Monte Carlo Algorithms[a]

- The randomized bipartite perfect matching algorithm is called a **Monte Carlo algorithm** in the sense that
  - If the algorithm finds that a matching exists, it is always correct (no **false positives**).
  - If the algorithm answers in the negative, then it may make an error (**false negative**).

[a] Metropolis and Ulam (1949).

Monte Carlo Algorithms (concluded)

- The algorithm makes a false negative with probability ≤ 0.5.
  - Note this probability refers to

$$\text{prob}[\,\text{algorithm answers "no"} \mid G \text{ has a perfect matching}\,]$$

not

$$\text{prob}[\,G \text{ has a perfect matching} \mid \text{algorithm answers "no"}\,].$$

- This probability is *not* over the space of all graphs or determinants, but *over* the algorithm's own coin flips.
  - It holds for *any* bipartite graph.

False Positives and False Negatives in Human Behavior?[a]

- "[Men] tend to misinterpret innocent friendliness as a sign that women are [$\cdots$] interested in them."
  - A false positive.
- "[Women] tend to undervalue signs that a man is interested in a committed relationship."
  - A false negative.

---

[a] "Don't misunderestimate yourself." *The Economist*, 2006.

The Markov Inequality[a]

Lemma 64 *Let x be a random variable taking nonnegative integer values. Then for any $k > 0$,*

$$\text{prob}[x \geq kE[x]] \leq 1/k.$$

- Let p_i denote the probability that $x = i$.

$$\begin{aligned} E[x] &= \sum_i i p_i \\ &= \sum_{i < kE[x]} i p_i + \sum_{i \geq kE[x]} i p_i \\ &\geq kE[x] \times \text{prob}[x \geq kE[x]]. \end{aligned}$$

[a] Andrei Andreyevich Markov (1856–1922).

Andrei Andreyevich Markov (1856–1922)

An Application of Markov's Inequality

- Algorithm C runs in expected time $T(n)$ and always gives the right answer.
- Consider an algorithm that runs C for time $kT(n)$ and rejects the input if C does not stop within the time bound.
- By Markov's inequality, this new algorithm runs in time $kT(n)$ and gives the wrong answer with probability $\leq 1/k$.
- By running this algorithm m times, we reduce the error probability to $\leq k^{-m}$.

An Application of Markov's Inequality (concluded)

- Suppose, instead, we run the algorithm for the same running time $mkT(n)$ once and rejects the input if it does not stop within the time bound.
- By Markov's inequality, this new algorithm gives the wrong answer with probability $\leq 1/(mk)$.
- This is a far cry from the previous algorithm's error probability of $\leq k^{-m}$.

FSAT for k-SAT Formulas (p. 427)

- Let $\phi(x_1, x_2, \ldots, x_n)$ be a k-SAT formula.
- If ϕ is satisfiable, then return a satisfying truth assignment.
- Otherwise, return "no."
- We next propose a randomized algorithm for this problem.

A Random Walk Algorithm for ϕ in CNF Form

Start with an *arbitrary* truth assignment T;
for $i = 1, 2, \ldots, r$ **do**
  if $T \models \phi$ **then**
    return "ϕ is satisfiable with T";
  else
    Let c be an unsatisfiable clause in ϕ under T; {All of its literals are false under T.}
    Pick any x of these literals *at random*;
    Modify T to make x true;
  end if
end for
return "ϕ is unsatisfiable";

3SAT vs. 2SAT Again

- Note that if ϕ is unsatisfiable, the algorithm will not refute it.
- The random walk algorithm needs expected exponential time for 3SAT.
  - In fact, it runs in expected $O((1.333\cdots+\epsilon)^n)$ time with $r = 3n$,[a] much better than $O(2^n)$.[b]
- We will show immediately that it works well for 2SAT.
- The state of the art as of 2006 is expected $O(1.322^n)$ time for 3SAT and expected $O(1.474^n)$ time for 4SAT.[c]

---

[a]Use this setting per run of the algorithm.
[b]Schöning (1999).
[c]Kwama and Tamaki (2004); Rolf (2006).

