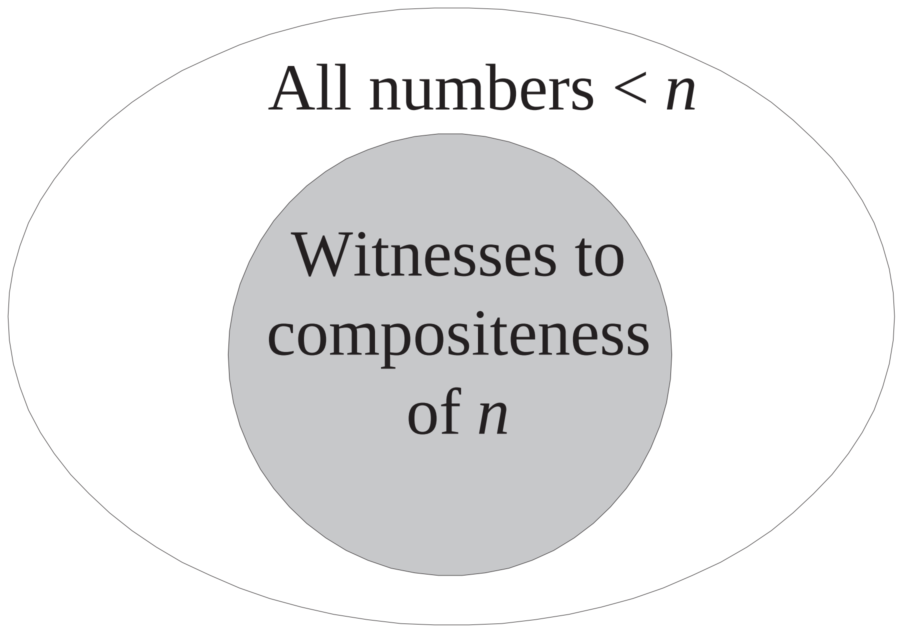
The Density Attack for PRIMES



The Density Attack for PRIMES

```
Pick k ∈ {1, ..., n} randomly;
if k | n and k ≠ 1 and k ≠ n then
    return "n is composite";
else
    return "n is (probably) a prime";
end if
```

The Density Attack for PRIMES (continued)

- It works, but does it work well?
- The ratio of numbers $\leq n$ relatively prime to n (the white ring) is
$$\frac{\phi(n)}{n}.$$
- When $n = pq$, where p and q are distinct primes,
$$\frac{\phi(n)}{n} = \frac{pq - p - q + 1}{pq} > 1 - \frac{1}{q} - \frac{1}{p}.$$

The Density Attack for PRIMES (concluded)

- So the ratio of numbers $\leq n$ *not* relatively prime to n (the grey area) is $< (1/q) + (1/p)$.
  - The "density attack" has probability about $2/\sqrt{n}$ of factoring $n = pq$ when $p \sim q = O(\sqrt{n}\,)$.
  - The "density attack" to factor $n = pq$ hence takes $\Omega(\sqrt{n})$ steps on average when $p \sim q = O(\sqrt{n}\,)$.
  - This running time is exponential: $\Omega(2^{0.5 \log_2 n})$.

The Chinese Remainder Theorem

- Let $n = n_1 n_2 \cdots n_k$, where n_i are pairwise relatively prime.
- For any integers $a_1, a_2, \ldots, a_k$, the set of simultaneous equations

$$\begin{aligned} x &= a_1 \bmod n_1, \\ x &= a_2 \bmod n_2, \\ &\vdots \\ x &= a_k \bmod n_k, \end{aligned}$$

has a unique solution modulo n for the unknown x.

Fermat's "Little" Theorem[a]

Lemma 56 *For all* $0 < a < p$, $a^{p-1} = 1 \bmod p$.

- Recall $\Phi(p) = \{1, 2, \ldots, p-1\}$.
- Consider $a\Phi(p) = \{am \bmod p : m \in \Phi(p)\}$.
- $a\Phi(p) = \Phi(p)$.
  - $a\Phi(p) \subseteq \Phi(p)$ as a remainder must be between 1 and $p - 1$.
  - Suppose $am \equiv am' \bmod p$ for $m > m'$, where $m, m' \in \Phi(p)$.
  - That means $a(m - m') = 0 \bmod p$, and p divides a or $m - m'$, which is impossible.

