MONOTONE CIRCUIT VALUE

- A **monotone** boolean circuit's output cannot change from true to false when one input changes from false to true.
- Monotone boolean circuits are hence less expressive than general circuits.
  - They can compute only *monotone* boolean functions.
- Monotone circuits do not contain $\neg$ gates (prove it).
- MONOTONE CIRCUIT VALUE is CIRCUIT VALUE applied to monotone circuits.

MONOTONE CIRCUIT VALUE Is P-Complete

Despite their limitations, MONOTONE CIRCUIT VALUE is as hard as CIRCUIT VALUE.

Corollary 32 MONOTONE CIRCUIT VALUE *is P-complete.*

- Given any general circuit, we can "move the $\neg$'s downwards" using de Morgan's laws. (Why?)

Cook's Theorem: the First NP-Complete Problem

Theorem 33 (Cook (1971)) SAT *is NP-complete.*

- SAT $\in$ NP (p. 86).
- CIRCUIT SAT reduces to SAT (p. 223).
- Now we only need to show that all languages in NP can be reduced to CIRCUIT SAT.

The Proof (continued)

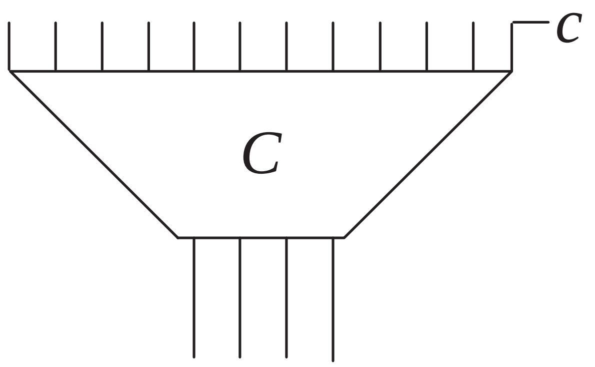
- Let single-string NTM M decide $L \in \text{NP}$ in time n^k.
- Assume M has exactly *two* nondeterministic choices at each step: choices 0 and 1.
- For each input x, we construct circuit $R(x)$ such that $x \in L$ if and only if $R(x)$ is satisfiable.
- A sequence of nondeterministic choices is a bit string

$$B = (c_1, c_2, \ldots, c_{|\,x\,|^k-1}) \in \{0,1\}^{|\,x\,|^k-1}.$$

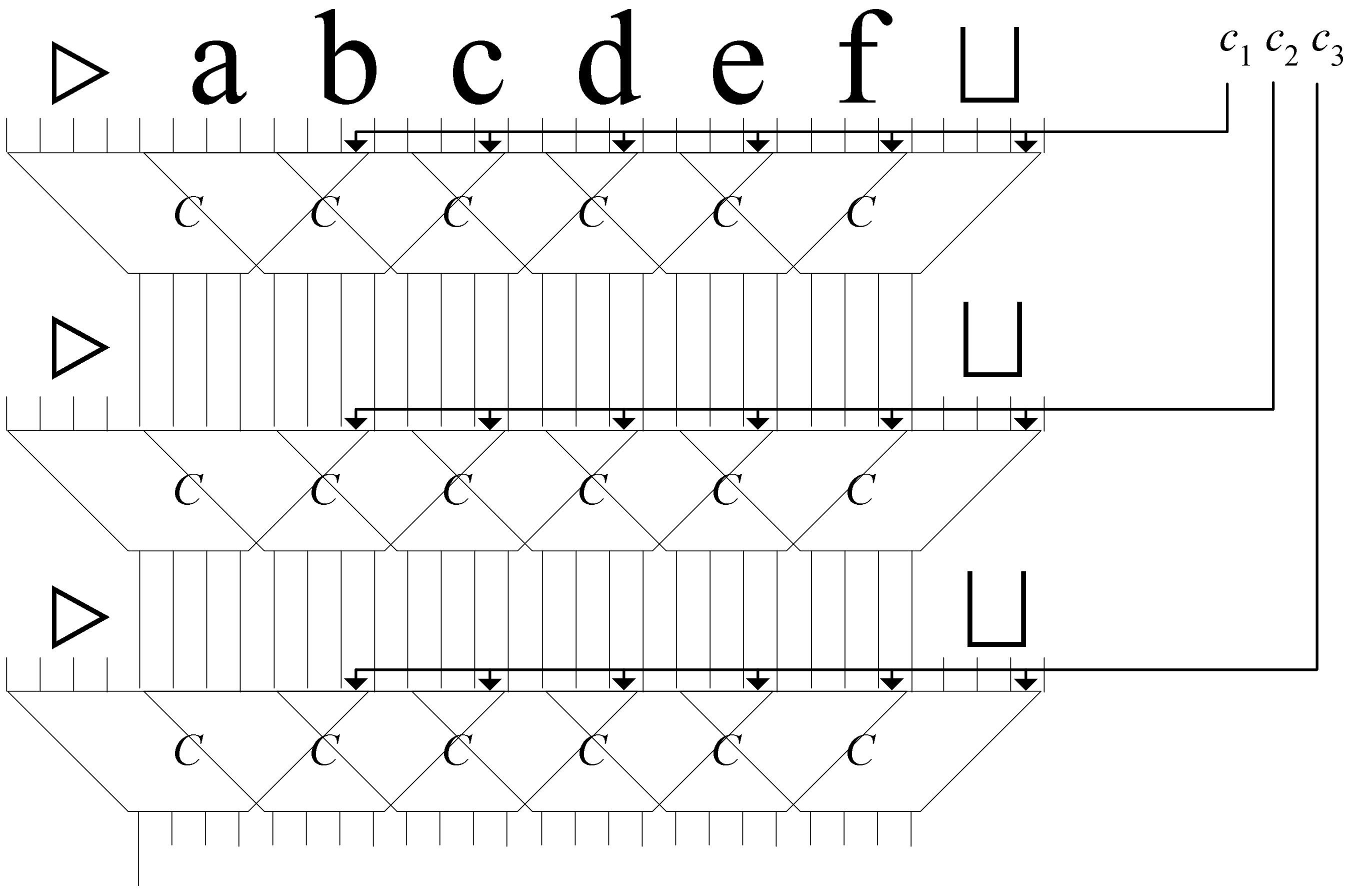
- Once B is given, the computation is *deterministic*.

