Theory of Computation Lecture Notes

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Class Information

- Papadimitriou. *Computational Complexity*. 2nd printing. Addison-Wesley. 1995.
  - We more or less follow the topics of the book.
  - Extra materials may be added.
- You may want to review discrete mathematics.

Class Information (concluded)

- More information and lecture notes can be found at

  `www.csie.ntu.edu.tw/~lyuu/complexity.html`

  - Homeworks, exams, solutions and teaching assistants will be announced there.

- Please ask many questions in class.

  - This is the best way for me to remember you in a large class.[a]

[a] "[A] science concentrator [...] said that in his eighth semester of [Harvard] college, there was not a single science professor who could identify him by name." (*New York Times*, September 3, 2003.)

Grading

- Homeworks.
  - Do not copy others' homeworks.
  - Do not give your homeworks for others to copy.
- Two to three exams.
- You must show up for the exams in person.
- If you cannot make it to an exam for a legitimate reason, please email me or a TA beforehand to the extent possible.
- Missing the final exam will automatically earn a "fail" grade.

Problems and Algorithms

I have never done anything "useful."
— Godfrey Harold Hardy (1877–1947),
A Mathematician's Apology (1940)

What This Course Is All About

Computation: What is computation?

Computability: What can be computed?

- There are *well-defined* problems that cannot be computed.
- In fact, most problems cannot be computed.

What This Course Is All About (continued)

Complexity: What is a computable problem's inherent complexity?

- Some computable problems require at least exponential time and/or space.
  - They are said to be **intractable**.
- Some practical problems require superpolynomial[a] resources unless certain conjectures are disproved.
- Resources besides time and space: Circuit size, circuit layout area, program size, number of random bits, etc.

[a] The prefix "super" means "above, beyond."

What This Course Is All About (concluded)

Applications: Intractability results can be very useful.

- Cryptography and security.
- Approximations.
- Conjectures about nature.

Tractability and Intractability

- Tractability means polynomial in terms of the input size n.
  - n, $n \log n$, n^2, n^{90}.
- It results in a fruitful and practical theory of complexity.
- Few practical, tractable problems require a large degree.
- Superpolynomial-time algorithms are seldom practical.
  - $n^{\log n}$, $2^{\sqrt{n}}$,[a] 2^n, $n! \sim \sqrt{2\pi n}\,(n/e)^n$.

[a] Size of depth-3 circuits to compute the majority function (Wolfovitz (2006)) and certain stochastic models used in finance (Dai (R86526008, D8852600) and Lyuu (2007), Lyuu and Wang (F95922018) (2011), and Chiu (R98723059) (2012)).

Exponential Growth of *E. Coli*[a]

- Under ideal conditions, *E. Coli* bacteria divide every 20 minutes.
- In two days, a single *E. Coli* bacterium would become 2^{144} bacteria.
- They would weigh 2,664 times the Earth!

[a] Nick Lane, *Power, Sex, Suicide: Mitochondria and the Meaning of Life* (2005).

Growth of Factorials

n	$n!$	n	$n!$
1	1	9	362,880
2	2	10	3,628,800
3	6	11	39,916,800
4	24	12	479,001,600
5	120	13	6,227,020,800
6	720	14	87,178,291,200
7	5040	15	1,307,674,368,000
8	40320	16	20,922,789,888,000

Moore's Law[a] to the Rescue?[b]

- Moore's law says the computing power doubles every 1.5 years.
- So the computing power grows like

$$4^{y/3},$$

  where y is the number of years from now.
- Assume Moore's law holds forever.
- Can you let the law take care of exponential complexity?

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[a] Moore (1965).

[b] Contributed by Ms. Amy Liu (J94922016) on May 15, 2006. Thanks also to a lively discussion on September 14, 2010.

Moore's Law to the Rescue (continued)?

