Undirected Graphs

- An **undirected graph** $G = (V, E)$ has a finite set of nodes, V, and a set of *undirected* edges, E.
- It is like a directed graph except that the edges have no directions and there are no self-loops.
- Use $[\,i, j\,]$ to denote the fact that there is an edge between node i and node j.

Independent Sets

- Let $G = (V, E)$ be an undirected graph.
- $I \subseteq V$.
- I is **independent** if there is no edge between any two nodes $i, j \in I$.
- The INDEPENDENT SET problem: Given an undirected graph and a goal K, is there an independent set of size K?
- Many applications.

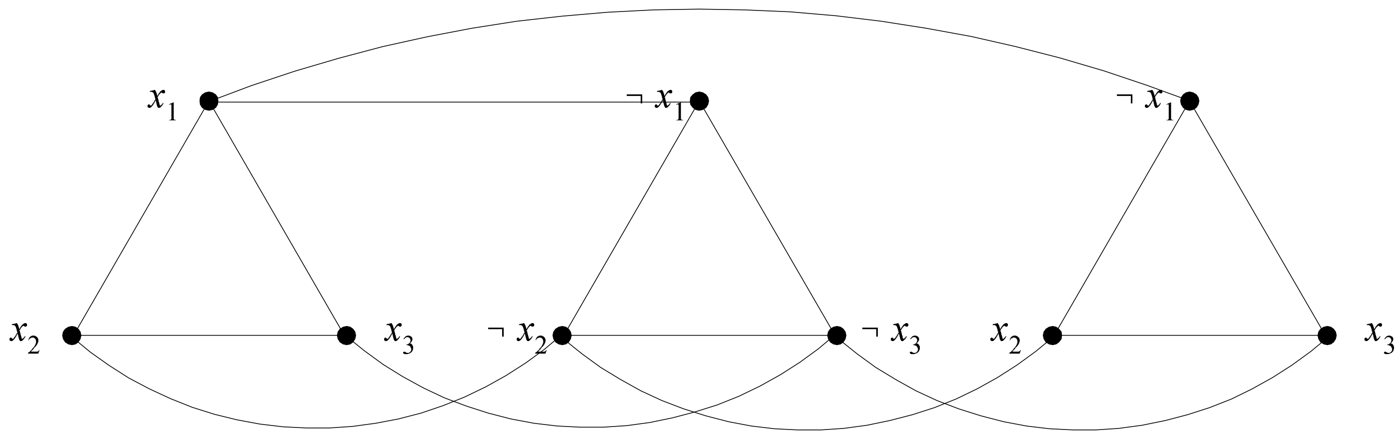
INDEPENDENT SET Is NP-Complete

- This problem is in NP: Guess a set of nodes and verify that it is independent and meets the count.
- We will reduce 3SAT to INDEPENDENT SET.
- If a graph contains a triangle, any independent set can contain at most one node of the triangle.
- The results of the reduction will be graphs whose nodes can be partitioned into m disjoint triangles.

The Proof (continued)

- Let ϕ be an instance of 3SAT with m clauses.
- We will construct graph G with $K = m$.
- Furthermore, ϕ is satisfiable if and only if G has an independent set of size K.
  - Here is the reduction:
  - There is a triangle for each clause with the literals as the nodes.
  - Add edges between x and $\neg x$ for every variable x.

$$(x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3)$$



Same literal labels that appear in different clauses yield *distinct* nodes.

The Proof (continued)

- Suppose G has an independent set I of size $K = m$.
  - An independent set can contain at most m nodes, one from each triangle.
  - So I contains exactly one node from each triangle.
  - Truth assignment T assigns true to those literals in I.
  - T is consistent because contradictory literals are connected by an edge; hence both cannot be in I.
  - T satisfies ϕ because it has a node from every triangle, thus satisfying every clause.[a]

[a] The variables without a truth value can be assigned arbitrarily. Contributed by Mr. Chun-Yuan Chen (`R99922119`) on November 2, 2010.

The Proof (concluded)

- Suppose a satisfying truth assignment T exists for ϕ.
  - Collect one node from each triangle whose literal is true under T.
  - The choice is arbitrary if there is more than one true literal.
  - This set of m nodes must be independent by construction.
    - Both literals x and $\neg x$ cannot be assigned true.

Other INDEPENDENT SET-Related NP-Complete Problems

Corollary 37 INDEPENDENT SET *is NP-complete for 4-degree graphs.*

Theorem 38 INDEPENDENT SET *is NP-complete for planar graphs.*

Theorem 39 (Garey and Johnson (1977)) INDEPENDENT SET *is NP-complete for 3-degree planar graphs.*