The Proof (continued)

- Each choice of B results in a deterministic polynomial-time computation.
- So each choice of B results in a table like the one on p. 254.
- Each circuit C at time i has an extra binary input c corresponding to the nondeterministic choice: $C(T_{i-1,j-1}, T_{i-1,j}, T_{i-1,j+1}, c) = T_{ij}$.



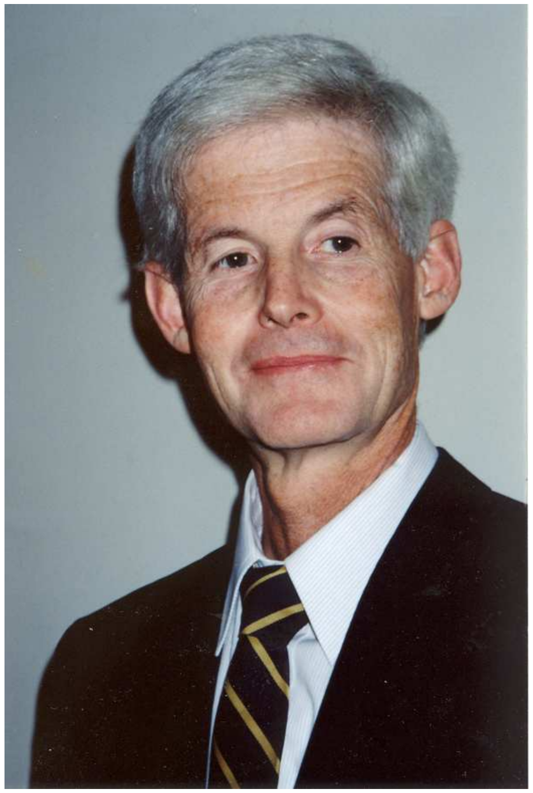
The Computation Tableau for NTMs and $R(x)$



The Proof (concluded)

- The overall circuit $R(x)$ (on p. 261) is satisfiable if there is a truth assignment B such that the computation table accepts.
- This happens if and only if M accepts x, i.e., $x \in L$.

Stephen Arthur Cook (1939–)



Richard Karp, “It is to our everlasting shame that we were unable to persuade the math department [of UC-Berkeley] to give him tenure.”

NP-Complete Problems

Wir müssen wissen, wir werden wissen.
(We must know, we shall know.)
— David Hilbert (1900)

I predict that scientists will one day adopt a new principle: "NP-complete problems are hard." That is, solving those problems efficiently is impossible on any device that could be built in the real world, whatever the final laws of physics turn out to be.
— Scott Aaronson (2008)

Two Notions

- Let $R \subseteq \Sigma^* \times \Sigma^*$ be a binary relation on strings.
- R is called **polynomially decidable** if
$$\{x; y : (x, y) \in R\}$$
is in P.[a]
- R is said to be **polynomially balanced** if $(x, y) \in R$ implies $|y| \leq |\, x\,|^k$ for some $k \geq 1$.

[a]Proposition 34 (p. 267) remains valid if P is replaced by NP. Contributed by Mr. Cheng-Yu Lee (`R95922035`) on October 26, 2006.

An Alternative Characterization of NP

Proposition 34 (Edmonds (1965)) *Let $L \subseteq \Sigma^*$ be a language. Then $L \in NP$ if and only if there is a polynomially decidable and polynomially balanced relation R such that*

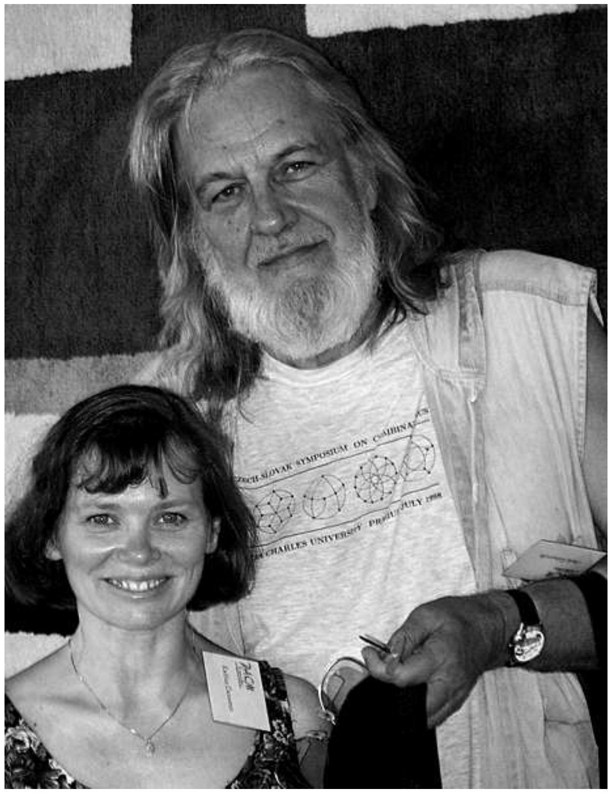
$$L = \{x : \exists y\,(x, y) \in R\}.$$

- Suppose such an R exists.
- L can be decided by this NTM:
  - On input x, the NTM guesses a y of length $\leq |\,x\,|^k$.
  - It then tests if $(x, y) \in R$ in polynomial time.
  - It returns "yes" if the test is positive.

The Proof (concluded)

- Now suppose $L \in \mathrm{NP}$.
- NTM N decides L in time $|\, x \,|^k$.
- Define R as follows: $(x, y) \in R$ if and only if y is the encoding of an accepting computation of N on input x.
- R is polynomially balanced as N is polynomially bounded.
- R is polynomially decidable because it can be efficiently verified by consulting N's transition function.
- Finally $L = \{x : (x, y) \in R \text{ for some } y\}$ because N decides L.