Random Walk Works for 2SAT[a]

Theorem 65 *Suppose the random walk algorithm with $r = 2n^2$ is applied to any satisfiable 2SAT problem with n variables. Then a satisfying truth assignment will be discovered with probability at least 0.5.*

- Let $\hat{T}$ be a truth assignment such that $\hat{T} \models \phi$.
- Let $t(i)$ denote the expected number of repetitions of the flipping step until a satisfying truth assignment is found if our starting T differs from $\hat{T}$ in i values.
  - Their Hamming distance is i.
  - Recall T is arbitrary.

[a] Papadimitriou (1991).

The Proof

- It can be shown that $t(i)$ is finite.
- $t(0) = 0$ because it means that $T = \hat{T}$ and hence $T \models \phi$.
- If $T \neq \hat{T}$ or T is not equal to any other satisfying truth assignment, then we need to flip at least once.
- We flip to pick among the 2 literals of a clause not satisfied by the present T.
- At least one of the 2 literals is true under $\hat{T}$ because $\hat{T}$ satisfies all clauses.
- So we have at least 0.5 chance of moving closer to $\hat{T}$.

The Proof (continued)

- Thus
$$t(i) \leq \frac{t(i-1)+t(i+1)}{2}+1$$
for $0 < i < n$.
  - Inequality is used because, for example, T may differ from $\hat{T}$ in both literals.
- It must also hold that
$$t(n) \leq t(n-1)+1$$
because at $i = n$, we can only decrease i.

The Proof (continued)

- As we are only interested in upper bounds, we solve

$$
\begin{aligned}
x(0) &= 0 \\
x(n) &= x(n-1)+1 \\
x(i) &= \frac{x(i-1)+x(i+1)}{2}+1, \quad 0<i<n
\end{aligned}
$$

- This is one-dimensional random walk with a reflecting and an absorbing barrier.

The Proof (continued)

- Add the equations up to obtain

$$\begin{aligned} & x(1) + x(2) + \cdots + x(n) \\ = \quad & \frac{x(0) + x(1) + 2x(2) + \cdots + 2x(n-2) + x(n-1) + x(n)}{2} \\ & +n + x(n-1). \end{aligned}$$

- Simplify to yield

$$\frac{x(1) + x(n) - x(n-1)}{2} = n.$$

- As $x(n) - x(n-1) = 1$, we have

$$x(1) = 2n - 1.$$

The Proof (continued)

- Iteratively, we obtain

$$\begin{aligned} x(2) &= 4n-4, \\ &\vdots \\ x(i) &= 2in-i^2. \end{aligned}$$

- The worst case happens when $i = n$, in which case

$$x(n) = n^2.$$

The Proof (concluded)

- We therefore reach the conclusion that

$$t(i) \leq x(i) \leq x(n) = n^2.$$

- So the expected number of steps is at most n^2.
- The algorithm picks a running time $2n^2$.
- This amounts to invoking the Markov inequality (p. 462) with $k = 2$, with the consequence of having a probability of 0.5.
- The proof does not yield a polynomial bound for 3SAT.[a]

[a]Contributed by Mr. Cheng-Yu Lee (R95922035) on November 8, 2006.

Boosting the Performance

- We can pick $r = 2mn^2$ to have an error probability of $\leq (2m)^{-1}$ by Markov's inequality.
- Alternatively, with the same running time, we can run the "$r = 2n^2$" algorithm m times.
- But the error probability is reduced to $\leq 2^{-m}$!

Primality Tests

- PRIMES asks if a number N is a prime.
- The classic algorithm tests if $k \mid N$ for $k = 2, 3, \ldots, \sqrt{N}$.
- But it runs in $\Omega(2^{n/2})$ steps, where $n = \mid N \mid = \log_2 N$.