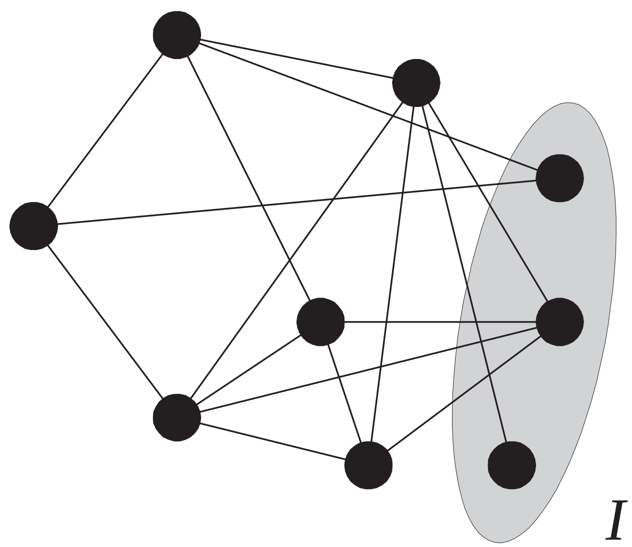
NODE COVER

- We are given an undirected graph G and a goal K.
- NODE COVER: Is there a set C with K or fewer nodes such that each edge of G has at least one of its endpoints (i.e., incident nodes) in C?
- Many applications.

NODE COVER Is NP-Complete

Corollary 40 (Karp (1972)) NODE COVER *is NP-complete.*

- I is an independent set of $G = (V, E)$ if and only if $V - I$ is a node cover of G.



CLIQUE

- We are given an undirected graph G and a goal K.
- CLIQUE asks if there is a set C with K nodes such that there is an edge between any two nodes $i, j \in C$.
- Many applications.

Remarks[a]

- Are INDEPENDENT SET and NODE COVER NP-complete if K is a constant?
  - No, because one can do an exhaustive search on all the possible node covers or independent sets (both $\binom{n}{k}$ of them, a polynomial).[b]
- Are INDEPENDENT SET and NODE COVER NP-complete if K is a linear function of n?
  - INDEPENDENT SET with $K = n/3$ and NODE COVER with $K = 2n/3$ remain NP-complete by our reductions.

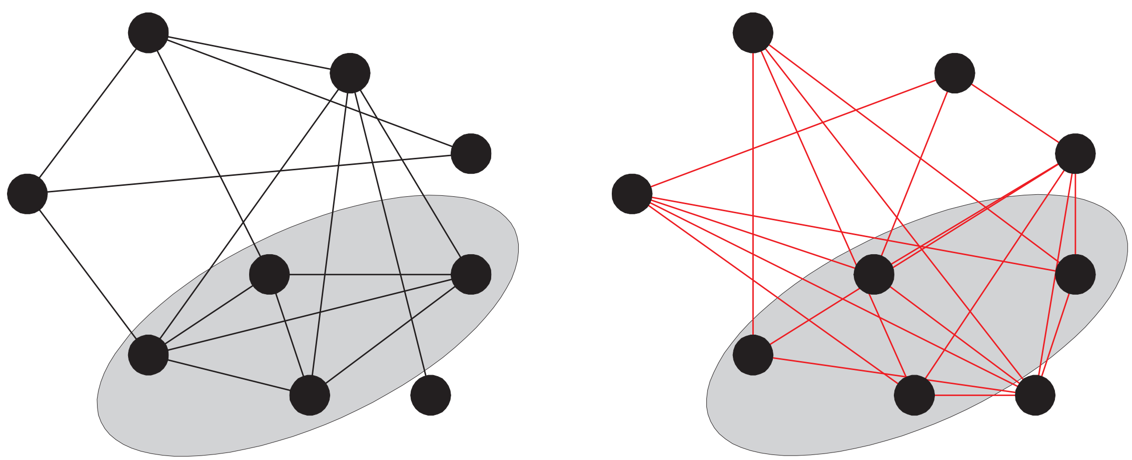
---

[a] Contributed by Mr. Ching-Hua Yu (`D00921025`) on October 30, 2012.
[b] $n = |\,V\,|$.

CLIQUE Is NP-Complete

Corollary 41 (Karp (1972)) CLIQUE *is NP-complete.*

- Let $\bar{G}$ be the **complement** of G, where $[x, y] \in \bar{G}$ if and only if $[x, y] \notin G$.
- I is a clique in $G \Leftrightarrow I$ is an independent set in $\bar{G}$.