Jack Edmonds



Comments

- Any "yes" instance x of an NP problem has at least one **succinct certificate** or **polynomial witness** y.
- "No" instances have none.
- Certificates are short and easy to verify.
  - An alleged satisfying truth assignment for SAT, an alleged Hamiltonian path for HAMILTONIAN PATH, etc.
- Certificates may be hard to generate,[a] but verification must be easy.
- NP is the class of *easy-to-verify* (i.e., in P) problems.

[a]Unless P equals NP.

Levin Reduction and Parsimonious Reductions

- The reduction R in Cook's theorem (p. 258) is such that
  - Each satisfying truth assignment for circuit $R(x)$ corresponds to an accepting computation path for $M(x)$.
- It actually yields an efficient way to transform a certificate for x to a satisfying assignment for $R(x)$, and vice versa.
- A reduction with this property is called a **Levin reduction**.[a]

[a]Levin is the co-inventor of NP-completeness, in 1973.

Leonid Levin (1948–)

Leonid Levin, “Mathematicians often think that historical evidence is that NP is exponential. Historical evidence is quite strongly in the other direction.”

Levin Reduction and Parsimonious Reductions (concluded)

- Furthermore, the proof gives a one-to-one and onto mapping between the set of certificates for x and the set of satisfying assignments for $R(x)$.
- So the number of satisfying truth assignments for $R(x)$ equals that of $M(x)$'s accepting computation paths.
- This kind of reduction is called **parsimonious**.
- We will loosen the timing requirement for parsimonious reduction: It runs in deterministic polynomial time.

You Have an NP-Complete Problem (for Your Thesis)

- From Propositions 26 (p. 236) and Proposition 29 (p. 239), it is the least likely to be in P.
- Your options are:
  - Approximations.
  - Special cases.
  - Average performance.
  - Randomized algorithms.
  - Exponential-time algorithms that work well in practice.
  - "Heuristics" (and pray that it works *for your thesis*).

I thought NP-completeness was an interesting idea:
I didn't quite realize its potential impact.
— Stephen Cook, in Shasha & Lazere (1998)

I was indeed surprised by Karp's work
since I did not expect so many
wonderful problems were NP-complete.
— Leonid Levin, in Shasha & Lazere (1998)

3SAT

- k-SAT, where $k \in \mathbb{Z}^+$, is the special case of SAT.
- The formula is in CNF and all clauses have *exactly* k literals (repetition of literals is allowed).
- For example,

$$(x_1 \lor x_2 \lor \neg x_3) \land (x_1 \lor x_1 \lor \neg x_2) \land (x_1 \lor \neg x_2 \lor \neg x_3).$$

3SAT Is NP-Complete

- Recall Cook's Theorem (p. 258) and the reduction of CIRCUIT SAT to SAT (p. 223).
- The resulting CNF has at most 3 literals for each clause.
  - This shows that 3SAT where each clause has at most 3 literals is NP-complete.
- Finally, duplicate one literal once or twice to make it a 3SAT formula.
- Note: The overall reduction remains parsimonious.

The Satisfiability of Random 3SAT Expressions

- Consider a random 3SAT expressions ϕ with n variables and cn clauses.
- Each clause is chosen independently and uniformly from the set of all possible clauses.
- Intuitively, the larger the c, the less likely ϕ is satisfiable as more constraints are added.
- Indeed, there is a c_n such that for $c < c_n(1-\epsilon)$, ϕ is satisfiable almost surely, and for $c > c_n(1+\epsilon)$, ϕ is unsatisfiable almost surely.[a]

[a] Friedgut and Bourgain (1999). As of 2006, $3.52 < c_n < 4.596$.

Another Variant of 3SAT

Proposition 35 3SAT *is NP-complete for expressions in which each variable is restricted to appear at most three times, and each literal at most twice. (*3SAT *here requires only that each clause has* at most *3 literals.)*

- Consider a general 3SAT expression in which x appears k times.
- Replace the first occurrence of x by x_1, the second by x_2, and so on, where $x_1, x_2, \ldots, x_k$ are k *new* variables.

The Proof (concluded)

- Add $(\neg x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge \cdots \wedge (\neg x_k \vee x_1)$ to the expression.
  - It is logically equivalent to
    $$x_1 \Rightarrow x_2 \Rightarrow \cdots \Rightarrow x_k \Rightarrow x_1.$$
  - Note that each clause above has only 2 literals.
- The resulting equivalent expression satisfies the condition for x.

An Example

- Suppose we are given the following 3SAT expression

$$\cdots(\neg x \vee w \vee g) \wedge \cdots \wedge (x \vee y \vee z) \cdots .$$

- The transformed expression is

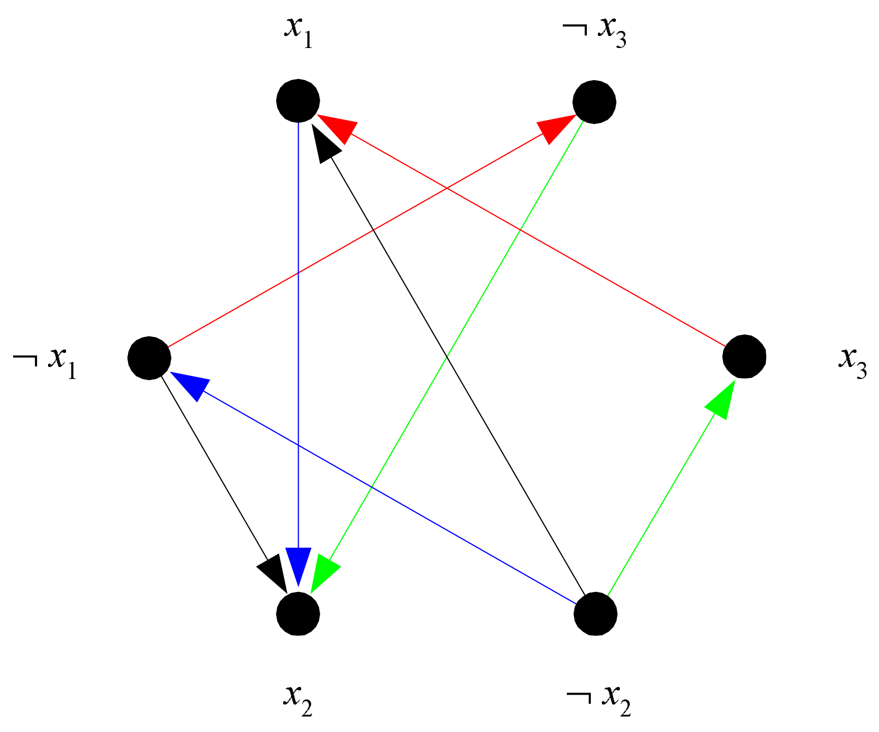
$$\cdots(\neg x_1 \vee w \vee g) \wedge \cdots \wedge (x_2 \vee y \vee z) \cdots (\neg x_1 \vee x_2) \wedge (\neg x_2 \vee x_1).$$

