than 0.
- The situation is maximally pessimistic for TSP, which cannot be approximated (p. 734).
  - The approximation threshold of TSP is 1.
    * The threshold is 1/3 if TSP satisfies the triangular inequality.
  - The same holds for INDEPENDENT SET (see the textbook).

Unapproximability of TSP[a]

Theorem 78 *The approximation threshold of* TSP *is 1 unless $P = NP$.*

- Suppose there is a polynomial-time ϵ-approximation algorithm for TSP for some $\epsilon < 1$.
- We shall construct a polynomial-time algorithm to solve the NP-complete HAMILTONIAN CYCLE.
- Given any graph $G = (V, E)$, construct a TSP with $|V|$ cities with distances

$$d_{ij} = \begin{cases} 1, & \text{if } \{i, j\} \in E \\ \frac{|V|}{1-\epsilon}, & \text{otherwise} \end{cases}$$

[a]Sahni and Gonzales (1976).

The Proof (concluded)

- Run the alleged approximation algorithm on this TSP.
- Suppose a tour of cost $|V|$ is returned.
  - This tour must be a Hamiltonian cycle.
- Suppose a tour that includes an edge of length $\frac{|V|}{1-\epsilon}$ is returned.
  - The total length of this tour is $> \frac{|V|}{1-\epsilon}$.
  - Because the algorithm is ϵ-approximate, the optimum is at least $1 - \epsilon$ times the returned tour's length.
  - The optimum tour has a cost exceeding $|V|$.
  - Hence G has no Hamiltonian cycles.

KNAPSACK Has an Approximation Threshold of Zero[a]

Theorem 79 *For any ϵ, there is a polynomial-time ϵ-approximation algorithm for* KNAPSACK.

- We have n weights $w_1, w_2, \ldots, w_n \in \mathbb{Z}^+$, a weight limit W, and n values $v_1, v_2, \ldots, v_n \in \mathbb{Z}^+$.[b]
- We must find an $S \subseteq \{1, 2, \ldots, n\}$ such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i$ is the largest possible.

---

[a]Ibarra and Kim (1975).

[b]If the values are fractional, the result is slightly messier, but the main conclusion remains correct. Contributed by Mr. Jr-Ben Tian (`B89902011`, `R93922045`) on December 29, 2004.

The Proof (continued)

- Let
$$V = \max\{v_1, v_2, \ldots, v_n\}.$$
- Clearly, $\sum_{i \in S} v_i \leq nV$.
- Let $0 \leq i \leq n$ and $0 \leq v \leq nV$.
- $W(i, v)$ is the minimum weight attainable by selecting only from the first i items and with a total value of v.
  - It is an $(n+1) \times (nV+1)$ table.
- Set $W(0, v) = \infty$ for $v \in \{1, 2, \ldots, nV\}$ and $W(i, 0) = 0$ for $i = 0, 1, \ldots, n$.[a]

[a] Contributed by Mr. Ren-Shuo Liu (`D98922016`) and Mr. Yen-Wei Wu (`D98922013`) on December 28, 2009.

The Proof (continued)

- Then, for $0 \leq i < n$,

$$W(i+1, v) = \min\{W(i, v), W(i, v - v_{i+1}) + w_{i+1}\}.$$

- Finally, pick the largest v such that $W(n, v) \leq W$.[a]
- The running time is $O(n^2 V)$, not polynomial time.
- Key idea: Limit the number of precision bits.

