The Kleene Star Operation $*$[a]

- Let A be a set.
- The **Kleene star** of A, denoted by A^*, is the set of all strings obtained by concatenating zero or more strings from A.
  - For example, suppose $A = \{ 0, 1 \}$.
  - Then

$$A^* = \{ \epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots \}.$$

  - Note that every string in A^* must be of finite length.

[a] Kleene (1956).

Decidability and Recursive Languages

- Let $L \subseteq (\Sigma - \{\sqcup\})^*$ be a **language**, i.e., a set of strings of non-$\sqcup$ symbols, with a *finite* length.
  - For example, $\{0, 01, 10, 210, 1010, \ldots\}$.
- Let M be a TM such that for any string x:
  - If $x \in L$, then $M(x) =$ "yes."
  - If $x \notin L$, then $M(x) =$ "no."
- We say M **decides** L.
- If there exists a TM that decides L, then L is **recursive**[a] or **decidable**.

[a]Little to do with the concept of "recursive" calls.

Recursive and Nonrecursive Languages: Examples

- The set of palindromes over any alphabet is recursive.[a]
- The set of prime numbers $\{2, 3, 5, 7, 11, 13, 17, \ldots\}$ is recursive.[b]
- The set of C programs that do not contain a `while`, a `for`, or a `goto` is recursive.[c]
- But, the set of C programs that do not contain an infinite loop is *not* recursive (see p. 155).

---

[a]Need a program that returns "yes" iff the input is a palindrome.

[b]Need a program that returns "yes" iff the input is a prime.

[c]Need a program that returns "yes" iff the input C code does not contain any of the keywords.

Acceptability and Recursively Enumerable Languages

- Let $L \subseteq (\Sigma - \{\sqcup\})^*$ be a language.
- Let M be a TM such that for any string x:
  - If $x \in L$, then $M(x) =$ "yes."
  - If $x \notin L$, then $M(x) = \nearrow$.[a]
- We say M **accepts** L.
- How to verify that a TM decides/accepts a language is a different matter.[b]

---

[a] This part is different from recursive languages.

[b] Thanks to a lively discussion on September 23, 2014.

Acceptability and Recursively Enumerable Languages (concluded)

- If L is accepted by some TM, then L is called a **recursively enumerable language**.[a]
  - A recursively enumerable language can be *generated* by a TM, thus the name.[b]
  - That is, there is an algorithm such that for every $x \in L$, it will be printed out eventually.
  - If L is infinite in size, this algorithm will not terminate.

---

[a] Post (1944).

[b] Thanks to a lively class discussion on September 20, 2011.

Emil Post (1897–1954)

Recursive and Recursively Enumerable Languages

Proposition 2 *If L is recursive, then it is recursively enumerable.*

- Let TM M decide L.
- Need to design a TM that accepts L.
- We will modify M to obtain an M' that accepts L.

The Proof (concluded)

- M' is identical to M except that when M is about to halt with a "no" state, M' goes into an infinite loop.
  - Simply replace any instruction that results in a "no" state with ones that move the cursor to the right forever and never halts.
- M' accepts L.
  - If $x \in L$, then $M'(x) = M(x) =$ "yes."
  - If $x \notin L$, then $M(x) =$ "no" and so $M'(x) = \nearrow$.

Recursively Enumerable Languages: Examples

- The set of C program-input pairs that do not run into an infinite loop is recursively enumerable.
  - Just run its binary code in a simulator environment.
  - Then the simulator will terminate if and only if the C program will terminate.
  - When the C program terminates, the simulator simply exits with a "yes" state.
- The set of C programs that contain an infinite loop is *not* recursively enumerable (see p. 155).