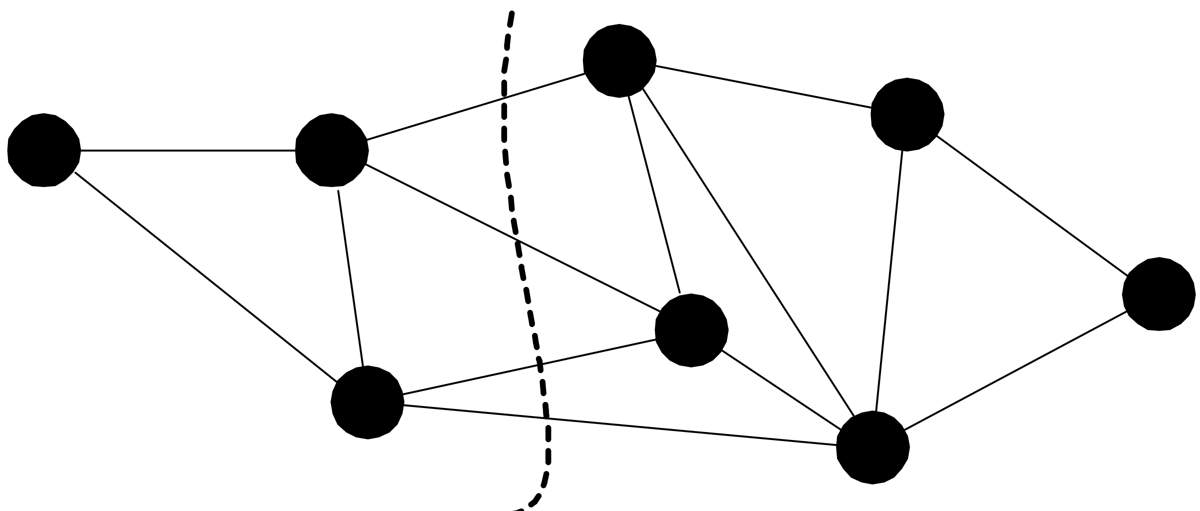


MIN CUT and MAX CUT

- A **cut** in an undirected graph $G = (V, E)$ is a partition of the nodes into two nonempty sets S and $V - S$.
- The size of a cut $(S, V - S)$ is the number of edges between S and $V - S$.
- MIN CUT $\in$ P by the maxflow algorithm.[a]
- MAX CUT asks if there is a cut of size at least K.
  - K is part of the input.

[a] In time $O(|V| \cdot |E|)$ by Orlin (2012).

A Cut of Size 4



MIN CUT and MAX CUT (concluded)

- MAX CUT has applications in circuit layout.
  - The minimum area of a VLSI layout of a graph is not less than the square of its maximum cut size.[a]

[a] Raspaud, Sýkora, and Vrťo (1995); Mak and Wong (2000).

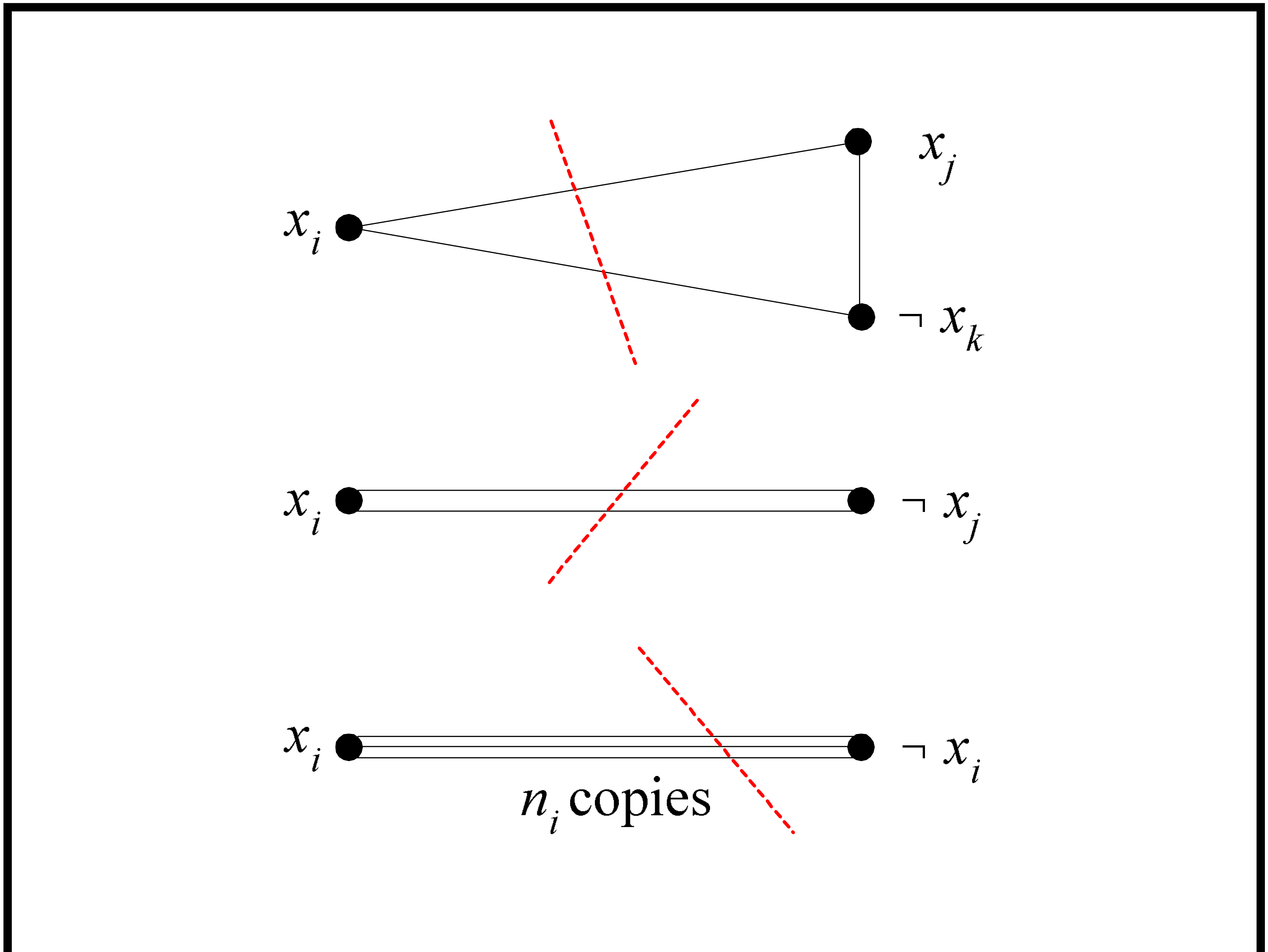
MAX CUT Is NP-Complete[a]