The Density Attack for PRIMES

Pick $k \in \{2, \ldots, N-1\}$ randomly; {Assume $N > 2$.}
if $k \mid N$ **then**
    return "N is composite";
else
    return "N is a prime";
end if

Analysis[a]

- Suppose $N = PQ$, a product of 2 primes.
- The probability of success is

$$< 1 - \frac{\phi(N)}{N} = 1 - \frac{(P-1)(Q-1)}{PQ} = \frac{P+Q-1}{PQ}.$$

- In the case where $P \approx Q$, this probability becomes

$$< \frac{1}{P} + \frac{1}{Q} \approx \frac{2}{\sqrt{N}}.$$

- This probability is exponentially small.

[a] See also p. 409.

The Fermat Test for Primality

Fermat's "little" theorem on p. 411 suggests the following primality test for any given number p:

```
Pick a number a randomly from {1, 2, ..., N − 1};
if a^(N−1) ≠ 1 mod N then
    return "N is composite";
else
    return "N is a prime";
end if
```

The Fermat Test for Primality (concluded)

- Unfortunately, there are composite numbers called **Carmichael numbers** that will pass the Fermat test for *all* $a \in \{1, 2, \ldots, N-1\}$.[a]
- There are infinitely many Carmichael numbers.[b]
- In fact, the number of Carmichael numbers less than n exceeds $n^{2/7}$ for n large enough.

---

[a] Carmichael (1910).
[b] Alford, Granville, and Pomerance (1992).

Square Roots Modulo a Prime

- Equation $x^2 = a \bmod p$ has at most two (distinct) roots by Lemma 59 (p. 416).
  - The roots are called **square roots**.
  - Numbers a with square roots *and* $\gcd(a, p) = 1$ are called **quadratic residues**.
    * They are $1^2 \bmod p, 2^2 \bmod p, \ldots, (p-1)^2 \bmod p$.
- We shall show that a number either has two roots or has none, and testing which one is true is trivial.
- There are no known efficient *deterministic* algorithms to find the roots, however.

Euler's Test

Lemma 66 (Euler) *Let p be an odd prime and* $a \neq 0 \bmod p$.

1. *If $a^{(p-1)/2} = 1 \bmod p$, then $x^2 = a \bmod p$ has two roots.*
2. *If $a^{(p-1)/2} \neq 1 \bmod p$, then $a^{(p-1)/2} = -1 \bmod p$ and $x^2 = a \bmod p$ has no roots.*

- Let r be a primitive root of p.
- By Fermat's "little" theorem, $r^{(p-1)/2}$ is a square root of 1, so $r^{(p-1)/2} = 1 \bmod p$ or $r^{(p-1)/2} = -1 \bmod p$.
- But as r is a primitive root, $r^{(p-1)/2} \neq 1 \bmod p$.
- Hence $r^{(p-1)/2} = -1 \bmod p$.

The Proof (continued)

- Let $a = r^k \bmod p$ for some k.
- Then
$$1 = a^{(p-1)/2} = r^{k(p-1)/2} = \left[r^{(p-1)/2} \right]^k = (-1)^k \bmod p.$$
- So k must be even.
- Suppose $a = r^{2j}$ for some $1 \leq j \leq (p-1)/2$.
- Then $a^{(p-1)/2} = r^{j(p-1)} = 1 \bmod p$ and its two *distinct* roots are $r^j, -r^j (= r^{j+(p-1)/2} \bmod p)$.
  - If $r^j = -r^j \bmod p$, then $2r^j = 0 \bmod p$, which implies $r^j = 0 \bmod p$, a contradiction.