  - Variable x_1 appears 3 times.
  - Literal x_1 appears once.
  - Literal $\neg x_1$ appears 2 times.

2SAT and Graphs

- Let ϕ be an instance of 2SAT: Each clause has 2 literals.
- Define graph $G(\phi)$ as follows:
  - The nodes are the variables and their negations.
  - Insert edges $(\neg\alpha, \beta)$ and $(\neg\beta, \alpha)$ for clause $\alpha \vee \beta$.
    * For example, if $x \vee \neg y \in \phi$, add $(\neg x, \neg y)$ and (y, x).
    * *Two* edges are added for each clause.
  - Think of the edges as $\neg\alpha \Rightarrow \beta$ and $\neg\beta \Rightarrow \alpha$.
- b is reachable from a iff $\neg a$ is reachable from $\neg b$.
- Paths in $G(\phi)$ are valid implications.

$$(x_1 \lor x_2) \land (x_1 \lor \neg x_3) \land (\neg x_1 \lor x_2) \land (x_2 \lor x_3)$$



Properties of $G(\phi)$

Theorem 36 *ϕ is unsatisfiable if and only if there is a variable x such that there are paths from x to $\neg x$ and from $\neg x$ to x in $G(\phi)$.*

- The expression on p. 283 can be satisfied by setting $x_1 = \texttt{true}, x_2 = \texttt{true}$.
- Note on p. 283, there is a path from $\neg x_2$ to x_2, but none from x_2 to $\neg x_2$.

2SAT Is in NL $\subseteq$ P

- NL is a subset of P (p. 194).
- By Eq. (2) on p. 204, coNL equals NL.
- We need to show only that recognizing unsatisfiable expressions is in NL.
- In nondeterministic logarithmic space, we can test the conditions of Theorem 36 (p. 284) by guessing a variable x and testing if $\neg x$ is reachable from x *and* if $\neg x$ can reach x.
  - See the algorithm for REACHABILITY (p. 94).

Generalized 2SAT: MAX2SAT

- Consider a 2SAT expression.
- Let $K \in \mathbb{N}$.
- MAX2SAT is the problem of whether there is a truth assignment that satisfies at least K of the clauses.
- MAX2SAT becomes 2SAT when K equals the number of clauses.
- MAX2SAT is an optimization problem.
- MAX2SAT $\in$ NP: Guess a truth assignment and verify the count.

MAX2SAT Is NP-Complete[a]

- Consider the following 10 clauses:

$$(x) \wedge (y) \wedge (z) \wedge (w)$$
$$(\neg x \vee \neg y) \wedge (\neg y \vee \neg z) \wedge (\neg z \vee \neg x)$$
$$(x \vee \neg w) \wedge (y \vee \neg w) \wedge (z \vee \neg w)$$

- Let the 2SAT formula $r(x, y, z, w)$ represent the conjunction of these clauses.
- The clauses are symmetric with respect to x, y, and z.
- How many clauses can we satisfy?

[a]Garey, Johnson, and Stockmeyer (1976).

The Proof (continued)

All of x, y, z are true: By setting w to true, we satisfy $4 + 0 + 3 = 7$ clauses, whereas by setting w to false, we satisfy only $3 + 0 + 3 = 6$ clauses.

Two of x, y, z are true: By setting w to true, we satisfy $3 + 2 + 2 = 7$ clauses, whereas by setting w to false, we satisfy $2 + 2 + 3 = 7$ clauses.

The Proof (continued)

One of x, y, z is true: By setting w to false, we satisfy $1 + 3 + 3 = 7$ clauses, whereas by setting w to true, we satisfy only $2 + 3 + 1 = 6$ clauses.

None of x, y, z is true: By setting w to false, we satisfy $0 + 3 + 3 = 6$ clauses, whereas by setting w to true, we satisfy only $1 + 3 + 0 = 4$ clauses.

The Proof (continued)

- Any truth assignment that satisfies $x \lor y \lor z$ can be *extended* to satisfy 7 of the 10 clauses and no more.
- Any other truth assignment can be extended to satisfy only 6 of them.
- The reduction from 3SAT ϕ to MAX2SAT $R(\phi)$:
  - For each clause $C_i = (\alpha \lor \beta \lor \gamma)$ of ϕ, add **group** $r(\alpha, \beta, \gamma, w_i)$ to $R(\phi)$.
- If ϕ has m clauses, then $R(\phi)$ has $10m$ clauses.
- Finally, set $K = 7m$.

The Proof (concluded)

- We now show that K clauses of $R(\phi)$ can be satisfied if and only if ϕ is satisfiable.
- Suppose $7m$ clauses of $R(\phi)$ can be satisfied.
  - 7 clauses must be satisfied in each group because each group can have at most 7 clauses satisfied.
  - Hence all clauses of ϕ must be satisfied.
- Suppose all clauses of ϕ are satisfied.
  - Each group can set its w_i appropriately to have 7 clauses satisfied.

Michael R. Garey (1945–)

David S. Johnson (1945–)

Larry Stockmeyer (1948–2004)

NAESAT

- The NAESAT (for “not-all-equal” SAT) is like 3SAT.
- But there must be a satisfying truth assignment under which no clauses have the three literals equal in truth value.
- Equivalently, there is a truth assignment such that each clause has one literal assigned true and one literal assigned false.

