Graph Isomorphism

- $V_1 = V_2 = \{1, 2, \ldots, n\}$.
- Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there exists a permutation π on $\{1, 2, \ldots, n\}$ so that $(u, v) \in E_1 \Leftrightarrow (\pi(u), \pi(v)) \in E_2$.
- The task is to answer if $G_1 \cong G_2$.
- No known polynomial-time algorithms.
- The problem is in NP (hence IP).
- But it is not likely to be NP-complete.[a]

[a] Schöning (1987).

GRAPH NONISOMORPHISM

- $V_1 = V_2 = \{1, 2, \ldots, n\}$.
- Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **nonisomorphic** if there exist no permutations π on $\{1, 2, \ldots, n\}$ so that $(u, v) \in E_1 \Leftrightarrow (\pi(u), \pi(v)) \in E_2$.
- The task is to answer if $G_1 \not\cong G_2$.
- Again, no known polynomial-time algorithms.
  - It is in coNP, but how about NP or BPP?
  - It is not likely to be coNP-complete.
- Surprisingly, GRAPH NONISOMORPHISM $\in$ IP.[a]

[a] Goldreich, Micali, and Wigderson (1986).

A 2-Round Algorithm

Victor selects a random $i \in \{1, 2\}$;
Victor selects a random permutation π on $\{1, 2, \ldots, n\}$;
Victor applies π on graph G_i to obtain graph H;
Victor sends (G_1, H) to Peggy;
if $G_1 \cong H$ **then**
    Peggy sends $j = 1$ to Victor;
else
    Peggy sends $j = 2$ to Victor;
end if
if $j = i$ **then**
    Victor accepts;
else
    Victor rejects;
end if

Analysis

- Victor runs in probabilistic polynomial time.
- Suppose $G_1 \not\cong G_2$.
  - Peggy is able to tell which G_i is isomorphic to H.
  - So Victor always accepts.
- Suppose $G_1 \cong G_2$.
  - No matter which i is picked by Victor, Peggy or any prover sees 2 identical graphs.
  - Peggy or any prover with exponential power has only probability one half of guessing i correctly.
  - So Victor erroneously accepts with probability $1/2$.
- Repeat the algorithm to obtain the desired probabilities.

Knowledge in Proofs

- Suppose I know a satisfying assignment to a satisfiable boolean expression.
- I can convince Alice of this by giving her the assignment.
- But then I give her more knowledge than necessary.
  - Alice can claim that she found the assignment!
  - Login authentication faces essentially the same issue.
  - See `www.wired.com/wired/archive/1.05/atm_pr.html` for a famous ATM fraud in the U.S.

Knowledge in Proofs (concluded)

- Digital signatures authenticate *documents* but not *individuals*.
- They hence do not solve the problem.
- Suppose I always give Alice random bits.
- Alice extracts no knowledge from me by any measure, but I prove nothing.
- Question 1: Can we design a protocol to convince Alice (the knowledge) of a secret without revealing anything extra?
- Question 2: How to define this idea rigorously?

Zero Knowledge Proofs[a]

An interactive proof protocol (P, V) for language L has the **perfect zero-knowledge** property if:

- For every verifier V', there is an algorithm M with expected polynomial running time.
- M on any input $x \in L$ generates the same probability distribution as the one that can be observed on the communication channel of (P, V') on input x.

[a] Goldwasser, Micali, and Rackoff (1985).

Comments

- Zero knowledge is a property of the prover.
  - It is the robustness of the prover against attempts of the verifier to extract knowledge via interaction.
  - The verifier may deviate arbitrarily (but in polynomial time) from the predetermined program.
  - A verifier cannot use the transcript of the interaction to convince a third-party of the validity of the claim.
  - The proof is hence not transferable.

Comments (continued)

- Whatever a verifier can "learn" from the specified prover P via the communication channel could as well be computed from the verifier alone.
- The verifier does not learn anything except "$x \in L$."
- Zero-knowledge proofs yield no knowledge in the sense that they can be constructed by the verifier who believes the statement, and yet these proofs do convince him.

Comments (continued)

- The "paradox" is resolved by noting that it is not the transcript of the conversation that convinces the verifier.
- But the fact that this conversation was held "on line."
- There is no zero-knowledge requirement when $x \notin L$.
- *Computational* zero-knowledge proofs are based on complexity assumptions.
  - M only needs to generate a distribution that is computationally indistinguishable from the verifier's view of the interaction.

Comments (concluded)

- It is known that if one-way functions exist, then zero-knowledge proofs exist for every problem in NP.[a]
- The verifier can be restricted to the honest one (i.e., it follows the protocol).[b]
- The coins can be public.[c]

[a] Goldreich, Micali, and Wigderson (1986).
[b] Vadhan (2006).
[c] Vadhan (2006).