- We will reduce NAESAT to MAX CUT.
- Given an instance ϕ of 3SAT with m clauses, we shall construct a graph $G = (V, E)$ and a goal K.
- Furthermore, there is a cut of size at least K if and only if ϕ is NAE-satisfiable.
- Our graph will have multiple edges between two nodes.
  - Each such edge contributes one to the cut if its nodes are separated.

[a]Karp (1972) and Garey, Johnson, and Stockmeyer (1976).

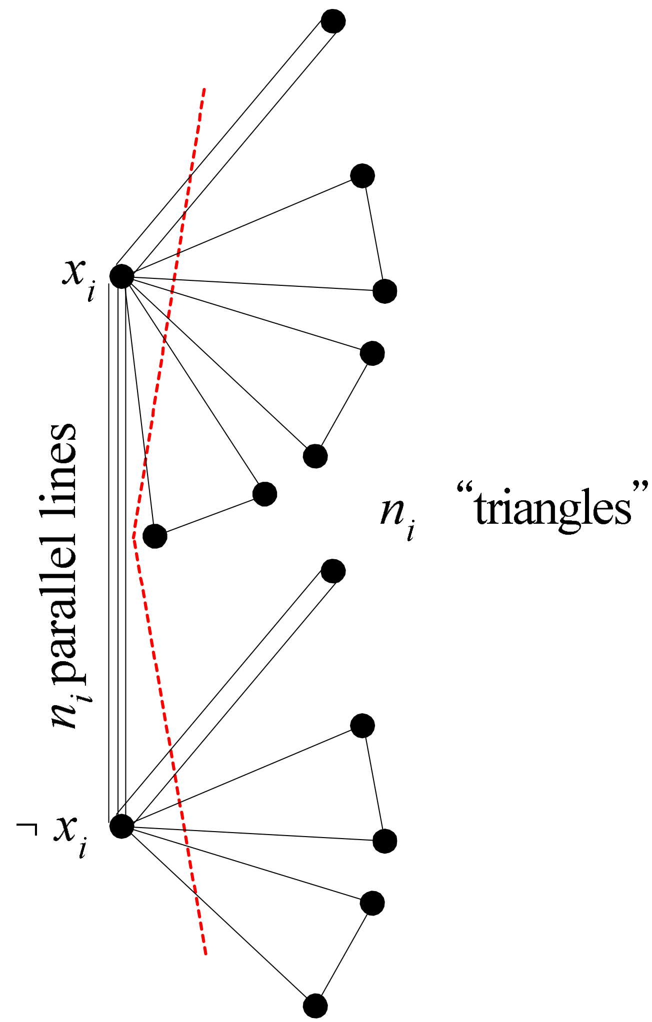
The Proof

- Suppose ϕ's m clauses are $C_1, C_2, \ldots, C_m$.
- The boolean variables are $x_1, x_2, \ldots, x_n$.
- G has $2n$ nodes: $x_1, x_2, \ldots, x_n, \neg x_1, \neg x_2, \ldots, \neg x_n$.
- Each clause with 3 distinct literals makes a triangle in G.
- For each clause with two identical literals, there are two parallel edges between the two distinct literals.
- No need to consider clauses with one literal (why?).
- For each variable x_i, add n_i copies of edge $[x_i, \neg x_i]$, where n_i is the number of occurrences of x_i and $\neg x_i$ in ϕ.


x_j
x_i
$\neg\ x_k$
x_i
$\neg\ x_j$
x_i
$\neg\ x_i$
n_i copies

The Proof (continued)

- Set $K = 5m$.
- Suppose there is a cut $(S, V - S)$ of size $5m$ or more.
- A clause (a triangle or two parallel edges) contributes at most 2 to a cut no matter how you split it.
- Suppose both x_i and $\neg x_i$ are on the same side of the cut.
- They *together* contribute at most $2n_i$ edges to the cut.
  - They appear in at most n_i different clauses.
  - A clause contributes at most 2 to a cut.



x_i
n_i parallel lines
$\neg x_i$
n_i "triangles"

The Proof (continued)