- Suppose a problem takes a^n seconds of CPU time to solve now, where n is the input length.
- The same problem will take

$$\frac{a^n}{4^{y/3}}$$

  seconds to solve y years from now.
- In particular, the hardware $3n \log_4 a$ years from now takes 1 second to solve it.
- The overall complexity becomes linear in n!

Moore's Law to the Rescue (concluded)?

- Potential objections:
  - Moore's law may not hold forever.
  - The total number of operations is the same; so the *algorithm* remains exponential in complexity.[a]
- What is a "good" theory on computational complexity?

[a]Contributed by Mr. Hung-Jr Shiu (D00921020) on September 14, 2011.

Turing Machines

Tarski has stressed in his lecture
(and I think justly)
the great importance of
the concept of general recursiveness
(or Turing's computability).
— Kurt Gödel (1946)

What Is Computation?

- That can be coded in an **algorithm**.[a]
- An algorithm is a detailed step-by-step method for solving a problem.
  - The Euclidean algorithm for the greatest common divisor is an algorithm.
  - "Let s be the least upper bound of compact set A" is not an algorithm.
  - "Let s be a smallest element of a finite-sized array" can be solved by an algorithm.

[a] Muhammad ibn Mūsā Al-Khwārizmī (780–850).

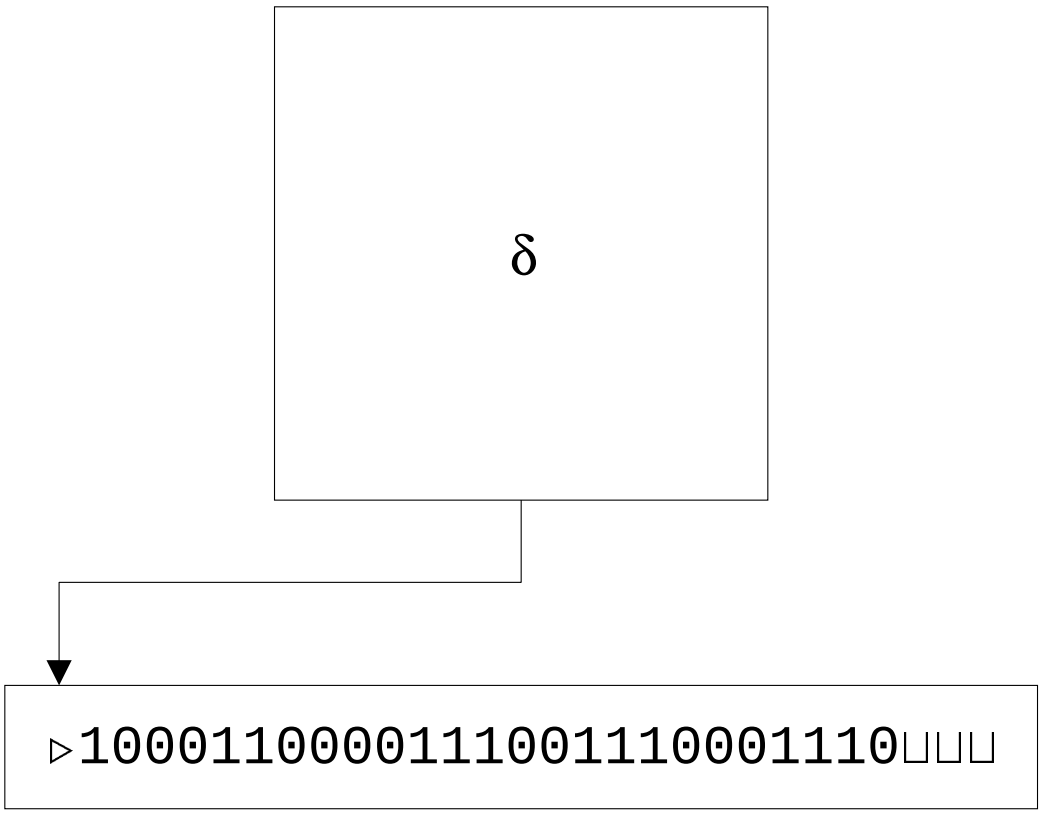
Turing Machines[a]