NAESAT Is NP-Complete[a]

- Recall the reduction of CIRCUIT SAT to SAT on p. 223.
- It produced a CNF ϕ in which each clause has at most 3 literals.
- Add the same variable z to all clauses with fewer than 3 literals to make it a 3SAT formula.
- Goal: The new formula $\phi(z)$ is NAE-satisfiable if and only if the original circuit is satisfiable.

[a]Karp (1972).

The Proof (continued)

- Suppose T NAE-satisfies $\phi(z)$.
  - $\bar{T}$ also NAE-satisfies $\phi(z)$.
  - Under T or $\bar{T}$, variable z takes the value false.
  - This truth assignment $\mathcal{T}$ must still satisfy all clauses of ϕ.
    - Note that $\mathcal{T} \models \phi$ with z being false.
  - So it satisfies the original circuit.

The Proof (concluded)

- Suppose there is a truth assignment that satisfies the circuit.
  - Then there is a truth assignment T that satisfies every clause of ϕ.
  - Extend T by adding $T(z) = \texttt{false}$ to obtain T'.
  - T' satisfies $\phi(z)$.
  - So in no clauses are all three literals false under T'.
  - Under T', in no clauses are all three literals true.
    * Need to review the detailed construction on p. 224 and p. 225.

Richard Karp (1935–)

Undirected Graphs

- An **undirected graph** $G = (V, E)$ has a finite set of nodes, V, and a set of *undirected* edges, E.
- It is like a directed graph except that the edges have no directions and there are no self-loops.
- Use $[\, i, j \,]$ to denote the fact that there is an edge between node i and node j.

Independent Sets

- Let $G = (V, E)$ be an undirected graph.
- $I \subseteq V$.
- I is **independent** if whenever $i, j \in I$, there is no edge between i and j.
- The INDEPENDENT SET problem: Given an undirected graph and a goal K, is there an independent set of size K?
  - Many applications.

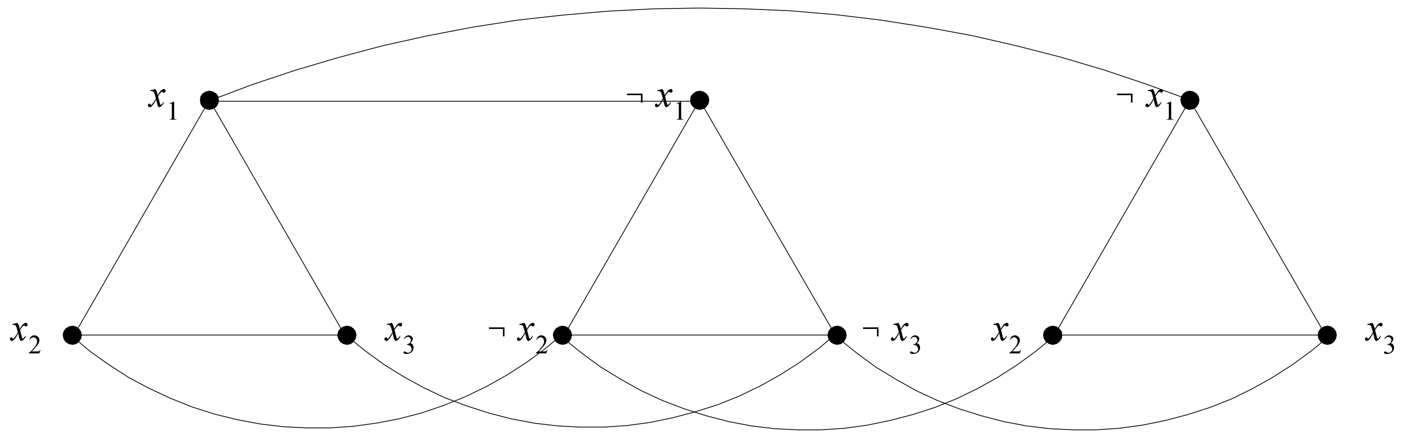
INDEPENDENT SET Is NP-Complete

- This problem is in NP: Guess a set of nodes and verify that it is independent and meets the count.
- If a graph contains a triangle, any independent set can contain at most one node of the triangle.
- We consider graphs whose nodes can be partitioned into m disjoint triangles.
  - If the special case of graphs is hard, the original problem must be at least as hard.
- We will reduce 3SAT to INDEPENDENT SET.

The Proof (continued)

- Let ϕ be an instance of 3SAT with m clauses.
- We will construct graph G (with constraints as said) with $K = m$ such that ϕ is satisfiable if and only if G has an independent set of size K.
- There is a triangle for each clause with the literals as the nodes.
- Add additional edges between x and $\neg x$ for every variable x.

$$(x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_1 \lor x_2 \lor x_3)$$



Same literals that appear in different clauses are on distinct nodes.

The Proof (continued)

- Suppose G has an independent set I of size $K = m$.
  - An independent set can contain at most m nodes, one from each triangle.
  - An independent set of size m exists if and only if it contains exactly one node from each triangle.
  - Truth assignment T assigns true to those literals in I.
  - T is consistent because contradictory literals are connected by an edge; hence both cannot be in I.
  - T satisfies ϕ because it has a node from every triangle, thus satisfying every clause.

The Proof (concluded)

- Suppose a satisfying truth assignment T exists for ϕ.
  - Collect one node from each triangle whose literal is true under T.
  - The choice is arbitrary if there is more than one true literal.
  - This set of m nodes must be independent by construction.
    - Both literals x and $\neg x$ cannot be assigned true.

