Theory of Computation

Solutions to Homework 5

Problem 1. Let A, B be finite nonempty sets, $f : A \times B \to \{0, 1\}$ and $\sum_{y \in B} f(x, y) < |\,B\,|/|\,A\,|$ for all $x \in A$. Prove the existence of a $y^* \in B$ with $\sum_{x \in A} f(x, y^*) = 0$. You may want to use the fact

$$\sum_{x \in A} \sum_{y \in B} f(x, y) = \sum_{y \in B} \sum_{x \in A} f(x, y).$$

Proof. As $\sum_{y \in B} f(x, y) < |\,B\,|/|\,A\,|$ for $x \in A$,

$$\sum_{x \in A} \sum_{y \in B} f(x, y) < \sum_{x \in A} \frac{|\,B\,|}{|\,A\,|} = |\,B\,|. \tag{1}$$

Suppose for contradiction that

$$\sum_{x \in A} f(x, y) \geq 1$$

for all $y \in B$. Then

$$\sum_{y \in B} \sum_{x \in A} f(x, y) \geq \sum_{y \in B} 1 = |\,B\,|,$$

contradicting inequality (1). □

Problem 2. Does IP contain all languages that have uniformly polynomial circuits?

Proof. Yes. P equals the class of languages with uniformly polynomial circuits. Furthermore, any language in P can be decided by an interactive proof system where the verifier simply decides the language itself and ignores the prover's messages. So P $\subseteq$ IP. □