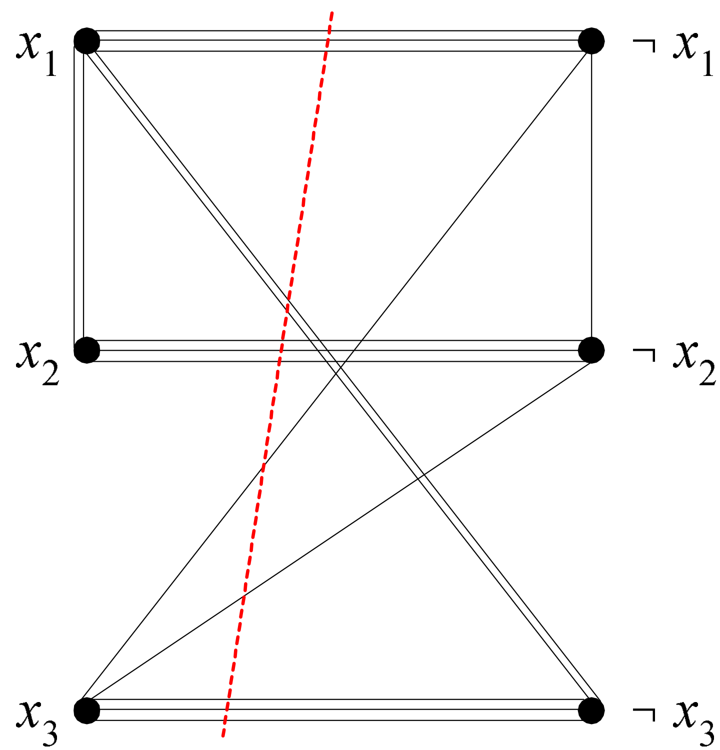
- Either x_i or $\neg x_i$ contributes at most n_i to the cut by the pigeonhole principle.
- Changing the side of that literal does *not decrease* the size of the cut.
- Hence we assume variables are separated from their negations.
- The total number of edges in the cut that join opposite literals x_i and $\neg x_i$ is $\sum_{i=1}^{n} n_i$.
- But $\sum_{i=1}^{n} n_i = 3m$ as it is simply the total number of literals.

The Proof (concluded)

- The *remaining* $K - 3m \geq 2m$ edges in the cut must come from the m triangles or parallel edges that correspond to the clauses.
- Each can contribute at most 2 to the cut.[a]
- So all are split.
- A split clause means at least one of its literals is true and at least one false.
- The other direction is left as an exercise.

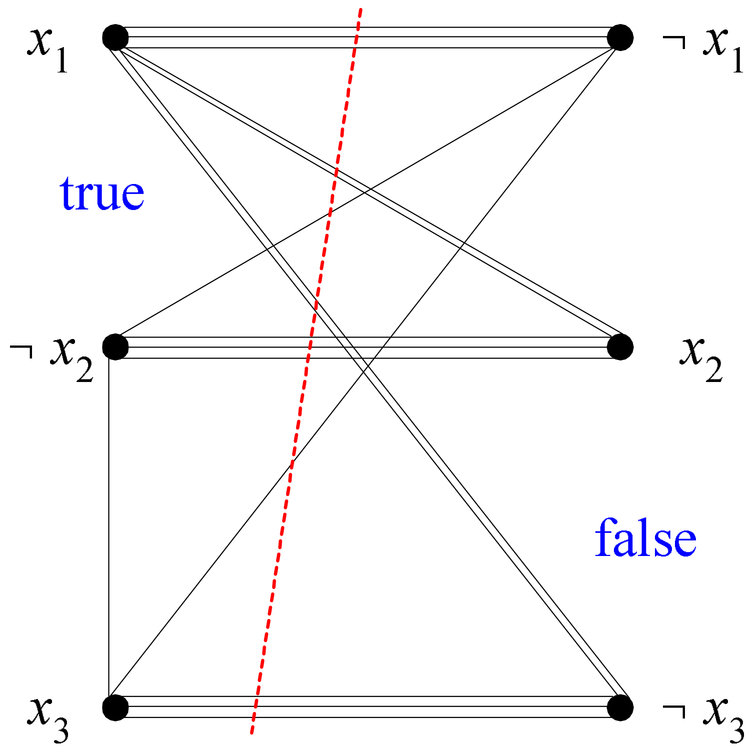
[a] So $K = 5m$.

A Cut That Does Not Meet the Goal $K = 5 \times 3 = 15$



- $(x_1 \lor x_2 \lor x_2) \land (x_1 \lor \neg x_3 \lor \neg x_3) \land (\neg x_1 \lor \neg x_2 \lor x_3)$.
- The cut size is $13 < 15$.

A Cut That Meets the Goal $K = 5 \times 3 = 15$



- $(x_1 \lor x_2 \lor x_2) \land (x_1 \lor \neg x_3 \lor \neg x_3) \land (\neg x_1 \lor \neg x_2 \lor x_3)$.
- The cut size is now 15.

Remarks

- We had proved that MAX CUT is NP-complete for multigraphs.
- How about proving the same thing for simple graphs?[a]
- How to modify the proof to reduce 4SAT to MAX CUT?[b]
- All NP-complete problems are mutually reducible by definition.[c]
  - So they are equally hard in this sense.[d]

[a] Contributed by Mr. Tai-Dai Chou (J93922005) on June 2, 2005.
[b] Contributed by Mr. Chien-Lin Chen (J94922015) on June 8, 2006.
[c] Contributed by Mr. Ren-Shuo Liu (D98922016) on October 27, 2009.
[d] Contributed by Mr. Ren-Shuo Liu (D98922016) on October 27, 2009.