The Proof (continued)

- As $1 \leq j \leq (p-1)/2$, there are $(p-1)/2$ such a's.
- Each such a has 2 distinct square roots.
- The square roots of all the a's are distinct.
  - The square roots of different a's must be different.
- Hence the set of *square roots* is $\{1, 2, \ldots, p-1\}$.
  - Because there are $(p-1)/2$ such a's and each a has two square roots.
- As a result, $a = r^{2j}$, $1 \leq j \leq (p-1)/2$, exhaust all the quadratic residues.

The Proof (concluded)

- If $a = r^{2j+1}$, then it has no roots because all the square roots have been taken.
- Now,

$$a^{(p-1)/2} = \left[r^{(p-1)/2} \right]^{2j+1} = (-1)^{2j+1} = -1 \bmod p.$$

The Legendre Symbol[a] and Quadratic Residuacity Test

- By Lemma 66 (p. 483) $a^{(p-1)/2} \bmod p = \pm 1$ for $a \neq 0 \bmod p$.
- For odd prime p, define the **Legendre symbol** $(a \,|\, p)$ as

$$(a \,|\, p) = \begin{cases} 0 & \text{if } p \,|\, a, \\ 1 & \text{if } a \text{ is a quadratic residue modulo } p, \\ -1 & \text{if } a \text{ is a } \textbf{quadratic nonresidue} \text{ modulo } p. \end{cases}$$

- Euler's test implies $a^{(p-1)/2} = (a \,|\, p) \bmod p$ for any odd prime p and any integer a.
- Note that $(ab|p) = (a|p)(b|p)$.

[a] Andrien-Marie Legendre (1752–1833).

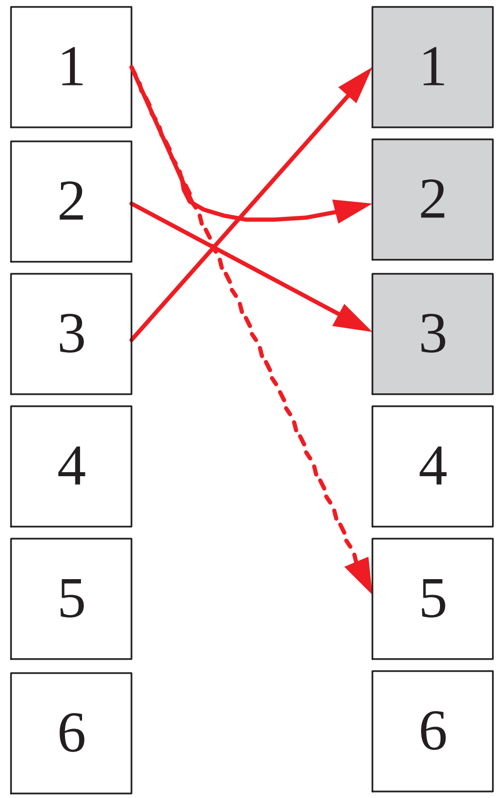
Gauss's Lemma

Lemma 67 (Gauss) *Let p and q be two odd primes. Then $(q|p) = (-1)^m$, where m is the number of residues in $R = \{\, iq \bmod p : 1 \leq i \leq (p-1)/2 \,\}$ that are greater than $(p-1)/2$.*

- All residues in R are distinct.
  - If $iq = jq \bmod p$, then $p|(j-i)\,q$ or $p|q$.
- No two elements of R add up to p.
  - If $iq + jq = 0 \bmod p$, then $p|(i+j)$ or $p|q$.
  - But neither is possible.

The Proof (continued)

- Consider the set R' of residues that result from R if we replace each of the m elements $a \in R$ such that $a > (p-1)/2$ by $p-a$.
  - This is equivalent to performing $-a \bmod p$.
- All residues in R' are now at most $(p-1)/2$.
- In fact, $R' = \{1, 2, \ldots, (p-1)/2\}$ (see illustration next page).
  - Otherwise, two elements of R would add up to p, which has been shown to be impossible.



$p = 7$ and $q = 5$.