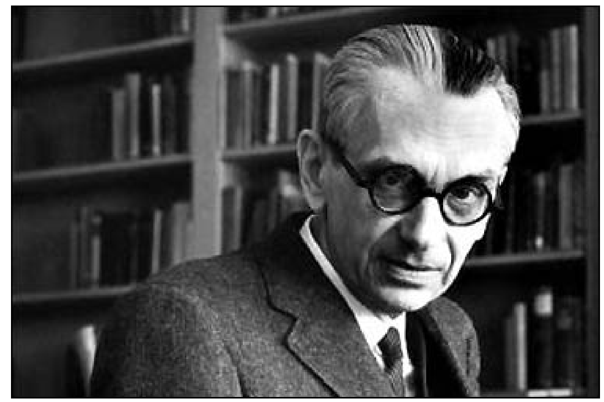
Turing-Computable Functions

- Let $f : (\Sigma - \{\sqcup\})^* \to \Sigma^*$.
  - Optimization problems, root finding problems, etc.
- Let M be a TM with alphabet Σ.
- M **computes** f if for any string $x \in (\Sigma - \{\sqcup\})^*$, $M(x) = f(x)$.
- We call f a **recursive function**[a] if such an M exists.

[a]Kurt Gödel (1931, 1934).

Kurt Gödel[a] (1906–1978)

Quine (1978), "this theorem [$\cdots$] sealed his immortality."



[a]This photo was taken by Alfred Eisenstaedt (1898–1995).

Church's Thesis or the Church-Turing Thesis

- What is computable is Turing-computable; TMs are algorithms.[a]
- No "intuitively computable" problems have been shown not to be Turing-computable, yet.[b]

---

[a] Church (1936); Kleene (1953).

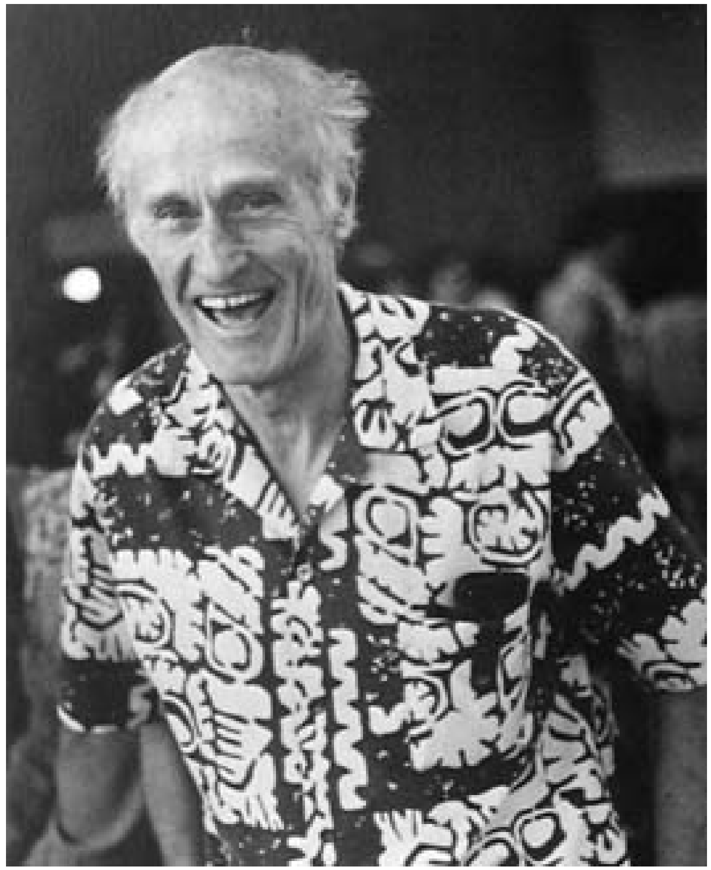
[b] Quantum computer of Manin (1980) and Feynman (1982) and DNA computer of Adleman (1994).

Church's Thesis or the Church-Turing Thesis (concluded)

- Many other computation models have been proposed.
  - Recursive function (Gödel), λ calculus (Church), formal language (Post), assembly language-like RAM (Shepherdson & Sturgis), boolean circuits (Shannon), extensions of the Turing machine (more strings, two-dimensional strings, and so on), etc.
- All have been proved to be equivalent.