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[a] Pierre de Fermat (1601–1665).

The Proof (concluded)

- Multiply all the numbers in $\Phi(p)$ to yield $(p-1)!$.
- Multiply all the numbers in $a\Phi(p)$ to yield $a^{p-1}(p-1)!$.
- As $a\Phi(p) = \Phi(p)$, we have

$$a^{p-1}(p-1)! \equiv (p-1)! \bmod p.$$

- Finally, $a^{p-1} = 1 \bmod p$ because $p \not| (p-1)!$.

The Fermat-Euler Theorem[a]

Corollary 57 *For all $a \in \Phi(n)$, $a^{\phi(n)} = 1 \bmod n$.*

- The proof is similar to that of Lemma 56 (p. 473).
- Consider $a\Phi(n) = \{am \bmod n : m \in \Phi(n)\}$.
- $a\Phi(n) = \Phi(n)$.
  - $a\Phi(n) \subseteq \Phi(n)$ as a remainder must be between 0 and $n-1$ and relatively prime to n.
  - Suppose $am \equiv am' \bmod n$ for $m' < m < n$, where $m, m' \in \Phi(n)$.
  - That means $a(m-m') = 0 \bmod n$, and n divides a or $m - m'$, which is impossible.

[a] Proof by Mr. Wei-Cheng Cheng (`R93922108`, `D95922011`) on November 24, 2004.

The Proof (concluded)[a]

- Multiply all the numbers in $\Phi(n)$ to yield $\prod_{m \in \Phi(n)} m$.
- Multiply all the numbers in $a\Phi(n)$ to yield $a^{\phi(n)} \prod_{m \in \Phi(n)} m$.
- As $a\Phi(n) = \Phi(n)$,

$$\prod_{m \in \Phi(n)} m \equiv a^{\phi(n)} \left(\prod_{m \in \Phi(n)} m \right) \bmod n.$$

- Finally, $a^{\phi(n)} = 1 \bmod n$ because $n \not| \prod_{m \in \Phi(n)} m$.

[a]Some typographical errors corrected by Mr. Jung-Ying Chen (`D95723006`) on November 18, 2008.

An Example

- As $12 = 2^2 \times 3$,

$$\phi(12) = 12 \times \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = 4.$$

- In fact, $\Phi(12) = \{1, 5, 7, 11\}$.
- For example,

$$5^4 = 625 = 1 \bmod 12.$$

Exponents

- The **exponent** of $m \in \Phi(p)$ is the least $k \in \mathbb{Z}^+$ such that
$$m^k = 1 \bmod p.$$
- Every residue $s \in \Phi(p)$ has an exponent.
  - $1, s, s^2, s^3, \ldots$ eventually repeats itself modulo p, say $s^i \equiv s^j \bmod p$, which means $s^{j-i} = 1 \bmod p$.
- If the exponent of m is k and $m^\ell = 1 \bmod p$, then $k | \ell$.
  - Otherwise, $\ell = qk + a$ for $0 < a < k$, and $m^\ell \equiv m^{qk+a} \equiv m^a \equiv 1 \bmod p$, a contradiction.

Lemma 58 *Any nonzero polynomial of degree k has at most k distinct roots modulo p.*

Exponents and Primitive Roots

- From Fermat's "little" theorem, all exponents divide $p-1$.
- A primitive root of p is thus a number with exponent $p-1$.
- Let $R(k)$ denote the total number of residues in $\Phi(p) = \{1, 2, \ldots, p-1\}$ that have exponent k.
- We already knew that $R(k) = 0$ for $k \not| (p-1)$.
- So
$$\sum_{k \mid (p-1)} R(k) = p - 1$$
as every number has an exponent.

Size of $R(k)$

- Any $a \in \Phi(p)$ of exponent k satisfies
$$x^k = 1 \bmod p.$$
- Hence there are at most k residues of exponent k, i.e., $R(k) \leq k$, by Lemma 58 (p. 478).
- Let s be a residue of exponent k.
- $1, s, s^2, \ldots, s^{k-1}$ are distinct modulo p.
  - Otherwise, $s^i \equiv s^j \bmod p$ with $i < j$.
  - Then $s^{j-i} = 1 \bmod p$ with $j - i < k$, a contradiction.
- As all these k distinct numbers satisfy $x^k = 1 \bmod p$, they comprise *all* the solutions of $x^k = 1 \bmod p$.