The Proof (concluded)

- Alternatively, $R' = \{\pm iq \bmod p : 1 \leq i \leq (p-1)/2\}$, where exactly m of the elements have the minus sign.
- Take the product of all elements in the two representations of R'.
- So $[(p-1)/2]! = (-1)^m q^{(p-1)/2} [(p-1)/2]! \bmod p$.
- Because $\gcd([(p-1)/2]!, p) = 1$, the above implies

$$1 = (-1)^m q^{(p-1)/2} \bmod p.$$

Legendre's Law of Quadratic Reciprocity[a]

- Let p and q be two odd primes.
- The next result says their Legendre symbols are distinct if and only if both numbers are 3 mod 4.

Lemma 68 (Legendre (1785), Gauss)

$$(p|q)(q|p) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}.$$

[a] First stated by Euler in 1751. Legendre (1785) did not give a correct proof. Gauss proved the theorem when he was 19. He gave at least 6 different proofs during his life. The 152nd proof appeared in 1963.

The Proof (continued)

- Sum the elements of R' in the previous proof in mod2.
- On one hand, this is just $\sum_{i=1}^{(p-1)/2} i \bmod 2$.
- On the other hand, the sum equals

$$\begin{aligned} & \sum_{i=1}^{(p-1)/2} \left(qi - p \left\lfloor \frac{iq}{p} \right\rfloor \right) + mp \bmod 2 \\ = & \left(q \sum_{i=1}^{(p-1)/2} i - p \sum_{i=1}^{(p-1)/2} \left\lfloor \frac{iq}{p} \right\rfloor \right) + mp \bmod 2. \end{aligned}$$

  - Signs are irrelevant under mod2.
  - m is as in Lemma 67 (p. 488).

The Proof (continued)

- Ignore odd multipliers to make the sum equal

$$\left(\sum_{i=1}^{(p-1)/2} i - \sum_{i=1}^{(p-1)/2} \left\lfloor \frac{iq}{p} \right\rfloor \right) + m \bmod 2.$$

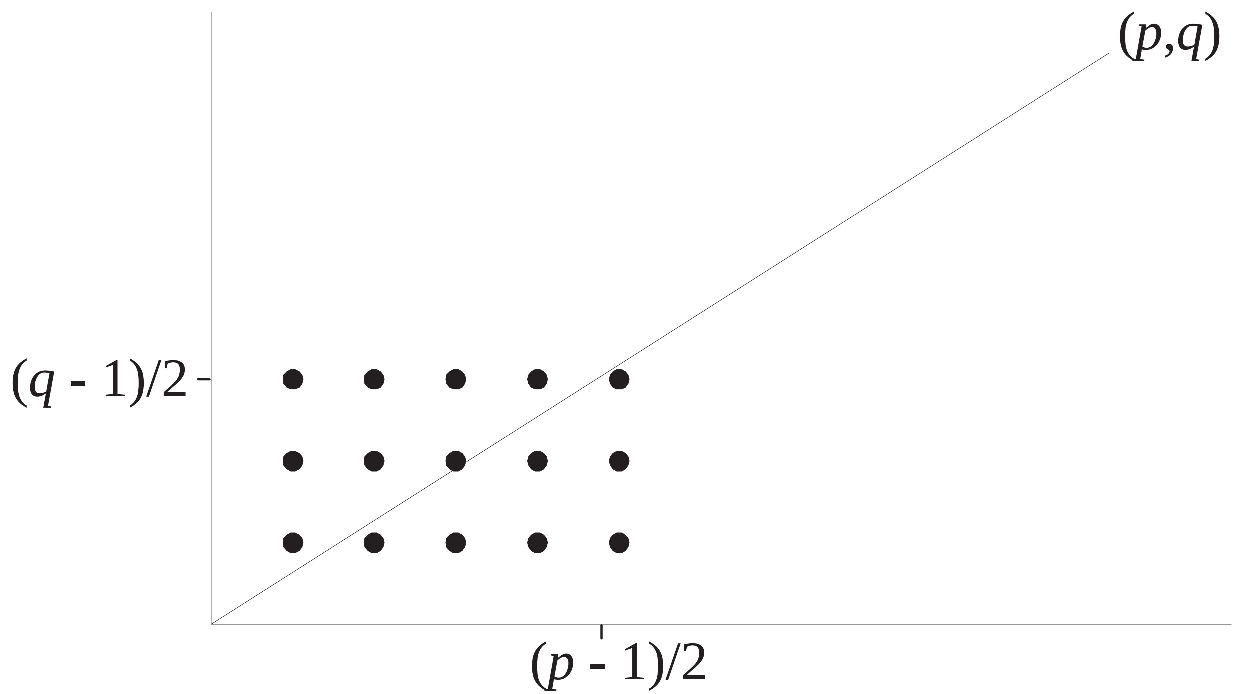
- Equate the above with $\sum_{i=1}^{(p-1)/2} i \bmod 2$ to obtain