Alonso Church (1903–1995)

Stephen Kleene (1909–1994)



Extended Church's Thesis[a]

- All "reasonably succinct encodings" of problems are *polynomially related* (e.g., n^2 vs. n^6).
  - Representations of a graph as an adjacency matrix and as a linked list are both succinct.
  - The *unary* representation of numbers is not succinct.
  - The *binary* representation of numbers is succinct.
    - 1001_2 vs. 111111111_1.
- All numbers for TMs will be binary from now on.

[a]Some call it "polynomial Church's thesis," which Lószló Lovász attributed to Leonid Levin.

Extended Church's Thesis (concluded)

- Representations that are not succinct may give misleadingly low complexities.
  - Consider an algorithm with binary inputs that runs in 2^n steps.
  - Suppose the input uses unary representation instead.
  - Then the same algorithm runs in linear time because the input length is now 2^n!
- So a succinct representation is for honest accounting.

Physical Church-Turing Thesis

- "[Church's thesis] is a profound claim about the physical laws of our universe, i.e.: any physical system that purports to be a 'computer' is not capable of any computational task that a Turing machine is incapable of."[a]
- "Anything computable in physics can also be computed on a Turing machine."[b]
- The universe is a Turing machine.[c]

---

[a] Warren Smith (1998).
[b] Cooper (2012).
[c] Edward Fredkin's (1992) digital physics.

The Strong Turing-Church Thesis[a]

- The **strong Turing-Church Thesis** states that:

  A Turing machine can compute *any* function computable by any "reasonable" physical device with only polynomial slowdown.

- A CPU and a DSP chip are good examples of physical devices.[b]

---

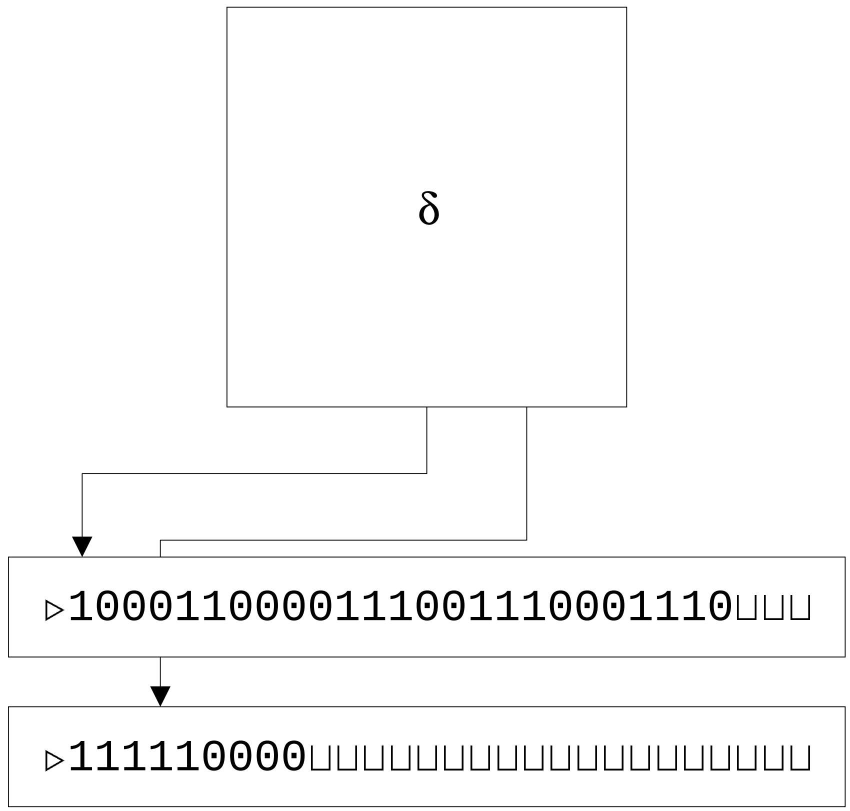
[a] Vergis, Steiglitz, and Dickinson (1986).