Size of $R(k)$ (continued)

- But do all of them have exponent k (i.e., $R(k) = k$)?
- And if not (i.e., $R(k) < k$), how many of them do?
- Pick s^ℓ, where $\ell < k$.
- Suppose $\ell \notin \Phi(k)$ with $\gcd(\ell, k) = d > 1$.
- Then
$$(s^\ell)^{k/d} = (s^k)^{\ell/d} = 1 \bmod p.$$
- Therefore, s^ℓ has exponent at most $k/d < k$.
- So s^ℓ has exponent k *only if* $\ell \in \Phi(k)$.
- We conclude that
$$R(k) \leq \phi(k).$$

Size of $R(k)$ (concluded)

- Because all $p-1$ residues have an exponent,

$$p-1=\sum_{k|(p-1)} R(k) \leq \sum_{k|(p-1)} \phi(k) = p-1$$

  by Lemma 55 (p. 465).

- Hence

$$R(k)=\begin{cases} \phi(k) & \text{when } k|(p-1) \\ 0 & \text{otherwise} \end{cases}$$

- In particular, $R(p-1)=\phi(p-1)>0$, and p has at least one primitive root.
- This proves one direction of Theorem 50 (p. 451).

A Few Calculations

- Let $p = 13$.
- From p. 475, we know $\phi(p-1) = 4$.
- Hence $R(12) = 4$.
- Indeed, there are 4 primitive roots of p.
- As
$$\Phi(p-1) = \{1, 5, 7, 11\},$$
the primitive roots are
$$g^1, g^5, g^7, g^{11},$$
where g is any primitive root.

The Other Direction of Theorem 50 (p. 451)

- We show p is a prime if there is a number r such that
  1. $r^{p-1} = 1 \bmod p$, and
  2. $r^{(p-1)/q} \neq 1 \bmod p$ for all prime divisors q of $p - 1$.
- Suppose p is not a prime.
- We proceed to show that no primitive roots exist.
- Suppose $r^{p-1} = 1 \bmod p$ (note $\gcd(r, p) = 1$).
- We will show that the 2nd condition must be violated.

The Proof (continued)

- So we proceed to show $r^{(p-1)/q} = 1 \bmod p$ for some prime divisor q of $p - 1$.
- $r^{\phi(p)} = 1 \bmod p$ by the Fermat-Euler theorem (p. 475).
- Because p is not a prime, $\phi(p) < p - 1$.
- Let k be the smallest integer such that $r^k = 1 \bmod p$.
- With the 1st condition, it is easy to show that $k \,|\, (p - 1)$ (similar to p. 478).
- Note that $k \,|\, \phi(p)$ (p. 478).
- As $k \leq \phi(p)$, $k < p - 1$.

The Proof (concluded)

- Let q be a prime divisor of $(p-1)/k > 1$.
- Then $k|(p-1)/q$.
- By the definition of k,

$$r^{(p-1)/q} = 1 \bmod p.$$

- But this violates the 2nd condition.

Function Problems

- Decision problems are yes/no problems (SAT, TSP (D), etc.).
- **Function problems** require a solution (a satisfying truth assignment, a best TSP tour, etc.).
- Optimization problems are clearly function problems.
- What is the relation between function and decision problems?
- Which one is harder?

Function Problems Cannot Be Easier than Decision Problems

- If we know how to generate a solution, we can solve the corresponding decision problem.
  - If you can find a satisfying truth assignment efficiently, then SAT is in P.
  - If you can find the best TSP tour efficiently, then TSP (D) is in P.
- But decision problems can be as hard as the corresponding function problems.

FSAT

- FSAT is this function problem:
  - Let $\phi(x_1, x_2, \ldots, x_n)$ be a boolean expression.
  - If ϕ is satisfiable, then return a satisfying truth assignment.
  - Otherwise, return "no."
- We next show that if SAT $\in$ P, then FSAT has a polynomial-time algorithm.
- SAT is a subroutine (black box) that returns "yes" or "no" on the satisfiability of the input.