Are You Convinced?

- A newspaper commercial for hair-growing products for men.
  - A (for all practical purposes) bald man has a full head of hair after 3 months.
- A TV commercial for weight-loss products.
  - A (by any reasonable measure) overweight woman loses 10 kilograms in 10 weeks.

Quadratic Residuacity

- Let n be a product of two distinct primes.
- Assume extracting the square root of a quadratic residue modulo n is hard without knowing the factors.
- We next present a zero-knowledge proof for $x \in Z_n^*$ being a quadratic residue.

Zero-Knowledge Proof of Quadratic Residuacity

1: **for** $m = 1, 2, \ldots, \log_2 n$ **do**

2: Peggy chooses a random $v \in Z_n^*$ and sends $y = v^2 \bmod n$ to Victor;

3: Victor chooses a random bit i and sends it to Peggy;

4: Peggy sends $z = u^i v \bmod n$, where u is a square root of x; $\{u^2 \equiv x \bmod n.\}$

5: Victor checks if $z^2 \equiv x^i y \bmod n$;

6: **end for**

7: Victor accepts x if Line 5 is confirmed every time;

A Useful Corollary

Corollary 76 *Let $n = pq$ be a product of two distinct primes. (1) If x and y are both quadratic residues modulo n, then $xy \in Z_n^*$ is a quadratic residue modulo n. (2) If x is a quadratic residue modulo n and y is a quadratic nonresidue modulo n, then $xy \in Z_n^*$ is a quadratic nonresidue modulo n.*

- Suppose x and y are both quadratic residues modulo n.
- Let $x \equiv a^2 \bmod n$ and $y \equiv b^2 \bmod n$.
- Now xy is a quadratic residue as $xy \equiv (ab)^2 \bmod n$.

The Proof (concluded)

- Suppose x is a quadratic residue modulo n and y is a quadratic nonresidue modulo n.
- By Lemma 75 (p. 586), $(x \mid p) = (x \mid q) = 1$ but, say, $(y \mid p) = -1$.
- Now xy is a quadratic nonresidue as $(xy \mid p) = -1$, again by Lemma 75 (p. 586).

Analysis

- Suppose x is a quadratic nonresidue.
  - Peggy can answer only one of the two possible challenges.
    - If a is a quadratic residue, then xa is a quadratic nonresidue by Corollary 76 (p. 613).
  - So Peggy will be caught in any given round with probability one half.

Analysis (continued)

- Suppose x is a quadratic residue.
  - Peggy can answer all challenges.
  - So Victor will accept x.
- How about the claim of zero knowledge?
- The transcript between Peggy and Victor when x is a quadratic residue can be generated without Peggy!
  - So interaction with Peggy is useless.
- Here is how.

Analysis (continued)

- Suppose x is a quadratic residue.[a]
- In each round of interaction with Peggy, the transcript is a triplet (y, i, z).
- We present an efficient Bob that generates (y, i, z) with the same probability *without* accessing Peggy.

[a] By definition, we do not need to consider the other case.

Analysis (concluded)

Bob chooses a random $z \in Z_n^*$;

Bob chooses a random bit i;

Bob calculates $y = z^2 x^{-i} \bmod n$;

Bob writes (y, i, z) into the transcript;

Comments

- Assume x is a quadratic residue.
- In both cases, for (y, i, z), y is a random quadratic residue, i is a random bit, and z is a random number.
- Bob cheats because (y, i, z) is *not* generated in the same order as in the original transcript.
  - Bob picks Victor's challenge first.
  - Bob then picks Peggy's answer.
  - Bob finally patches the transcript.

Comments (concluded)

- So it is not the transcript that convinces Victor, but that conversation with Peggy is held "on line."
- The same holds even if the transcript was generated by a cheating Victor's interaction with (honest) Peggy.
- But we skip the details.

Does the Following Work, Too?[a]

```
1: for m = 1, 2, ..., log_2 n do
2:     Peggy chooses a random v ∈ Z_n^* and sends
       y = v^2 mod n to Victor;
3:     Peggy sends z = uv mod n, where u is a square root of
       x; {u^2 ≡ x mod n.}
4:     Victor checks if z^2 ≡ xy mod n;
5: end for
6: Victor accepts x if Line 4 is confirmed every time;
```

[a]Contributed by Mr. Chih-Duo Hong (R95922079) on December 13, 2006. It is like always choosing $i = 1$ in the original protocol.

Does the Following Work, Too?[a] (concluded)

- Suppose x is a quadratic nonresidue.
- But Peggy can mislead Victor into accepting x as a quadratic residue.
- She simply sends $y = x$ and $z = x$ to Victor.
- This pair will satisfy $z^2 \equiv xy \bmod n$ by construction.
- The protocol is hence not even an IP protocol!