[b] Thanks to a lively discussion on September 23, 2014.

Turing Machines with Multiple Strings

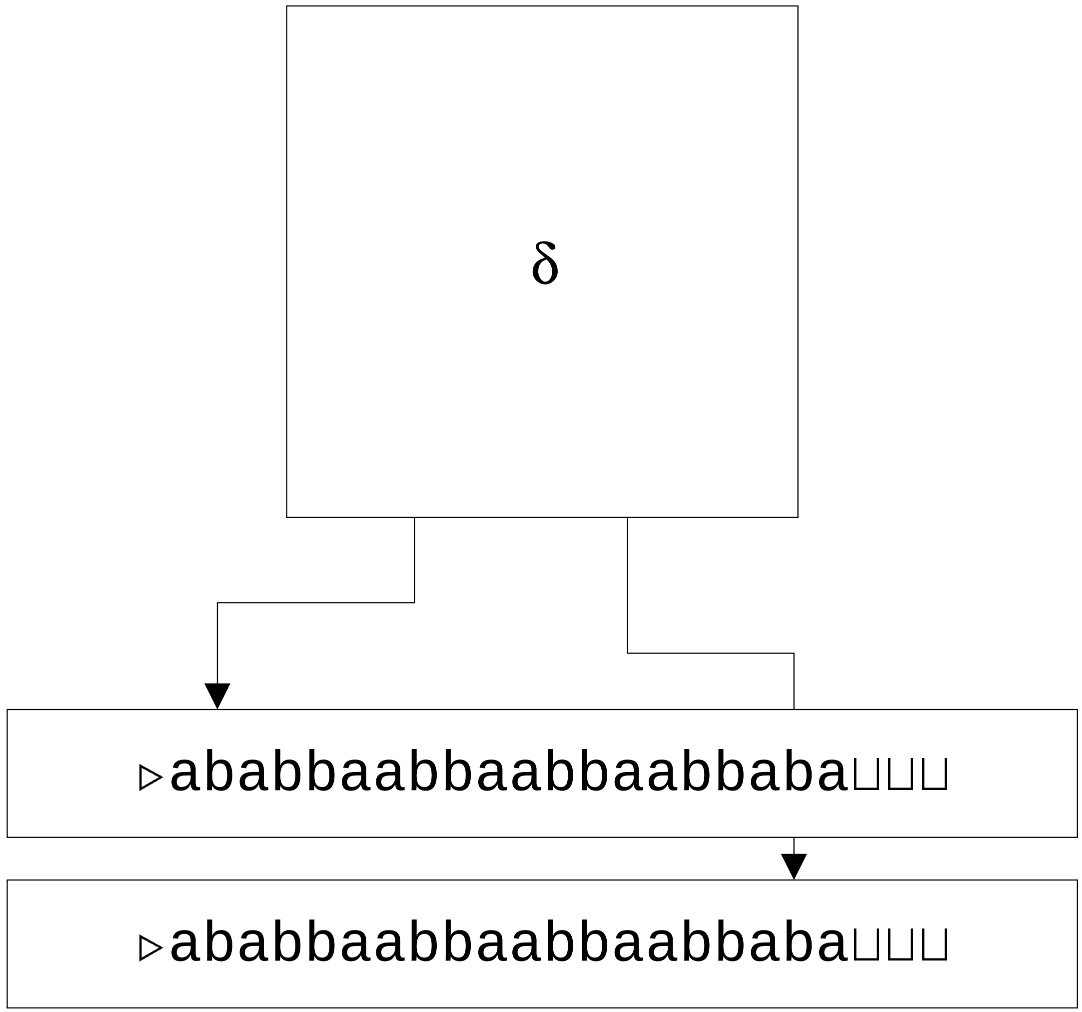
- A k-string Turing machine (TM) is a quadruple $M = (K, \Sigma, \delta, s)$.
- K, Σ, s are as before.
- $\delta : K \times \Sigma^k \to (K \cup \{h, \text{"yes"}, \text{"no"}\}) \times (\Sigma \times \{\leftarrow, \rightarrow, -\})^k$.
- All strings start with a $\triangleright$.
- The first string contains the input.
- Decidability and acceptability are the same as before.
- When TMs compute functions, the output is the last (kth) string.