MAX BISECTION

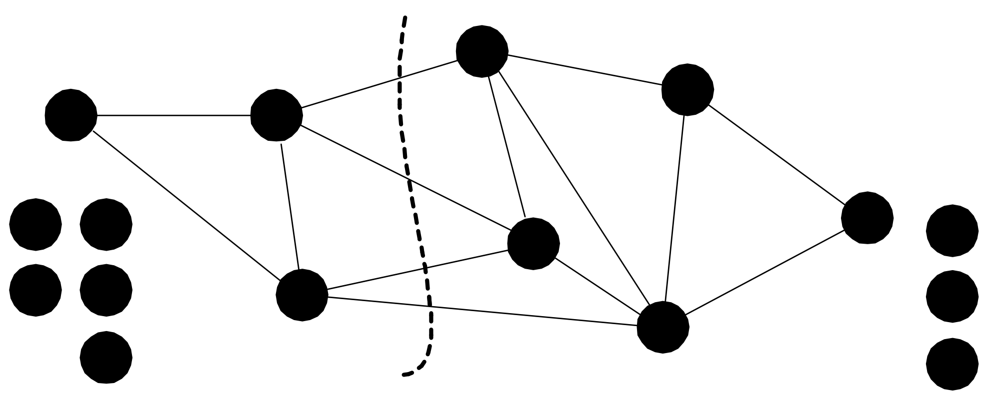
- MAX CUT becomes MAX BISECTION if we require that $|S| = |V - S|$.
- It has many applications, especially in VLSI layout.

MAX BISECTION Is NP-Complete

- We shall reduce the more general MAX CUT to MAX BISECTION.
- Add $|V| = n$ **isolated nodes** to G to yield G'.
- G' has $2n$ nodes.
- G''s goal K is identical to G's
  - As the new nodes have no edges, they contribute nothing to the cut.
- This completes the reduction.

The Proof (concluded)

- Every cut $(S, V - S)$ of $G = (V, E)$ can be made into a bisection by appropriately allocating the new nodes between S and $V - S$.
- Hence each cut of G can be made a cut of G' of the same size, and vice versa.



BISECTION WIDTH

- BISECTION WIDTH is like MAX BISECTION except that it asks if there is a bisection of size *at most* K (sort of MIN BISECTION).
- Unlike MIN CUT, BISECTION WIDTH is NP-complete.
- We reduce MAX BISECTION to BISECTION WIDTH.
- Given a graph $G = (V, E)$, where $|V|$ is even, we generate the complement of G.
- Given a goal of K, we generate a goal of $n^2 - K$.

The Proof (concluded)

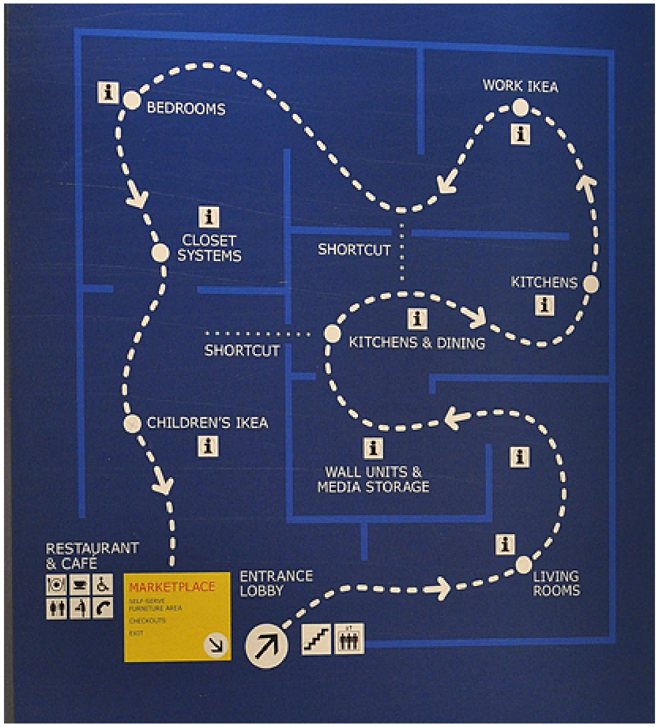
- To show the reduction works, simply notice the following easily verifiable claims.
  - A graph $G = (V, E)$, where $|V| = 2n$, has a bisection of size K if and only if the complement of G has a bisection of size $n^2 - K$.
  - So G has a bisection of size $\geq K$ if and only if its complement has a bisection of size $\leq n^2 - K$.

HAMILTONIAN PATH Is NP-Complete[a]

Theorem 42 *Given an* undirected *graph, the question whether it has a Hamiltonian path is NP-complete.*

[a]Karp (1972).

A Hamiltonian Path at IKEA, Covina, California?