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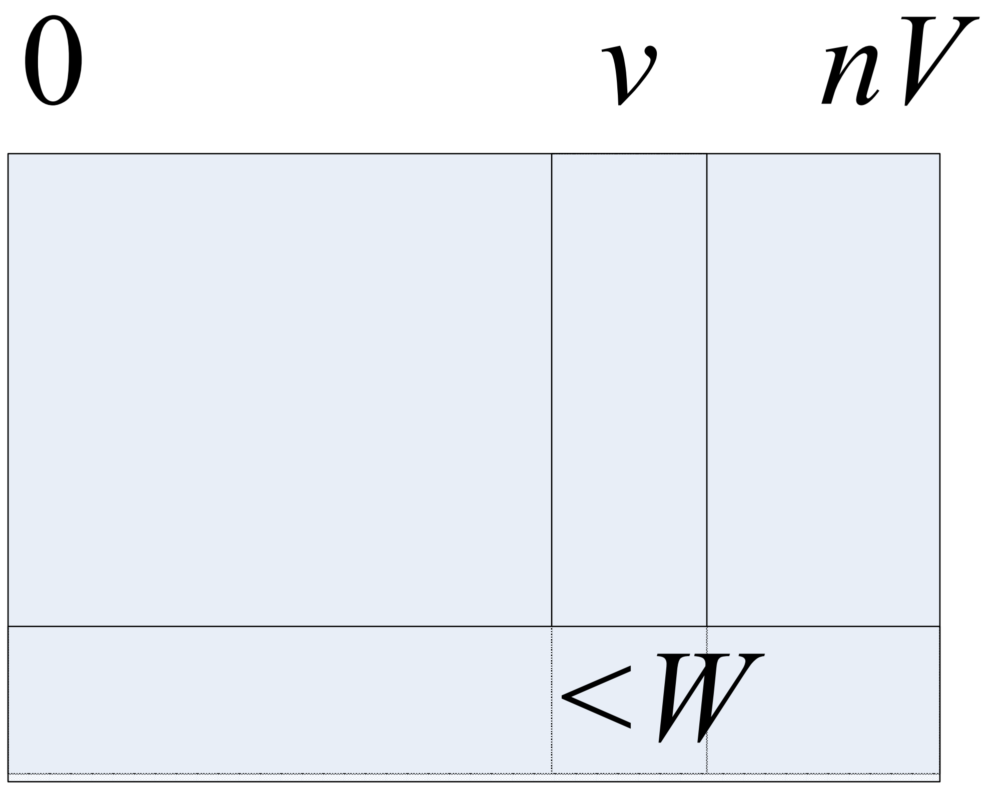
[a] Lawler (1979).


0
v
nV
<W

The Proof (continued)

- Define
$$v_i' = 2^b \left\lfloor \frac{v_i}{2^b} \right\rfloor.$$
  - This is equivalent to zeroing each v_i's last b bits.
- Call the original instance
$$x = (w_1, \ldots, w_n, W, v_1, \ldots, v_n).$$
- Call the approximate instance
$$x' = (w_1, \ldots, w_n, W, v_1', \ldots, v_n').$$

The Proof (continued)

- Solving x' takes time $O(n^2 V/2^b)$.
  - The algorithm only performs subtractions on the v_i-related values.
  - So the b last bits can be *removed* from the calculations.
  - That is, use $v_i'' = \left\lfloor \frac{v_i}{2^b} \right\rfloor$ and $V = \max(v_1'', v_2'', \ldots, v_n'')$ in the calculations.
  - Then multiply the returned value by 2^b.
  - It is an $(n+1) \times (nV+1)/2^b$ table.

The Proof (continued)

- The solution S' is close to the optimum solution S:

$$\sum_{i \in S'} v_i \geq \sum_{i \in S'} v_i' \geq \sum_{i \in S} v_i' \geq \sum_{i \in S} (v_i - 2^b) \geq \sum_{i \in S} v_i - n2^b.$$

- Hence

$$\sum_{i \in S'} v_i \geq \sum_{i \in S} v_i - n2^b.$$

- Without loss of generality, assume $w_i \leq W$ for all i.
  - Otherwise, item i is redundant.
- V is a lower bound on OPT.
  - Picking an item with value V is a legitimate choice.

The Proof (concluded)

- The relative error from the optimum is:

$$\frac{\sum_{i\in S} v_i - \sum_{i\in S'} v_i}{\sum_{i\in S} v_i} \leq \frac{\sum_{i\in S} v_i - \sum_{i\in S'} v_i}{V} \leq \frac{n2^b}{V}.$$

- Suppose we pick $b = \lfloor \log_2 \frac{\epsilon V}{n} \rfloor$.
- The algorithm becomes ϵ-approximate.[a]
- The running time is then $O(n^2 V/2^b) = O(n^3/\epsilon)$, a polynomial in n and $1/\epsilon$.[b]

---

[a]See Eq. (17) on p. 706.

[b]It hence depends on the *value* of $1/\epsilon$. Thanks to a lively class discussion on December 20, 2006. If we fix ϵ and let the problem size increase, then the complexity is cubic. Contributed by Mr. Ren-Shan Luoh (`D97922014`) on December 23, 2008.

Comments

- INDEPENDENT SET and NODE COVER are reducible to each other (Corollary 41, p. 368).
- NODE COVER has an approximation threshold at most 0.5 (p. 714).
- But INDEPENDENT SET is unapproximable (see the textbook).
- INDEPENDENT SET limited to graphs with degree $\leq k$ is called k-DEGREE INDEPENDENT SET.
- k-DEGREE INDEPENDENT SET is approximable (see the textbook).