$$m = \sum_{i=1}^{(p-1)/2} \left\lfloor \frac{iq}{p} \right\rfloor \bmod 2.$$

The Proof (concluded)

- $\sum_{i=1}^{(p-1)/2} \lfloor \frac{iq}{p} \rfloor$ is the number of integral points under the line $y = (q/p)\, x$ for $1 \leq x \leq (p-1)/2$.
- Gauss's lemma (p. 488) says $(q|p) = (-1)^m$.
- Repeat the proof with p and q reversed.
- So $(p|q) = (-1)^{m'}$, where m' is the number of integral points *above* the line $y = (q/p)\, x$ for $1 \leq y \leq (q-1)/2$.
- As a result, $(p|q)(q|p) = (-1)^{m+m'}$.
- But $m + m'$ is the total number of integral points in the $\frac{p-1}{2} \times \frac{q-1}{2}$ rectangle, which is $\frac{p-1}{2} \frac{q-1}{2}$.

Eisenstein's Rectangle



$p = 11$ and $q = 7$.

The Jacobi Symbol[a]

- The Legendre symbol only works for odd *prime* moduli.
- The **Jacobi symbol** $(a \,|\, m)$ extends it to cases where m is not prime.
- Let $m = p_1 p_2 \cdots p_k$ be the prime factorization of m.
- When $m > 1$ is odd and $\gcd(a, m) = 1$, then

$$(a|m) = \prod_{i=1}^{k} (a \,|\, p_i).$$

  - Note that the Jacobi symbol equals ± 1.
  - It reduces to the Legendre symbol when m is a prime.
- Define $(a \,|\, 1) = 1$.

[a] Carl Jacobi (1804–1851).

Properties of the Jacobi Symbol

The Jacobi symbol has the following properties, for arguments for which it is defined.

1. $(ab \mid m) = (a \mid m)(b \mid m)$.
2. $(a \mid m_1 m_2) = (a \mid m_1)(a \mid m_2)$.
3. If $a = b \bmod m$, then $(a \mid m) = (b \mid m)$.
4. $(-1 \mid m) = (-1)^{(m-1)/2}$ (by Lemma 67 on p. 488).
5. $(2 \mid m) = (-1)^{(m^2-1)/8}$.[a]
6. If a and m are both odd, then $(a \mid m)(m \mid a) = (-1)^{(a-1)(m-1)/4}$.

[a] By Lemma 67 (p. 488) and some parity arguments.