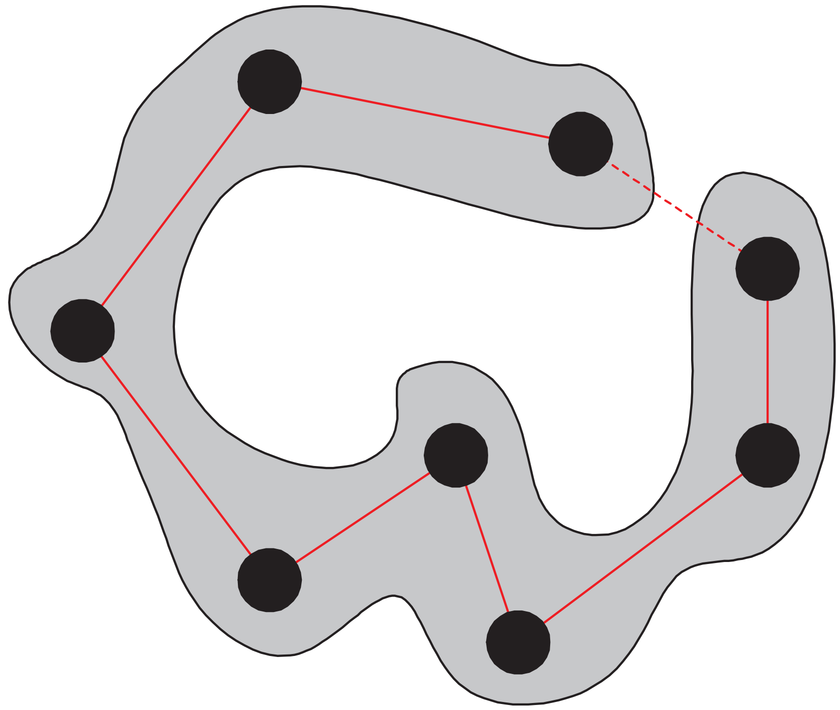
TSP (D) Is NP-Complete

Corollary 43 TSP (D) *is NP-complete.*

- Consider a graph G with n nodes.
- Create a weighted complete graph G' with the same nodes as from G follows.
- Set $d_{ij} = 1$ on G' if $[\, i, j\,] \in G$ and $d_{ij} = 2$ on G' if $[\, i, j\,] \notin G$.
  - Note that G' is a complete graph.
- Set the budget $B = n + 1$.
- This completes the reduction.

TSP (D) Is NP-Complete (continued)

- Suppose G' has a tour of distance at most $n + 1$.
- Then that tour on G' must contain at most one edge with weight 2.
- If a tour on G' contains 1 edge with weight 2, remove that edge to arrive at a Hamiltonian path for G.
- If, on the other hand, a tour on G' contains no edge with weight 2.
- Remove any edge to arrive at a Hamiltonian path for G.



TSP (D) Is NP-Complete (concluded)

- On the other hand, suppose G has Hamiltonian paths.
- Then there is a tour on G' containing at most one edge with weight 2.
- The total cost is then at most $(n-1)+2 = n+1 = B$.
- We conclude that there is a tour of length B or less on G' if and only if G has a Hamiltonian path.

Graph Coloring

- k-COLORING: Can the nodes of a graph be colored with $\leq k$ colors such that no two adjacent nodes have the same color?[a]
- 2-COLORING is in P (why?).
- But 3-COLORING is NP-complete (see next page).
- k-COLORING is NP-complete for $k \geq 3$ (why?).
- EXACT-k-COLORING asks if the nodes of a graph can be colored using exactly k colors.
- It remains NP-complete for $k \geq 3$ (why?).

[a] k is *not* part of the input; k is part of the problem statement.

3-COLORING Is NP-Complete[a]

- We will reduce NAESAT to 3-COLORING.
- We are given a set of clauses $C_1, C_2, \ldots, C_m$ each with 3 literals.
- The boolean variables are $x_1, x_2, \ldots, x_n$.
- We shall construct a graph G such that it can be colored with colors $\{0, 1, 2\}$ if and only if all the clauses can be NAE-satisfied.