- A Turing machine (TM) is a quadruple $M = (K, \Sigma, \delta, s)$.
- K is a finite set of **states**.[b]
- $s \in K$ is the **initial state**.
- Σ is a finite set of **symbols** (disjoint from K).
  - Σ includes $\sqcup$ (blank) and $\triangleright$ (first symbol).
- $\delta : K \times \Sigma \to (K \cup \{h, \text{“yes”}, \text{“no”}\}) \times \Sigma \times \{\leftarrow, \rightarrow, -\}$ is a **transition function**.
  - $\leftarrow$ (left), $\rightarrow$ (right), and $-$ (stay) signify cursor movements.

[a] Turing (1936).

[b] Turing (1936), “If we admitted an infinity of states of mind, some of them will be ‘arbitrarily close’ and will be confused.”

A TM Schema



More about δ

- The program has the **halting state** (h), the **accepting state** ("yes"), and the **rejecting state** ("no").
- Given current state $q \in K$ and current symbol $\sigma \in \Sigma$,

$$\delta(q, \sigma) = (p, \rho, D).$$

  - It specifies:
    - The next state p;
    - The symbol ρ to be written over σ;
    - The direction D the cursor will move *afterwards*.
- Assume $\delta(q, \triangleright) = (p, \triangleright, \rightarrow)$.
  - So the cursor never falls off the left end of the string.

More about δ (concluded)

- Think of the program as lines of codes:

$$\begin{aligned}
\delta(q_1, \sigma_1) &= (p_1, \rho_1, D_1), \\
\delta(q_2, \sigma_2) &= (p_2, \rho_2, D_2), \\
&\vdots \\
\delta(q_n, \sigma_n) &= (p_n, \rho_n, D_n).
\end{aligned}$$

- Assume the state is q and the symbol under the cursor σ.
- The line of code that matches (q, σ) is executed.[a]
- Then the process is repeated.

[a] So there should be one and only one instruction for every possible pair (q, σ). Contributed by Mr. Ya-Hsun Chang (B96902025, R00922044) on September 13, 2011.

The Operations of TMs

- Initially the state is s.
- The string on the tape is initialized to a $\triangleright$, followed by a *finite-length* string $x \in (\Sigma - \{\sqcup\})^*$.
- x is the **input** of the TM.
  - The input must not contain $\sqcup$s (why?)!
- The cursor is pointing to the first symbol, always a $\triangleright$.
- The TM takes each step according to δ.
- The cursor may overwrite $\sqcup$ to make the string longer during the computation.

"Physical" Interpretations

- The tape: computer memory and registers.
  - Except that the tape can be lengthened on demand.
- δ: program.
  - A program has a *finite* size.
- K: instruction numbers.
- s: "`main()`" in the C programming language.
- Σ: **alphabet**, much like the ASCII code.

The Halting of a TM

- A TM M may **halt** in three cases.

  "yes": M **accepts** its input x, and $M(x) =$ "yes".

  "no": M **rejects** its input x, and $M(x) =$ "no".

  h**:** $M(x) = y$ means the string (tape) consists of a $\triangleright$, followed by a finite string y, whose last symbol is not $\sqcup$, followed by a string of $\sqcup$s.
  - y is the **output** of the computation.
  - y may be empty denoted by ϵ.

- If M never halts on x, then write $M(x) = \nearrow$.

Why Turing Machines?

- Because of the simplicity of the TM, the model has the advantage when it comes to complexity issues.
- One can conceivably develop a complexity theory based on something similar to C, C++ or Java.
- But the added complexity does not yield additional fundamental insights.
- We will describe TMs in pseudocode only.[a]

[a] But students are strongly encouraged to read and understand the TM codes in the textbook to gain insight on this programming language.

