Theory of Computation

Mid-Term Exam, 2014 Fall Semester,
11/11/2014

Note: Unless stated otherwise, you may use any results proved in class

Problem 1 (25 points) A Boolean function $f : \{0,1\}^m \to \{0,1\}$ is symmetric if $f(x_1, x_2, \ldots, x_m)$ depends only on $\sum_i x_i$. How many distinct symmetric Boolean functions of m variables are there?

Ans: 2^{m+1}. ■

Problem 2 (20 points) Let A and B be two complexity classes. We say that the inclusion is proper if A $\subsetneq$ B. Consider the following chain of class inclusions introduced in class:

$$\text{L} \subseteq \text{NL} \subseteq \text{P} \subseteq \text{NP} \subseteq \text{PSPACE}.$$

We can be sure that (at least) two pairs of classes have proper inclusions. Which are they and why?

Ans: L $\subsetneq$ PSPACE (see slide p. 234) and NL $\subsetneq$ PSPACE (see homework 3 problem 1). ■

Problem 3 (25 points) (a) Denote $L(M)$ as the language L accepted by Turing machine M. Is the language

$$L = \{(M) \mid M \text{ is a Turing machine and } L(M) \text{ is countable}\}$$

decidable? Why?

(b) Does there exist a language which is not recursively enumerable? If your answer is "NO", justify your answer; otherwise, give an example.

Ans: (a) Yes, L is decidable. In fact, L is the language of all TM's, which can be easily checked in polynomial time.
(b) Yes, there exist languages which are not recursively enumerable, for example,

$$\{(M, x) \mid M \text{ is a TM and it does not halt on string } x\}.$$

■

Problem 4 (30 points) Reduce k-SAT to 3SAT, where $k > 3$. (Hint: Consider the Boolean expressions A, B and C and the variable y. It is known that the expression

$$(y \vee A) \wedge (\neg y \vee B) \wedge C$$

is satisfiable if and only if

$$(A \vee B) \wedge C$$

is too.)

Ans: Consider a k-SAT expression Φ with n variables, m clauses and k literals in every clause, where $n > k$. Let $c_1, c_2, \ldots, c_m$ be the clauses of Φ. For each c_j of the form

$$c_j = (w_1 \vee w_2 \vee \cdots \vee w_{k-1} \vee w_k),\ j = 1, 2, \ldots, m,$$

where $w_1, w_2, \ldots, w_k$ are the literals, we introduce new variables $y_{j,1}, y_{j,2}, \ldots, y_{j,k-3}$ to form a new clause c'_j to replace c_j:

$$\begin{aligned} c'_j = (w_1 \vee w_2 \vee y_{j,1}) \wedge (\neg y_{j,1} \vee w_3 \vee y_{j,2}) \wedge (\neg y_{j,2} \vee w_4 \vee y_{j,3}) \wedge \cdots \\ \wedge (\neg y_{j,k-4} \vee w_{k-2} \vee y_{j,k-3}) \wedge (\neg y_{j,k-3} \vee w_{k-1} \vee w_k). \end{aligned}$$

The above replacement is clearly a polynomial-time reduction.

Note that the results of the hint can be easily extended inductively such that c'_j is satisfiable if and only if c_j is also satisfiable.

Now, we show that $c'_1 \wedge c'_2 \wedge \cdots \wedge c'_m$ is satisfiable if Φ is. Suppose Φ is satisfied by a truth assignment T. We extend T by assigning the values of the new variables arbitrarily to form a new truth assignment T'. With the extended results of the hint, $c'_1 \wedge c'_2 \wedge \cdots \wedge c'_m$ must be satisfied by T'

because the new variables do not affect the result. Hence, $c'_1 \wedge c'_2 \wedge \cdots \wedge c'_m$ is satisfiable if Φ is.

Conversely, suppose $c'_1 \wedge c'_2 \wedge \cdots \wedge c'_m$ is satisfied by a truth assignment T'. Again, from the extended results of the hint, it is obvious that Φ is also satisfied by T' by ignoring the values of all the new variables $y_{j,1}, y_{j,2}, \ldots, y_{j,k-3}$. Hence, Φ is satisfiable if $c'_1 \wedge c'_2 \wedge \cdots \wedge c'_m$ is.