A 2-String TM



PALINDROME Revisited

- A 2-string TM can decide PALINDROME in $O(n)$ steps.
  - It copies the input to the second string.
  - The cursor of the first string is positioned at the first symbol of the input.
  - The cursor of the second string is positioned at the last symbol of the input.
  - The symbols under the cursors are then compared.
  - The two cursors are then moved in opposite directions until the ends are reached.
  - The machine accepts if and only if the symbols under the two cursors are identical at all steps.


δ
▷ababbaabbaabbaabbaba⊔⊔⊔
▷ababbaabbaabbaabbaba⊔⊔⊔

PALINDROME Revisited (concluded)

- The running times of a 2-string TM and a single-string TM are quadratically related: n^2 vs. n.
- This is consistent with extended Church's thesis.

Configurations and Yielding

- The concept of configuration and yielding is the same as before except that a configuration is a $(2k+1)$-tuple

$$(q, w_1, u_1, w_2, u_2, \ldots, w_k, u_k).$$

  - $w_i u_i$ is the ith string.
  - The ith cursor is reading the last symbol of w_i.
  - Recall that $\rhd$ is each w_i's first symbol.

- The k-string TM's initial configuration is

$$(s, \overbrace{\underbrace{\rhd, x}_{1}, \underbrace{\rhd, \epsilon}_{2}, \underbrace{\rhd, \epsilon}_{3}, \ldots, \underbrace{\rhd, \epsilon}_{k}}^{2k}).$$

Time seemed to be
the most obvious measure
of complexity.
— Stephen Arthur Cook (1939–)

Time Complexity

- The multistring TM is the basis of our notion of the time expended by TMs.
- If a k-string TM M halts after t steps on input x, then the **time required by M on input x** is t.
- If $M(x) = \nearrow$, then the time required by M on x is ∞.

Time Complexity (concluded)

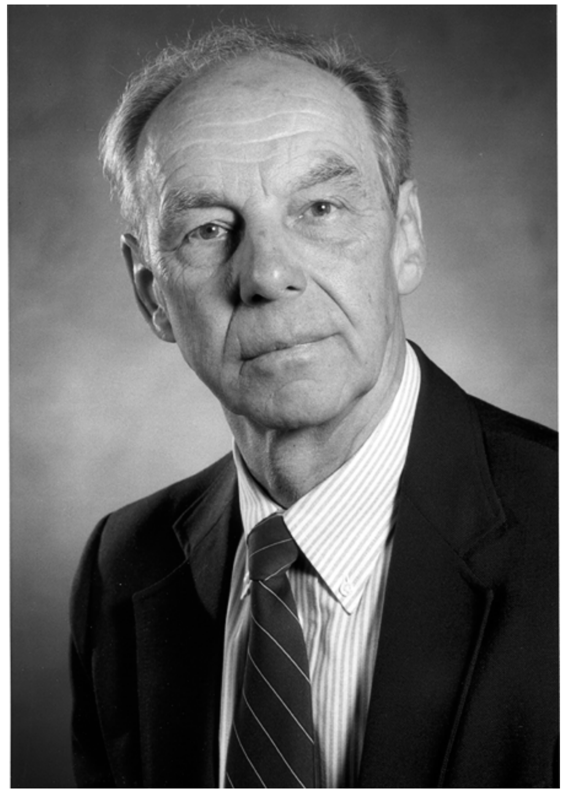
- Machine M **operates within time** $f(n)$ for $f : \mathbb{N} \to \mathbb{N}$ if for any input string x, the time required by M on x is at most $f(|x|)$.
  - $|x|$ is the length of string x.
- Function $f(n)$ is a **time bound** for M.