Remarks

- A computation model should be "physically" realizable.
  - E.g., our brain, at least as powerful as a Turing machine, is physical.
- Although a TM requires a tape of potentially infinite length, which is not realizable, it is not a major *conceptual* issue.[a]
  - Imagine you ("the program") are living next to a paper mill while carrying out a TM code using pencil ("the cursor") and paper ("the tape").
  - The mill will produce extra paper if needed.

[a] Thanks to a lively discussion on September 20, 2006.

Remarks (concluded)

- Even our computer is only an approximation of a TM for the same reason.
  - But it is easy to imagine our computer with more and more address space, memory space, and disk space.

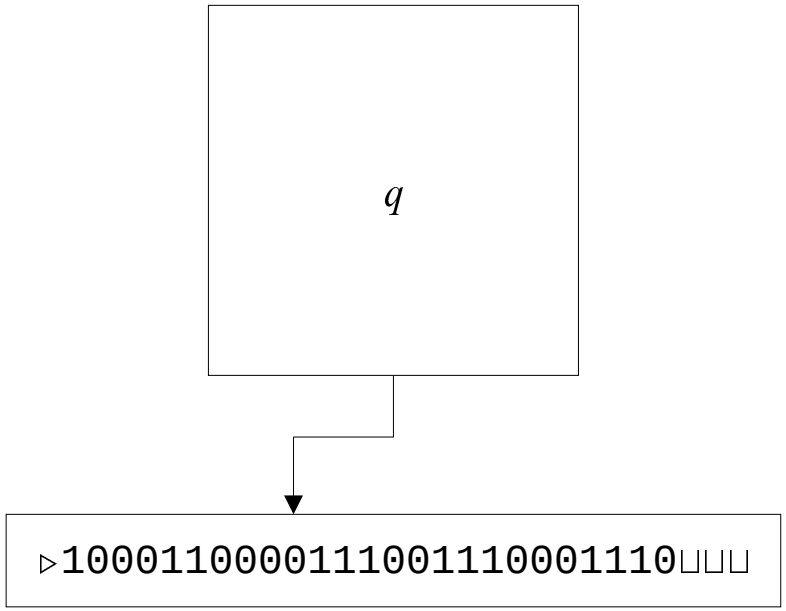
The Concept of Configuration

- A **configuration**[a] is a complete description of the current state of the computation.
- The specification of a configuration is sufficient for the computation to continue as if it had not been stopped.
  - What does your PC save before it sleeps?
  - Enough for it to resume work later.
- Similar to the concept of state in Markov process.

[a] This term was due to Turing (1936).

Configurations (concluded)

- A configuration is a triple (q, w, u):
  - $q \in K$.
  - $w \in \Sigma^*$ is the string to the left of the cursor (inclusive).
  - $u \in \Sigma^*$ is the string to the right of the cursor.
- Note that (w, u) describes both the string and the cursor position.



- $w = \triangleright 1000110000$.
- $u = 111001110001110$.

Yielding

- Fix a TM M.
- Configuration (q, w, u) **yields** configuration (q', w', u') in one step,

  $$(q, w, u) \xrightarrow{M} (q', w', u'),$$

  if a step of M from configuration (q, w, u) results in configuration (q', w', u').
- $(q, w, u) \xrightarrow{M^k} (q', w', u')$: Configuration (q, w, u) yields configuration (q', w', u') after $k \in \mathbb{N}$ steps.
- $(q, w, u) \xrightarrow{M^*} (q', w', u')$: Configuration (q, w, u) yields configuration (q', w', u').

Alan Turing (1912–1954)

Richard Dawkins (2006), "Turing arguably made a greater contribution to defeating the Nazis than Eisenhower or Churchill."