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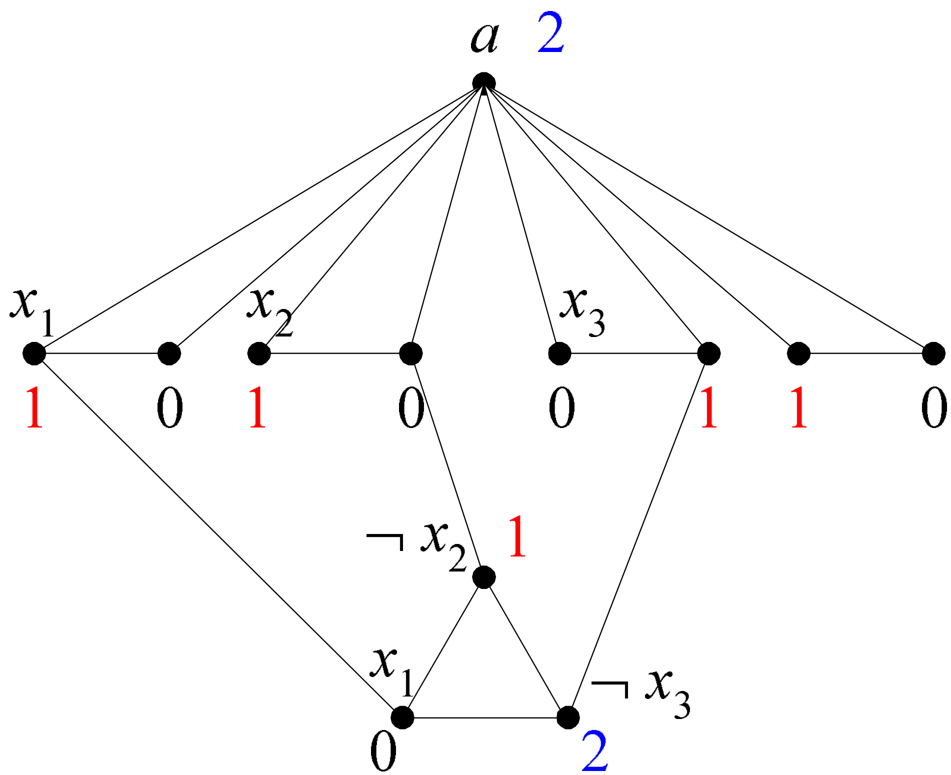
[a]Karp (1972).

The Proof (continued)

- Every variable x_i is involved in a triangle $[\,a, x_i, \neg x_i\,]$ with a common node a.
- Each clause $C_i = (c_{i1} \vee c_{i2} \vee c_{i3})$ is also represented by a triangle
$$[\,c_{i1}, c_{i2}, c_{i3}\,].$$
  - Node c_{ij} with the same label as one in some triangle $[\,a, x_k, \neg x_k\,]$ represent *distinct* nodes.
- There is an edge between c_{ij} and the node that represents the jth literal of C_i.
  - Alternative proof: there is an edge between $\neg c_{ij}$ and the node that represents the jth literal of C_i.[a]

[a] Contributed by Mr. Ren-Shuo Liu (`D98922016`) on October 27, 2009.

Construction for $\cdots \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \wedge \cdots$



The Proof (continued)

Suppose the graph is 3-colorable.

- Assume without loss of generality that node a takes the color 2.
- A triangle must use up all 3 colors.
- As a result, one of x_i and $\neg x_i$ must take the color 0 and the other 1.

The Proof (continued)

- Treat 1 as `true` and 0 as `false`.[a]
  - We were dealing only with those triangles with the "a" node, not the clause triangles.
- The resulting truth assignment is clearly contradiction free.
- As each clause triangle contains one color 1 and one color 0, the clauses are NAE-satisfied.

[a] The opposite also works.

The Proof (continued)

Suppose the clauses are NAE-satisfiable.

- Color node a with color 2.
- Color the nodes representing literals by their truth values (color 0 for `false` and color 1 for `true`).
  - We were dealing only with those triangles with the "a" node, not the clause triangles.

The Proof (continued)

- For each clause triangle:
  - Pick any two literals with opposite truth values.
  - Color the corresponding nodes with 0 if the literal is `true` and 1 if it is `false`.
  - Color the remaining node with color 2.

The Proof (concluded)

- The coloring is legitimate.
  - If literal w of a clause triangle has color 2, then its color will never be an issue.
  - If literal w of a clause triangle has color 1, then it must be connected up to literal w with color 0.
  - If literal w of a clause triangle has color 0, then it must be connected up to literal w with color 1.

Algorithms for 3-COLORING and the Chromatic Number $\chi(G)$

- Assume G is 3-colorable.
- There is an algorithm to find a 3-coloring in time $O(3^{n/3}) = 1.4422^n$.[a]
- It has been improved to $O(1.3289^n)$.[b]

---

[a] Lawler (1976).
[b] Beigel and Eppstein (2000).

Algorithms for 3-COLORING and the Chromatic Number $\chi(G)$ (concluded)

- The **chromatic number** $\chi(G)$ is the smallest number of colors needed to color a graph G.
- There is an algorithm to find $\chi(G)$ in time $O((4/3)^{n/3}) = 2.4422^n$.[a]
- It can be improved to $O((4/3 + 3^{4/3}/4)^n) = O(2.4150^n)$[b] and $2^n n^{O(1)}$.[c]
- Computing $\chi(G)$ cannot be easier than 3-COLORING.[d]

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[a] Lawler (1976).
[b] Eppstein (2003).
[c] Koivisto (2006).
[d] Contributed by Mr. Ching-Hua Yu (`D00921025`) on October 30, 2012.

TRIPARTITE MATCHING

- We are given three sets B, G, and H, each containing n elements.
- Let $T \subseteq B \times G \times H$ be a ternary relation.
- TRIPARTITE MATCHING asks if there is a set of n triples in T, none of which has a component in common.
  - Each element in B is matched to a different element in G and different element in H.

Theorem 44 (Karp (1972)) TRIPARTITE MATCHING *is NP-complete.*