[a]Contributed by Mr. Chin-Luei Chang (`D95922007`) on June 16, 2008.

Zero-Knowledge Proof of 3 Colorability[a]

for $i = 1, 2, \ldots, |E|^2$ **do**

  Peggy chooses a random permutation π of the 3-coloring ϕ;

  Peggy samples an encryption scheme randomly and sends $\pi(\phi(1)), \pi(\phi(2)), \ldots, \pi(\phi(|V|))$ encrypted to Victor;

  Victor chooses at random an edge $e \in E$ and sends it to Peggy for the coloring of the endpoints of e;

  if $e = (u, v) \in E$ **then**

    Peggy reveals the coloring of u and v and "proves" that they correspond to their encryption;

  else

    Peggy stops;

  end if

---

[a]Goldreich, Micali, and Wigderson (1986).


```
10:     if the "proof" provided in Line 6 is not valid then
11:         Victor rejects and stops;
12:     end if
13:     if π(φ(u)) = π(φ(v)) or π(φ(u)), π(φ(v)) ∉ {1, 2, 3} then
14:         Victor rejects and stops;
15:     end if
16: end for
17: Victor accepts;
```

Analysis

- If the graph is 3-colorable and both Peggy and Victor follow the protocol, then Victor always accepts.
- If the graph is not 3-colorable and Victor follows the protocol, then however Peggy plays, Victor will accept with probability $\leq (1 - m^{-1})^{m^2} \leq e^{-m}$, where $m = |\, E \,|$.
- Thus the protocol is valid.
- This protocol yields no knowledge to Victor as all he gets is a bunch of random pairs.
- The proof that the protocol is zero-knowledge to *any* verifier is intricate.

Comments

- Each $\pi(\phi(i))$ is encrypted by a different cryptosystem.[a]
  - Otherwise, all the colors will be revealed in Step 6.
- Each edge e must be picked randomly.[b]
  - Otherwise, Peggy will know Victor's game plan and plot accordingly.

---

[a] Contributed by Ms. Yui-Huei Chang (R96922060) on May 22, 2008

[b] Contributed by Mr. Chang-Rong Hung (R96922028) on May 22, 2008

Approximability

Tackling Intractable Problems

- Many important problems are NP-complete or worse.
- **Heuristics** have been developed to attack them.
- They are **approximation algorithms**.
- How good are the approximations?
  - We are looking for theoretically *guaranteed* bounds, not "empirical" bounds.
- Are there NP problems that cannot be approximated well (assuming NP $\neq$ P)?
- Are there NP problems that cannot be approximated at all (assuming NP $\neq$ P)?

Some Definitions

- Given an **optimization problem**, each problem instance x has a set of **feasible solutions** $F(x)$.
- Each feasible solution $s \in F(x)$ has a cost $c(s) \in \mathbb{Z}^+$.
  - Here, cost refers to the quality of the feasible solution, not the time required to obtain it.
  - It is our **objective function**, e.g., total distance, satisfaction, or cut size.
- The **optimum cost** is $\textsc{opt}(x) = \min_{s \in F(x)} c(s)$ for a minimization problem.
- It is $\textsc{opt}(x) = \max_{s \in F(x)} c(s)$ for a maximization problem.

Approximation Algorithms

- Let algorithm M on x returns a feasible solution.
- M is an **ϵ-approximation algorithm**, where $\epsilon \geq 0$, if for all x,

$$\frac{|c(M(x)) - \text{OPT}(x)|}{\max(\text{OPT}(x), c(M(x)))} \leq \epsilon.$$

  - For a minimization problem,

$$\frac{c(M(x)) - \min_{s \in F(x)} c(s)}{c(M(x))} \leq \epsilon.$$

  - For a maximization problem,

$$\frac{\max_{s \in F(x)} c(s) - c(M(x))}{\max_{s \in F(x)} c(s)} \leq \epsilon. \tag{10}$$

Lower and Upper Bounds

- For a minimization problem,

$$\min_{s \in F(x)} c(s) \leq c(M(x)) \leq \frac{\min_{s \in F(x)} c(s)}{1 - \epsilon}.$$

  - So **approximation ratio** $\frac{\min_{s \in F(x)} c(s)}{c(M(x))} \geq 1 - \epsilon$.

- For a maximization problem,

$$(1 - \epsilon) \times \max_{s \in F(x)} c(s) \leq c(M(x)) \leq \max_{s \in F(x)} c(s). \tag{11}$$

  - So approximation ratio $\frac{c(M(x))}{\max_{s \in F(x)} c(s)} \geq 1 - \epsilon$.

- They are alternative definitions of ϵ-approximation.