Other INDEPENDENT SET-Related NP-Complete Problems

Corollary 37 INDEPENDENT SET *is NP-complete for 4-degree graphs.*

Theorem 38 INDEPENDENT SET *is NP-complete for planar graphs.*

Theorem 39 (Garey and Johnson (1977)) INDEPENDENT SET *is NP-complete for 3-degree planar graphs.*

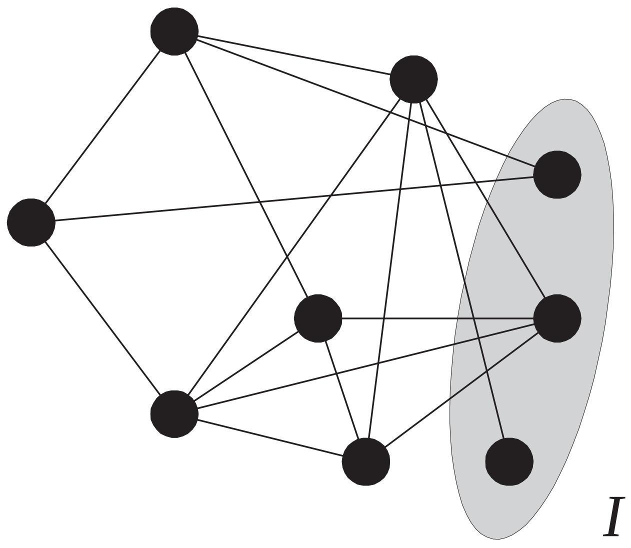
NODE COVER

- We are given an undirected graph G and a goal K.
- NODE COVER: Is there is a set C with K or fewer nodes such that each edge of G has at least one of its endpoints in C?

NODE COVER Is NP-Complete

Corollary 40 NODE COVER *is NP-complete.*

- I is an independent set of $G = (V, E)$ if and only if $V - I$ is a node cover of G.



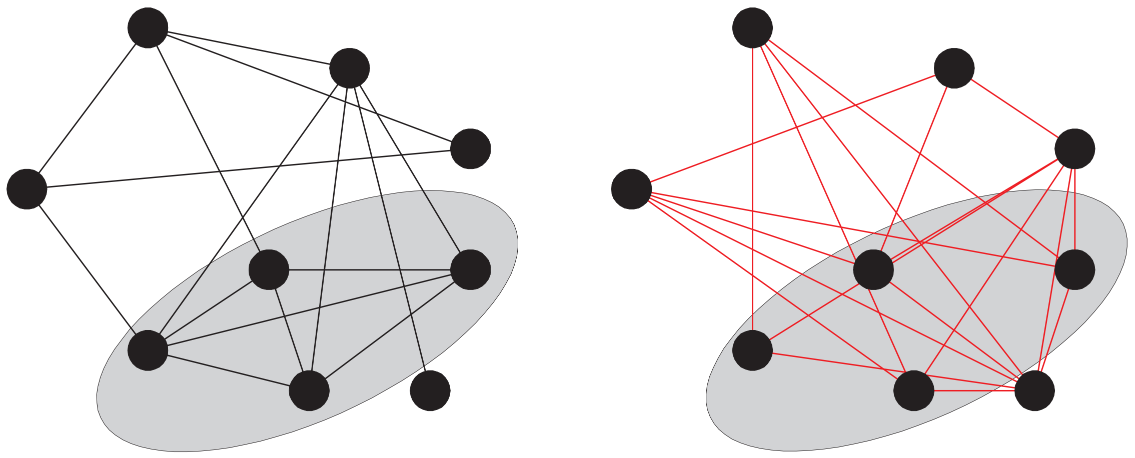
CLIQUE

- We are given an undirected graph G and a goal K.
- CLIQUE asks if there is a set C with K nodes such that whenever $i, j \in C$, there is an edge between i and j.

CLIQUE Is NP-Complete

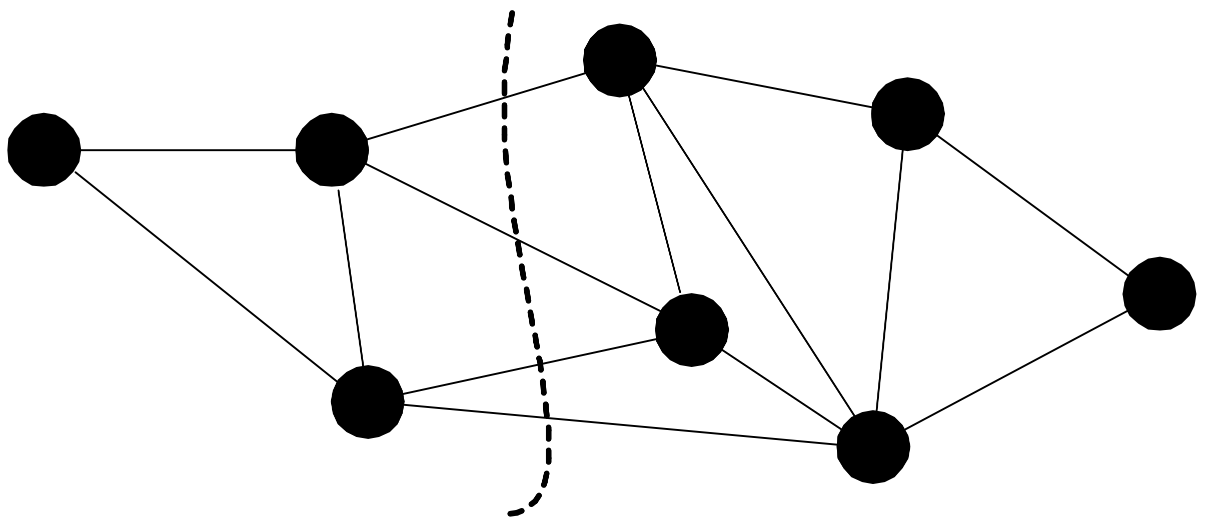
Corollary 41 CLIQUE *is NP-complete.*

- Let $\bar{G}$ be the **complement** of G, where $[x, y] \in \bar{G}$ if and only if $[x, y] \notin G$.
- I is a clique in $G \Leftrightarrow I$ is an independent set in $\bar{G}$.



MIN CUT and MAX CUT

- A **cut** in an undirected graph $G = (V, E)$ is a partition of the nodes into two nonempty sets S and $V - S$.
- The size of a cut $(S, V - S)$ is the number of edges between S and $V - S$.
- MIN CUT $\in$ P by the maxflow algorithm.
- MAX CUT asks if there is a cut of size at least K.
  - K is part of the input.