Calculation of $(2200|999)$

Similar to the Euclidean algorithm and does *not* require factorization.

$$
\begin{aligned}
& (202|999) = (-1)^{(999^2-1)/8}(101|999) \\
= \quad & (-1)^{124750}(101|999) = (101|999) \\
= \quad & (-1)^{(100)(998)/4}(999|101) = (-1)^{24950}(999|101) \\
= \quad & (999|101) = (90|101) = (-1)^{(101^2-1)/8}(45|101) \\
= \quad & (-1)^{1275}(45|101) = -(45|101) \\
= \quad & -(-1)^{(44)(100)/4}(101|45) = -(101|45) = -(11|45) \\
= \quad & -(-1)^{(10)(44)/4}(45|11) = -(45|11) \\
= \quad & -(1|11) = -1.
\end{aligned}
$$

A Result Generalizing Proposition 10.3 in the Textbook

Theorem 69 *The group of set* $\Phi(n)$ *under multiplication* mod n *has a primitive root if and only if* n *is either 1, 2, 4,* p^k*, or* $2p^k$ *for some nonnegative integer* k *and and odd prime* p*.*

This result is essential in the proof of the next lemma.

The Jacobi Symbol and Primality Test[a]

Lemma 70 *If* $(M|N) = M^{(N-1)/2} \bmod N$ *for all* $M \in \Phi(N)$*, then* N *is prime. (Assume* N *is odd.)*

- Assume $N = mp$, where p is an odd prime, $\gcd(m, p) = 1$, and $m > 1$ (not necessarily prime).
- Let $r \in \Phi(p)$ such that $(r \,|\, p) = -1$.
- The Chinese remainder theorem says that there is an $M \in \Phi(N)$ such that

$$
\begin{aligned}
M &= r \bmod p, \\
M &= 1 \bmod m.
\end{aligned}
$$

---

[a]Mr. Clement Hsiao (`R88526067`) pointed out that the textbook's proof for Lemma 11.8 is incorrect while he was a senior in January 1999.

The Proof (continued)

- By the hypothesis,

$$M^{(N-1)/2} = (M \mid N) = (M \mid p)(M \mid m) = -1 \bmod N.$$

- Hence

$$M^{(N-1)/2} = -1 \bmod m.$$

- But because $M = 1 \bmod m$,

$$M^{(N-1)/2} = 1 \bmod m,$$

  a contradiction.

The Proof (continued)

- Second, assume that $N = p^a$, where p is an odd prime and $a \geq 2$.
- By Theorem 69 (p. 500), there exists a primitive root r modulo p^a.
- From the assumption,

$$M^{N-1} = \left[M^{(N-1)/2} \right]^2 = (M|N)^2 = 1 \bmod N$$

for all $M \in \Phi(N)$.

The Proof (continued)

- As $r \in \Phi(N)$ (prove it), we have

$$r^{N-1} = 1 \bmod N.$$

- As r's exponent modulo $N = p^a$ is $\phi(N) = p^{a-1}(p-1)$,

$$p^{a-1}(p-1) \mid N-1,$$

which implies that $p \mid N-1$.

- But this is impossible given that $p \mid N$.

The Proof (continued)

- Third, assume that $N = mp^a$, where p is an odd prime, $\gcd(m, p) = 1$, $m > 1$ (not necessarily prime), and a is even.
- The proof mimics that of the second case.
- By Theorem 69 (p. 500), there exists a primitive root r modulo p^a.
- From the assumption,

$$M^{N-1} = \left[M^{(N-1)/2} \right]^2 = (M|N)^2 = 1 \bmod N$$

for all $M \in \Phi(N)$.

The Proof (continued)

- In particular,
$$M^{N-1} = 1 \bmod p^a \tag{7}$$
for all $M \in \Phi(N)$.

- The Chinese remainder theorem says that there is an $M \in \Phi(N)$ such that
$$\begin{aligned} M &= r \bmod p^a, \\ M &= 1 \bmod m. \end{aligned}$$

- Because $M = r \bmod p^a$ and Eq. (7),
$$r^{N-1} = 1 \bmod p^a.$$

The Proof (concluded)

- As r's exponent modulo $N = p^a$ is $\phi(N) = p^{a-1}(p-1)$,

$$p^{a-1}(p-1) \mid N-1,$$

  which implies that $p \mid N-1$.

- But this is impossible given that $p \mid N$.