An Algorithm for FSAT Using SAT

```
t := ϵ; {Truth assignment.}
if ϕ ∈ SAT then
    for i = 1, 2, ..., n do
        if ϕ[ x_i = true ] ∈ SAT then
            t := t ∪ { x_i = true };
            ϕ := ϕ[ x_i = true ];
        else
            t := t ∪ { x_i = false };
            ϕ := ϕ[ x_i = false ];
        end if
    end for
    return t;
else
    return "no";
end if
```

Analysis

- If SAT can be solved in polynomial time, so can FSAT.
  - There are $\leq n+1$ calls to the algorithm for SAT.[a]
  - Boolean expressions shorter than ϕ are used in each call to the algorithm for SAT.
- Hence SAT and FSAT are equally hard (or easy).
- Note that this reduction from FSAT to SAT is not a Karp reduction (recall p. 265).
- Instead, it calls SAT multiple times as a subroutine and moves on SAT's outputs.

[a] Contributed by Ms. Eva Ou (R93922132) on November 24, 2004.

TSP and TSP (D) Revisited

- We are given n cities $1, 2, \ldots, n$ and integer distances $d_{ij} = d_{ji}$ between any two cities i and j.
- TSP (D) asks if there is a tour with a total distance at most B.
- TSP asks for a tour with the shortest total distance.
  - The shortest total distance is at most $\sum_{i,j} d_{ij}$.
    * Recall that the input string contains $d_{11}, \ldots, d_{nn}$.
    * Thus the shortest total distance is less than $2^{|\, x\, |}$ in magnitude, where x is the input (why?).
- We next show that if TSP (D) $\in$ P, then TSP has a polynomial-time algorithm.

An Algorithm for TSP Using TSP (D)

Perform a binary search over interval $[0, 2^{|x|}]$ by calling TSP (D) to obtain the shortest distance, C;

for $i, j = 1, 2, \ldots, n$ **do**

  Call TSP (D) with $B = C$ and $d_{ij} = C + 1$;

  if "no" **then**

    Restore d_{ij} to old value; {Edge $[i, j]$ is critical.}

  end if

end for

return the tour with edges whose $d_{ij} \leq C$;

Analysis

- An edge that is not on *any* optimal tour will be eliminated, with its d_{ij} set to $C + 1$.
- In fact, an edge which is not on *all* remaining optimal tours will also be eliminated.
- So the algorithm ends with n edges which are not eliminated (why?).
- This is true even if there are multiple optimal tours![a]

[a]Thanks to a lively class discussion on November 12, 2013.

Analysis (concluded)

- There are $O(|x| + n^2)$ calls to the algorithm for TSP (D).
- Each call has an input length of $O(|x|)$.
- So if TSP (D) can be solved in polynomial time, so can TSP.
- Hence TSP (D) and TSP are equally hard (or easy).

Randomized Computation

I know that half my advertising works,
I just don't know which half.
— John Wanamaker

I know that half my advertising is
a waste of money,
I just don't know which half!
— McGraw-Hill ad.

Randomized Algorithms[a]

- Randomized algorithms flip unbiased coins.
- There are important problems for which there are no known efficient *deterministic* algorithms but for which very efficient randomized algorithms exist.
  - Extraction of square roots, for instance.
- There are problems where randomization is *necessary*.
  - Secure protocols.
- Randomized version can be more efficient.
  - Parallel algorithm for maximal independent set.[b]

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[a] Rabin (1976); Solovay and Strassen (1977).

[b] "Maximal" (a local maximum) not "maximum" (a global maximum).

"Four Most Important Randomized Algorithms"[a]

1. Primality testing.[b]
2. Graph connectivity using random walks.[c]
3. Polynomial identity testing.[d]
4. Algorithms for approximate counting.[e]

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[a] Trevisan (2006).

[b] Rabin (1976); Solovay and Strassen (1977).

[c] Aleliunas, Karp, Lipton, Lovász, and Rackoff (1979).