Our First TM Program: How To Insert a Symbol

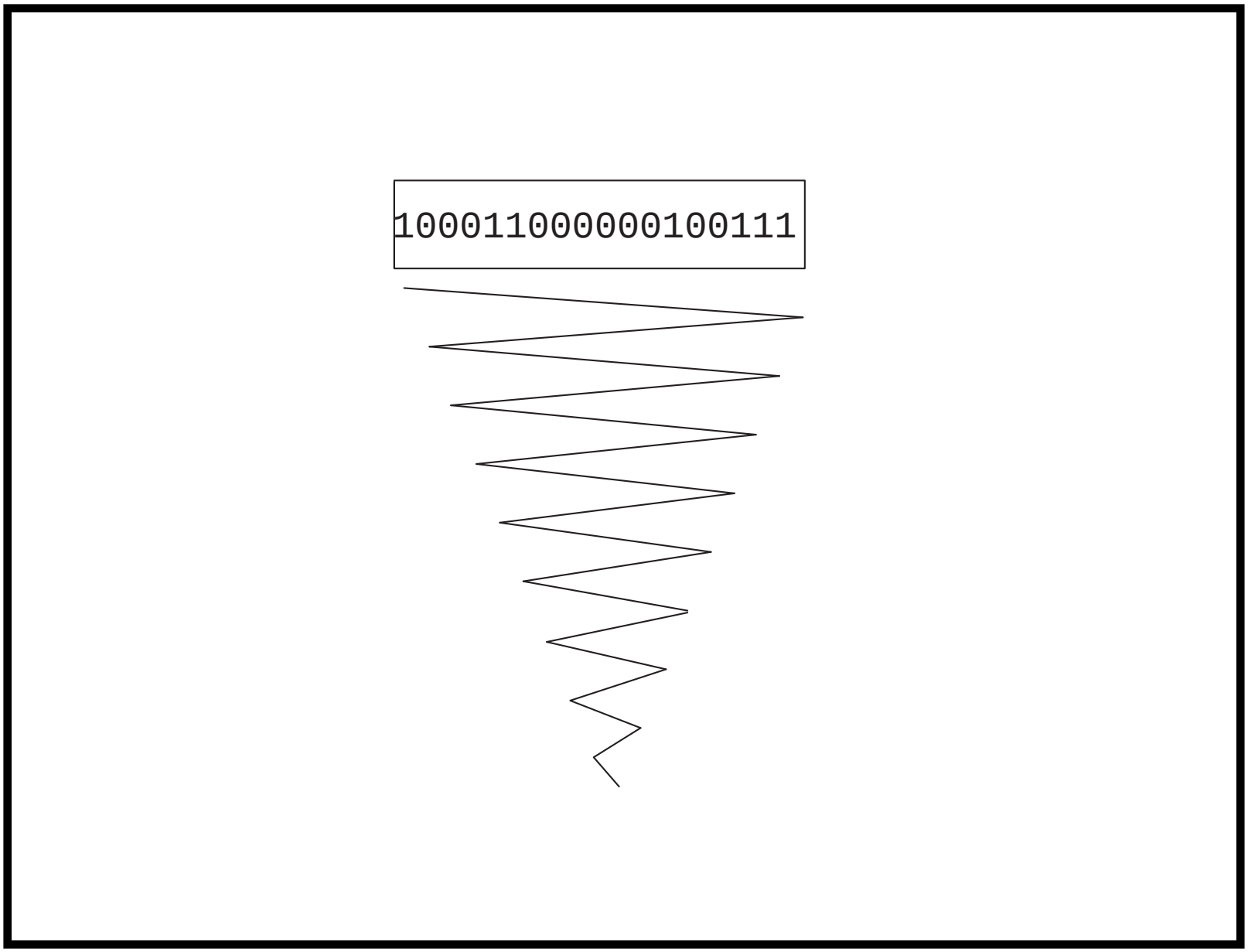
- We want to compute $f(x) = ax$.
  - The TM moves its cursor to the last symbol.[a]
  - It moves the last symbol of x to the right by one position.
  - It moves the next to last symbol to the right, and so on.
  - The TM finally writes a in the first position.
- The total number of steps is $O(n)$, where n is the length of x.