Time Complexity Classes[a]

- Suppose language $L \subseteq (\Sigma - \{\bigsqcup\})^*$ is decided by a multistring TM operating in time $f(n)$.
- We say $L \in \text{TIME}(f(n))$.
- $\text{TIME}(f(n))$ is the set of languages decided by TMs with multiple strings operating within time bound $f(n)$.
- $\text{TIME}(f(n))$ is a **complexity class**.
  - PALINDROME is in $\text{TIME}(f(n))$, where $f(n) = O(n)$.

[a]Hartmanis and Stearns (1965); Hartmanis, Lewis, and Stearns (1965).

Juris Hartmanis[a] (1928–)



[a]Turing Award (1993).

Richard Edwin Stearns[a] (1936–)

[a]Turing Award (1993).

The Simulation Technique

Theorem 3 *Given any k-string M operating within time $f(n)$, there exists a (single-string) M' operating within time $O(f(n)^2)$ such that $M(x) = M'(x)$ for any input x.*

- The single string of M' implements the k strings of M.

The Proof

- Represent configuration $(q, w_1, u_1, w_2, u_2, \ldots, w_k, u_k)$ of M by this string of M':

$$(q, \rhd w_1' u_1 \lhd w_2' u_2 \lhd \cdots \lhd w_k' u_k \lhd \lhd).$$

  - $\lhd$ is a special delimiter.
  - w_i' is w_i with the first[a] and last symbols "primed."
  - It serves the purpose of "," in a configuration.[b]

---

[a] The first symbol is always $\rhd$.

[b] An alternative is to use $(q, \rhd w_1' u_1,' \lhd w_2',' u_2 \lhd \cdots \lhd w_k',' u_k \lhd \lhd)$ by priming only $\rhd$ in w_i, where ",′" is a new symbol.

The Proof (continued)

- The "priming" of the last symbol of w_i ensures that M' knows which symbol is under each cursor of M.[a]
- The first symbol of w_i is the primed version of $\triangleright$: $\triangleright'$.
  - Recall TM cursors are not allowed to move to the left of $\triangleright$ (p. 21).
  - Now the cursor of M' can move *between* the simulated strings of M.[b]

---

[a] Added because of comments made by Mr. Che-Wei Chang (R95922093) on September 27, 2006.

[b] Thanks to a lively discussion on September 22, 2009.

The Proof (continued)

- The initial configuration of M' is

$$(s, \triangleright \triangleright'' \, x \lhd \overbrace{\triangleright'' \lhd \cdots \triangleright'' \lhd}^{k-1 \text{ pairs}} \lhd).$$

  - $\triangleright$ is double-primed because it is the beginning and the ending symbol as the cursor is reading it.[a]

[a] Added after the class discussion on September 20, 2011.

The Proof (continued)

- We simulate each move of M thus:
  1. M' scans the string to pick up the k symbols under the cursors.
     - The states of M' must be enlarged to include $K \times \Sigma^k$ to remember them.[a]
     - The transition functions of M' must also reflect it.
  2. M' then changes the string to reflect the overwriting of symbols and cursor movements of M.