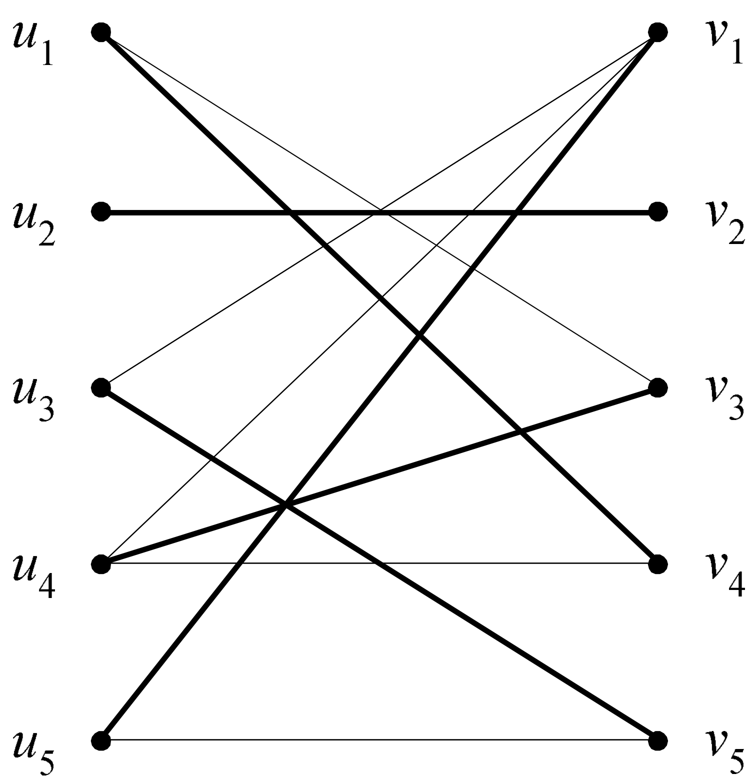
[d] Schwartz (1980); Zippel (1979).

[e] Sinclair and Jerrum (1989).

Bipartite Perfect Matching

- We are given a **bipartite graph** $G = (U, V, E)$.
  - $U = \{u_1, u_2, \ldots, u_n\}$.
  - $V = \{v_1, v_2, \ldots, v_n\}$.
  - $E \subseteq U \times V$.
- We are asked if there is a **perfect matching**.
  - A permutation π of $\{1, 2, \ldots, n\}$ such that
$$(u_i, v_{\pi(i)}) \in E$$
    for all $i \in \{1, 2, \ldots, n\}$.
- A perfect matching contains n edges.

A Perfect Matching in a Bipartite Graph



Symbolic Determinants

- We are given a bipartite graph G.
- Construct the $n \times n$ matrix A^G whose (i, j)th entry A^G_{ij} is a symbolic variable x_{ij} if $(u_i, v_j) \in E$ and 0 otherwise:

$$A^G_{ij} = \begin{cases} x_{ij}, & \text{if } (u_i, v_j) \in E, \\ 0, & \text{othersie.} \end{cases}$$

Symbolic Determinants (continued)

- The matrix for the bipartite graph G on p. 501 is[a]

$$A^G = \begin{bmatrix} 0 & 0 & x_{13} & x_{14} & 0 \\ 0 & x_{22} & 0 & 0 & 0 \\ x_{31} & 0 & 0 & 0 & x_{35} \\ x_{41} & 0 & x_{43} & x_{44} & 0 \\ x_{51} & 0 & 0 & 0 & x_{55} \end{bmatrix}. \tag{7}$$

[a]The idea is similar to the Tanner graph in coding theory by Tanner (1981).

Symbolic Determinants (concluded)

- The **determinant** of A^G is

$$\det(A^G) = \sum_{\pi} \text{sgn}(\pi) \prod_{i=1}^{n} A^G_{i,\pi(i)}. \tag{8}$$

  – π ranges over all permutations of n elements.

  – $\text{sgn}(\pi)$ is 1 if π is the product of an even number of transpositions and -1 otherwise.

  – Equivalently, $\text{sgn}(\pi) = 1$ if the number of (i, j)s such that $i < j$ and $\pi(i) > \pi(j)$ is even.[a]

- $\det(A^G)$ contains $n!$ terms, many of which may be 0s.