[a] Had we allowed the input to contain ␣s, the last symbol would not be well-defined!

Palindromes

- A string is a **palindrome** if it reads the same forwards and backwards (e.g., `001100`).
- A TM program can be written to recognize palindromes:
  - It matches the first character with the last character.
  - It matches the second character with the next to last character, etc. (see next page).
  - "yes" for palindromes and "no" for nonpalindromes.
- This program takes $O(n^2)$ steps.
- We cannot do better.[a]

[a] Hennie (1965).


1000110000000100111

The Kleene Star Operation $*$[a]

- Let A be a set.
- The **Kleene star** of A, denoted by A^*, is the set of all strings obtained by concatenating zero or more strings from A.
  - For example, suppose $A = \{ 0, 1 \}$.
  - Then

$$A^* = \{ \epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots \}.$$

  - Note that every string in A^* must be of finite length.

[a] Kleene (1956).

Decidability and Recursive Languages

- Let $L \subseteq (\Sigma - \{\sqcup\})^*$ be a **language**, i.e., a set of strings of non-$\sqcup$ symbols, with a *finite* length.
  - For example, $\{0, 01, 10, 210, 1010, \ldots\}$.
- Let M be a TM such that for any string x:
  - If $x \in L$, then $M(x) =$ "yes."
  - If $x \notin L$, then $M(x) =$ "no."
- We say M **decides** L.
- If there exists a TM that decides L, then L is **recursive**[a] or **decidable**.

[a] Little to do with the concept of recursive calls.

Recursive and Nonrecursive Languages: Examples

- The set of palindromes over any alphabet is recursive.[a]
- The set of prime numbers $\{2, 3, 5, 7, 11, 13, 17, \ldots\}$ is recursive.[b]
- The set of C programs that do not contain a `while`, a `for`, or a `goto` is recursive.[c]
- But, the set of C programs that do not contain an infinite loop is *not* recursive (see p. 137).

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[a] Need a program that returns "yes" iff the input is a palindrome.

[b] Need a program that returns "yes" iff the input is a prime.

[c] Need a program that returns "yes" iff the input C code does not contain any of the keywords.

Acceptability and Recursively Enumerable Languages

- Let $L \subseteq (\Sigma - \{\sqcup\})^*$ be a language.
- Let M be a TM such that for any string x:
  - If $x \in L$, then $M(x) =$ "yes."
  - If $x \notin L$, then $M(x) = \nearrow$.[a]
- We say M **accepts** L.

[a] This part is different from recursive languages.

Acceptability and Recursively Enumerable Languages (concluded)

- If L is accepted by some TM, then L is called a **recursively enumerable language**.[a]
  - A recursively enumerable language can be *generated* by a TM, thus the name.[b]
  - That is, there is an algorithm such that for every $x \in L$, it will be printed out eventually.
  - If L is infinite in size, this algorithm will not terminate.

[a] Post (1944).

[b] Thanks to a lively class discussion on September 20, 2011.

Emil Post (1897–1954)

Recursive and Recursively Enumerable Languages

Proposition 1 *If L is recursive, then it is recursively enumerable.*

- Let TM M decide L.
- Need to design a TM that accepts L.
- We will modify M to obtain an M' that accepts L.

The Proof (concluded)

- M' is identical to M except that when M is about to halt with a "no" state, M' goes into an infinite loop.
  - Simply replace any instruction that results in a "no" state with ones that move the cursor to the right forever and never halts.
- M' accepts L.
  - If $x \in L$, then $M'(x) = M(x) =$ "yes."
  - If $x \notin L$, then $M(x) =$ "no" and so $M'(x) = \nearrow$.