MIN CUT and MAX CUT (concluded)

- MAX CUT has applications in VLSI layout.
  - The minimum area of a VLSI layout of a graph is not less than the square of its maximum cut size.[a]

[a]Raspaud, Sýkora, and Vrťo (1995); Mak and Wong (2000).

MAX CUT Is NP-Complete[a]

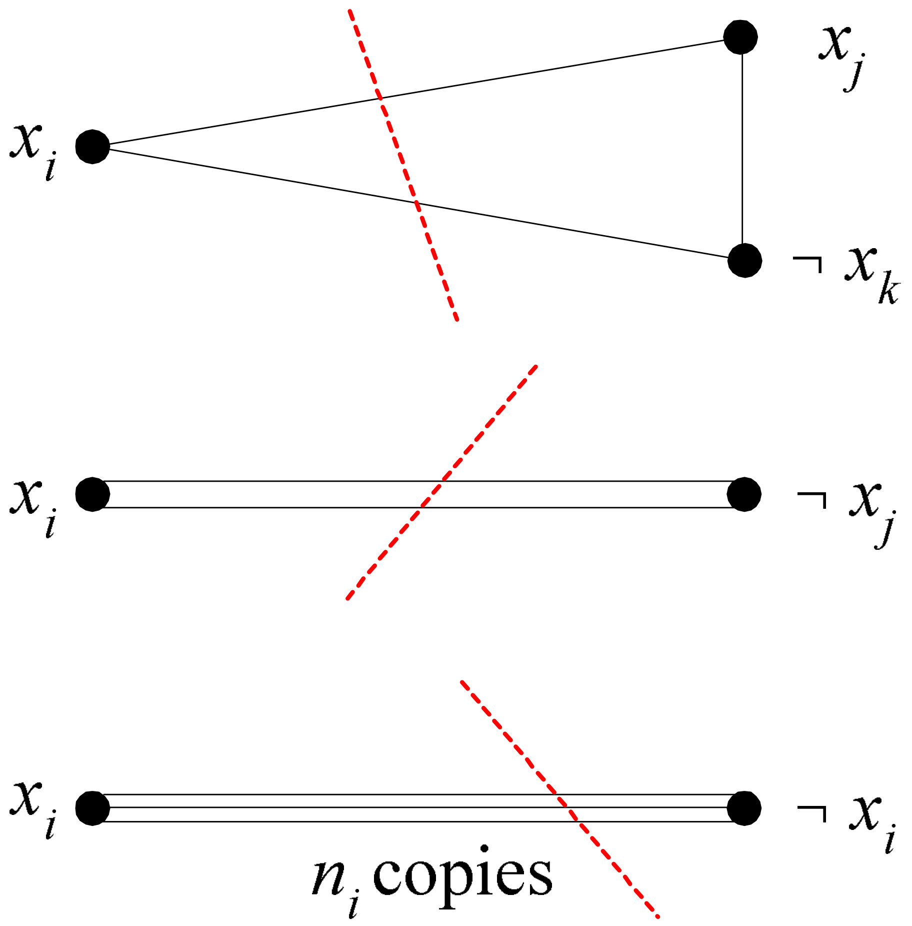
- We will reduce NAESAT to MAX CUT.
- Given an instance ϕ of 3SAT with m clauses, we shall construct a graph $G = (V, E)$ and a goal K such that:
  - There is a cut of size at least K if and only if ϕ is NAE-satisfiable.
- Our graph will have multiple edges between two nodes.
  - Each such edge contributes one to the cut if its nodes are separated.

[a]Garey, Johnson, and Stockmeyer (1976).

The Proof

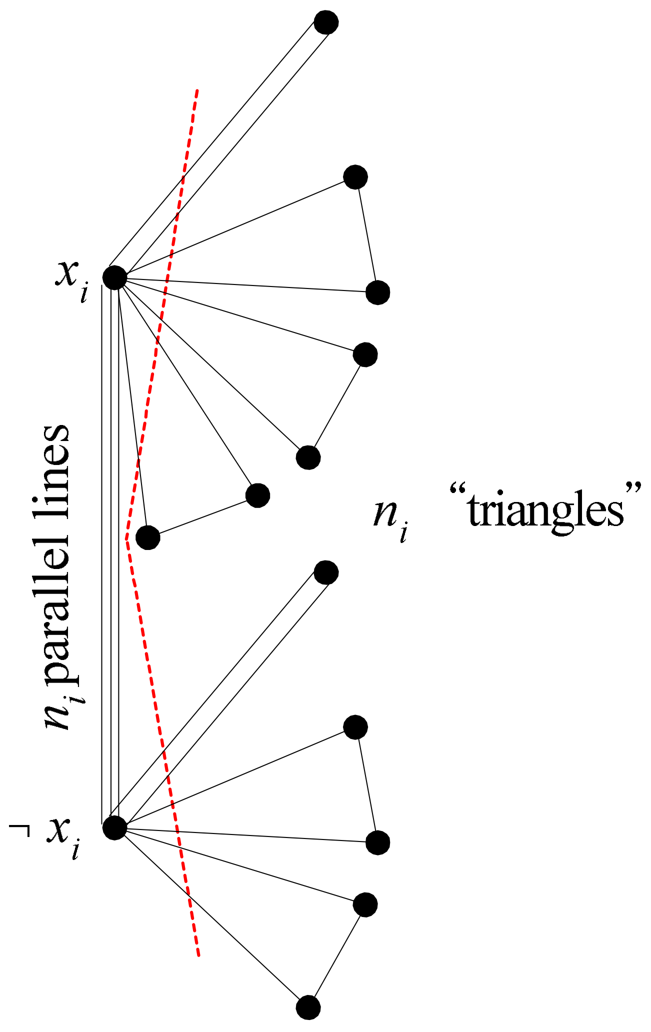
- Suppose ϕ's m clauses are $C_1, C_2, \ldots, C_m$.
- The boolean variables are $x_1, x_2, \ldots, x_n$.
- G has $2n$ nodes: $x_1, x_2, \ldots, x_n, \neg x_1, \neg x_2, \ldots, \neg x_n$.
- Each clause with 3 distinct literals makes a triangle in G.
- For each clause with two identical literals, there are two parallel edges between the two distinct literals.
- No need to consider clauses with one literal (why?).
- For each variable x_i, add n_i copies of edge $[x_i, \neg x_i]$, where n_i is the number of occurrences of x_i and $\neg x_i$ in ϕ.[a]