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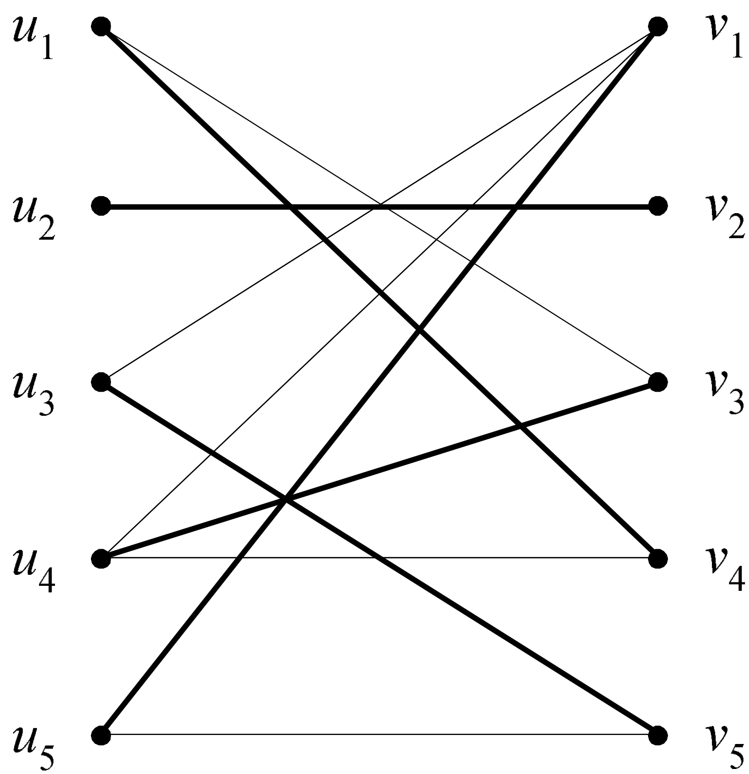
[a]Contributed by Mr. Hwan-Jeu Yu (`D95922028`) on May 1, 2008.

Determinant and Bipartite Perfect Matching

- In $\sum_\pi \text{sgn}(\pi) \prod_{i=1}^n A^G_{i,\pi(i)}$, note the following:
  - Each summand corresponds to a possible perfect matching π.
  - All of the nonzero summands $\prod_{i=1}^n A^G_{i,\pi(i)}$ are distinct monomials and *will not cancel.*
- $\det(A^G)$ is essentially an exhaustive enumeration.

Proposition 59 (Edmonds (1967)) *G has a perfect matching if and only if* $\det(A^G)$ *is not identically zero.*

Perfect Matching and Determinant (p. 501)



Perfect Matching and Determinant (concluded)

- The matrix is (p. 503)

$$A^G = \begin{bmatrix} 0 & 0 & x_{13} & \boxed{x_{14}} & 0 \\ 0 & \boxed{x_{22}} & 0 & 0 & 0 \\ x_{31} & 0 & 0 & 0 & \boxed{x_{35}} \\ x_{41} & 0 & \boxed{x_{43}} & x_{44} & 0 \\ \boxed{x_{51}} & 0 & 0 & 0 & x_{55} \end{bmatrix}.$$

- $\det(A^G) = -x_{14}x_{22}x_{35}x_{43}x_{51} + x_{13}x_{22}x_{35}x_{44}x_{51} + x_{14}x_{22}x_{31}x_{43}x_{55} - x_{13}x_{22}x_{31}x_{44}x_{55}$.
- Each nonzero term denotes a perfect matching, and vice versa.

How To Test If a Polynomial Is Identically Zero?

- $\det(A^G)$ is a polynomial in n^2 variables.
- There are exponentially many terms in $\det(A^G)$.
- Expanding the determinant polynomial is not feasible.
  - Too many terms.
- If $\det(A^G) \equiv 0$, then it remains zero if we substitute *arbitrary* integers for the variables $x_{11}, \ldots, x_{nn}$.
- When $\det(A^G) \not\equiv 0$, what is the likelihood of obtaining a zero?

Number of Roots of a Polynomial

Lemma 60 (Schwartz (1980)) *Let* $p(x_1, x_2, \ldots, x_m) \not\equiv 0$ *be a polynomial in* m *variables each of degree at most* d. *Let* $M \in \mathbb{Z}^+$. *Then the number of* m*-tuples*

$$(x_1, x_2, \ldots, x_m) \in \{0, 1, \ldots, M-1\}^m$$

such that $p(x_1, x_2, \ldots, x_m) = 0$ *is*

$$\leq mdM^{m-1}.$$

- By induction on m (consult the textbook).

