Theory of Computation

Solutions to Homework 1

Problem 1. Consider a deterministic k-tape Turing machine with q states and σ alphabetic symbols. Suppose this Turing machine halts after using a maximum of h cells on each of the tapes. What is the maximum running time?

Proof. $q \times \sigma^{hk} \times h^k$ $\square$

Problem 2. Cantor's theorem says that the set of all subsets of $\mathbb{N}$ (i.e. $2^{\mathbb{N}}$) is infinite and not countable. But consider the following counterargument. Let $p_1 < p_2 < p_3 < \cdots$ be all the prime numbers. Define the following function from $2^{\mathbb{N}}$ to $\mathbb{N}$:

$$f(X) = p_1^{n_1} p_2^{n_2} p_3^{n_3} \ldots,$$

where $X = \{n_1, n_2, n_3, \ldots\}$ and $n_1 < n_2 < n_3 < \cdots$. Clearly, f maps every subset of $\mathbb{N}$ into some number of $\mathbb{N}$. So, $2^{\mathbb{N}}$ is countable, contradicting Cantor's theorem. What is wrong with the argument? [1]

Proof. Suppose $X = \mathbb{N} = \{1, 2, 3, \ldots\}$. Then $f(X) = f(\mathbb{N}) = p_1^1 p_2^2 p_3^3 \cdots$, an infinite number. Since infinity is not a natural number, there is a trivial subset of $\mathbb{N}$ mapped by f into a number not in $\mathbb{N}$. Hence f is not a valid mapping. $\square$

[1] This problem was contributed by Mr. Wen-Chie Yang (`R92922069`) on October 3, 2003.