[a]Regardless of whether both x_i and $\neg x_i$ occur in ϕ.


x_i
x_j
$\neg x_k$
x_i
$\neg x_j$
x_i
$\neg x_i$
n_i copies

The Proof (continued)

- Set $K = 5m$.
- Suppose there is a cut $(S, V - S)$ of size $5m$ or more.
- A clause (a triangle or two parallel edges) contributes at most 2 to a cut no matter how you split it.
- Suppose both x_i and $\neg x_i$ are on the same side of the cut.
- Then they *together* contribute at most $2n_i$ edges to the cut as they appear in at most n_i different clauses.

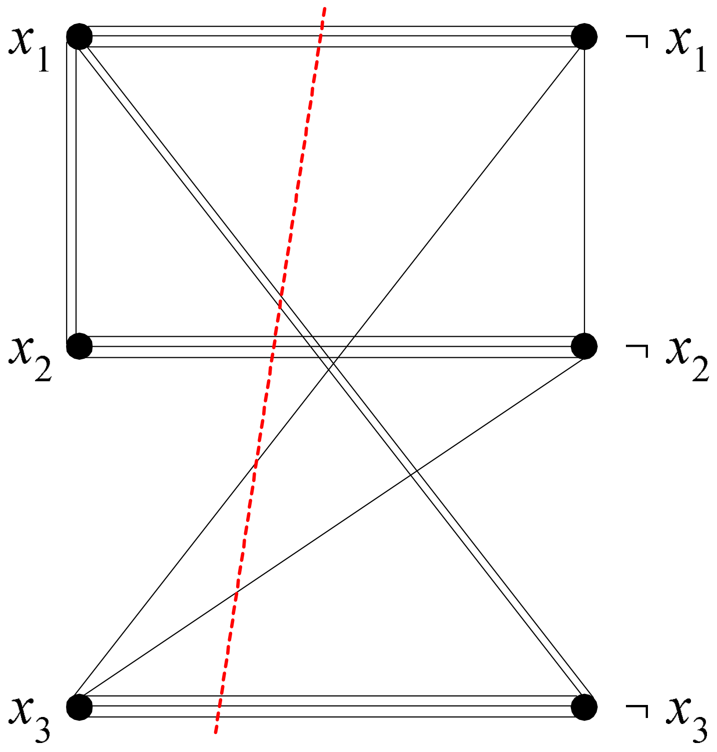

x_i
$\neg\ x_i$
n_i parallel lines
n_i "triangles"

The Proof (continued)

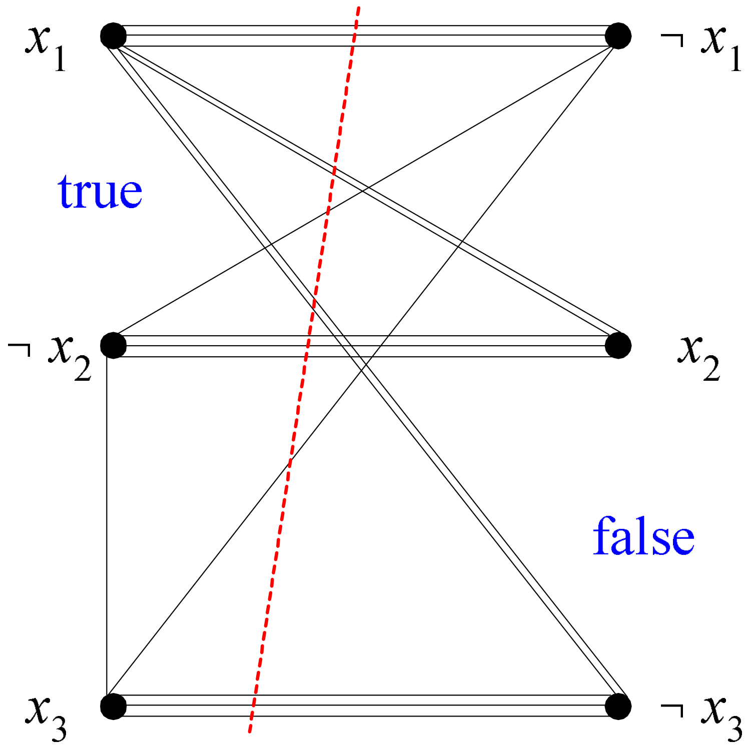
- Either x_i or $\neg x_i$ contributes at most n_i to the cut by the pigeonhole principle.
- Changing the side of that literal does not decrease the size of the cut.
- Hence we assume variables are separated from their negations.
- The total number of edges in the cut that join opposite literals is $\sum_i n_i = 3m$.
  - $\sum_i n_i = 3m$ as the total number of literals is $3m$.

The Proof (concluded)

- The *remaining* $2m$ edges in the cut must come from the m triangles or parallel edges that correspond to the clauses.
- As each can contribute at most 2 to the cut, all are split.
- A split clause means at least one of its literals is true and at least one false.
- The other direction is left as an exercise.



- $(x_1 \lor x_2 \lor x_2) \land (x_1 \lor \neg x_3 \lor \neg x_3) \land (\neg x_1 \lor \neg x_2 \lor x_3)$.
- The cut size is $13 < 5 \times 3 = 15$.



- $(x_1 \vee x_2 \vee x_2) \wedge (x_1 \vee \neg x_3 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$.
- The cut size is now 15.

Remarks

- We had proved that MAX CUT is NP-complete for multigraphs.
- How about proving the same thing for simple graphs?[a]
- For 4SAT, how do you modify the proof?[b]

[a]Contributed by Mr. Tai-Dai Chou (`J93922005`) on June 2, 2005.
[b]Contributed by Mr. Chien-Lin Chen (`J94922015`) on June 8, 2006.