Density Attack

- The density of roots in the domain is at most
$$\frac{mdM^{m-1}}{M^m} = \frac{md}{M}. \tag{9}$$
- So suppose $p(x_1, x_2, \ldots, x_m) \not\equiv 0$.
- Then a random
$$(x_1, x_2, \ldots, x_m) \in \{\, 0, 1, \ldots, M-1 \,\}^m$$
has a probability of $\leq md/M$ of being a root of p.
- Note that M is under our control!
  - One can raise M to lower the error probability, e.g.

Density Attack (concluded)

Here is a sampling algorithm to test if $p(x_1, x_2, \ldots, x_m) \not\equiv 0$.

Choose $i_1, \ldots, i_m$ from $\{0, 1, \ldots, M-1\}$ randomly;
if $p(i_1, i_2, \ldots, i_m) \neq 0$ **then**
    return "p is not identically zero";
else
    return "p is (probably) identically zero";
end if

Analysis

- If $p(x_1, x_2, \ldots, x_m) \equiv 0$, the algorithm will always be correct as $p(i_1, i_2, \ldots, i_m) = 0$.
- Suppose $p(x_1, x_2, \ldots, x_m) \not\equiv 0$.
  - The algorithm will answer incorrectly with probability at most md/M by Eq. (9) on p. 510.
- We next return to the original problem of bipartite perfect matching.

A Randomized Bipartite Perfect Matching Algorithm[a]

Choose n^2 integers $i_{11}, \ldots, i_{nn}$ from $\{0, 1, \ldots, 2n^2 - 1\}$ randomly; {So $M = 2n^2$.}
Calculate $\det(A^G(i_{11}, \ldots, i_{nn}))$ by Gaussian elimination;
if $\det(A^G(i_{11}, \ldots, i_{nn})) \neq 0$ **then**
    return "G has a perfect matching";
else
    return "G has no perfect matchings";
end if

[a]Lovász (1979). According to Paul Erdős, Lovász wrote his first significant paper "at the ripe old age of 17."

Analysis

- If G has no perfect matchings, the algorithm will always be correct as $\det(A^G(i_{11}, \ldots, i_{nn})) = 0$.
- Suppose G has a perfect matching.
  - The algorithm will answer incorrectly with probability at most $md/M = 0.5$ with $m = n^2$, $d = 1$ and $M = 2n^2$ in Eq. (9) on p. 510.
- Run the algorithm *independently* k times.
- Output "G has no perfect matchings" if and only if *all* say "no perfect matchings."
- The error probability is now reduced to at most 2^{-k}.

Lószló Lovász (1948–)

Remarks[a]

- Note that we are calculating

$$\text{prob}[\,\text{algorithm answers "no"} \mid G \text{ has no perfect matchings}\,],$$
$$\text{prob}[\,\text{algorithm answers "yes"} \mid G \text{ has a perfect matching}\,].$$

- We are *not* calculating[b]

$$\text{prob}[\,G \text{ has no perfect matchings} \mid \text{algorithm answers "no"}\,],$$
$$\text{prob}[\,G \text{ has a perfect matching} \mid \text{algorithm answers "yes"}\,].$$

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[a] Thanks to a lively class discussion on May 1, 2008.

[b] *Numerical Recipes in C* (1988), "[As] we already remarked, statistics is *not* a branch of mathematics!"

But How Large Can $\det(A^G(i_{11}, \ldots, i_{nn}))$ Be?

- It is at most
$$n!\left(2n^2\right)^n.$$
- Stirling's formula says $n! \sim \sqrt{2\pi n}\,(n/e)^n$.
- Hence
$$\log_2 \det(A^G(i_{11}, \ldots, i_{nn})) = O(n \log_2 n)$$
bits are sufficient for representing the determinant.
- We skip the details about how to make sure that all *intermediate* results are of polynomial sizes.

An Intriguing Question[a]

- Is there an $(i_{11}, \ldots, i_{nn})$ that will always give correct answers for the algorithm on p. 513?
- A theorem on p. 612 shows that such an $(i_{11}, \ldots, i_{nn})$ exists!
  - Whether it can be found efficiently is another matter.
- Once $(i_{11}, \ldots, i_{nn})$ is available, the algorithm can be made deterministic.

[a]Thanks to a lively class discussion on November 24, 2004.