---

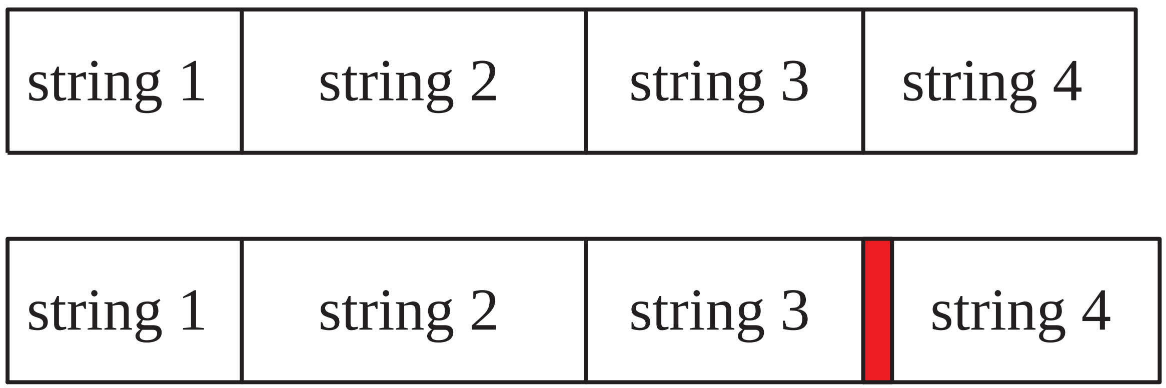
[a]Recall the TM program on p. 27.

The Proof (continued)

- It is possible that some strings of M need to be lengthened (see next page).
  - The linear-time algorithm on p. 40 can be used for each such string.
- The simulation continues until M halts.
- M' then erases all strings of M except the last one.[a]

---

[a]Because whatever appears on the string of M' will be considered the output. So those $\triangleright'$s and $\triangleright''$s need to be removed.


string 1
string 2
string 3
string 4
string 1
string 2
string 3
string 4

The Proof (continued)

- Since M halts within time $f(|x|)$, none of its strings ever becomes longer than $f(|x|)$.[a]
- The length of the string of M' at any time is $O(kf(|x|))$.
- Simulating each step of M takes, *per string of* M, $O(kf(|x|))$ steps.
  - $O(f(|x|))$ steps to collect information from this string.
  - $O(kf(|x|))$ steps to write and, if needed, to lengthen the string.

[a] We tacitly assume $f(n) \geq n$.

The Proof (concluded)

- M' takes $O(k^2 f(|x|))$ steps to simulate each step of M because there are k strings.
- As there are $f(|x|)$ steps of M to simulate, M' operates within time $O(k^2 f(|x|)^2)$.

Linear Speedup[a]

Theorem 4 *Let* $L \in TIME(f(n))$*. Then for any* $\epsilon > 0$*,* $L \in TIME(f'(n))$*, where* $f'(n) = \epsilon f(n) + n + 2$*.*

[a] Hartmanis and Stearns (1965).

Implications of the Speedup Theorem

- State size can be traded for speed.[a]
- If $f(n) = cn$ with $c > 1$, then c can be made arbitrarily close to 1.
- If $f(n)$ is superlinear, say $f(n) = 14n^2 + 31n$, then the constant in the leading term (14 in this example) can be made arbitrarily small.
  - *Arbitrary* linear speedup can be achieved.[b]
  - This justifies the big-O notation in the analysis of algorithms.

---

[a] $m^k \cdot |\Sigma|^{3mk}$-fold increase to gain a speedup of $O(m)$. No free lunch.

[b] Can you apply the theorem multiple times to achieve superlinear speedup? Thanks to a question by a student on September 21, 2010.

P

- By the linear speedup theorem, any polynomial time bound can be represented by its leading term n^k for some $k \geq 1$.
- If L is a **polynomially decidable language**, it is in $\text{TIME}(n^k)$ for some $k \in \mathbb{N}$.
  - Clearly, $\text{TIME}(n^k) \subseteq \text{TIME}(n^{k+1})$.
- The union of all polynomially decidable languages is denoted by P:

$$\text{P} = \bigcup_{k>0} \text{TIME}(n^k).$$

- P contains problems that can be efficiently solved.

Philosophers have explained space.
They have not explained time.
— Arnold Bennett (1867–1931),
How To Live on 24 Hours a Day (1910)

I keep bumping into that silly quotation
attributed to me that says
640K of memory is enough.
— Bill Gates (1996)