Range Bounds

- ϵ takes values between 0 and 1.
- For maximization problems, an ϵ-approximation algorithm returns solutions within $[\,(1-\epsilon)\times \text{OPT}, \text{OPT}\,]$.
- For minimization problems, an ϵ-approximation algorithm returns solutions within $[\,\text{OPT}, \frac{\text{OPT}}{1-\epsilon}\,]$.
- For each NP-complete optimization problem, we shall be interested in determining the *smallest* ϵ for which there is a polynomial-time ϵ-approximation algorithm.
- Sometimes ϵ has no minimum value.

Approximation Thresholds

- The **approximation threshold** is the greatest lower bound of all $\epsilon \geq 0$ such that there is a polynomial-time ϵ-approximation algorithm.
- The approximation threshold of an optimization problem can be anywhere between 0 (approximation to any desired degree) and 1 (no approximation is possible).
- If P = NP, then all optimization problems in NP have an approximation threshold of 0.
- So we assume P $\neq$ NP for the rest of the discussion.

NODE COVER

- NODE COVER seeks the smallest $C \subseteq V$ in graph $G = (V, E)$ such that for each edge in E, at least one of its endpoints is in C.
- A heuristic to obtain a good node cover is to iteratively move a node with the highest degree to the cover.
- This turns out to produce

$$\frac{c(M(x))}{\text{OPT}(x)} = \Theta(\log n).$$

- Hence the approximation ratio is $\Theta(\log^{-1} n)$.
- It is not an ϵ-approximation algorithm for any $\epsilon < 1$.

A 0.5-Approximation Algorithm[a]

$C := \emptyset$;
while $E \neq \emptyset$ **do**
    Delete an arbitrary edge $\{ u, v \}$ from E;
    Delete edges incident with u and v from E;
    Add u and v to C; {Add 2 nodes to C each time.}
end while
return C;

---

[a] Johnson (1974).

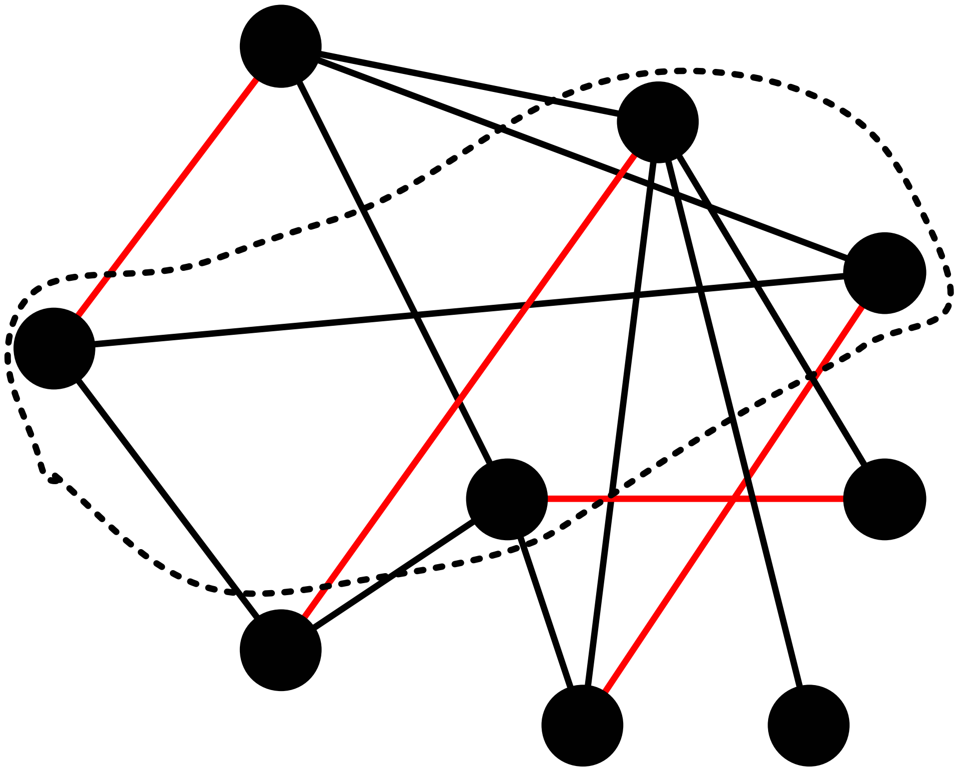
Analysis

- C contains $|C|/2$ edges.
- No two edges of C share a node.[a]
- *Any* node cover must contain at least one node from each of these edges.
- This means that $\text{OPT}(G) \geq |C|/2$.
- So
$$\frac{\text{OPT}(G)}{|C|} \geq 1/2.$$
- The approximation threshold is ≤ 0.5.[b]

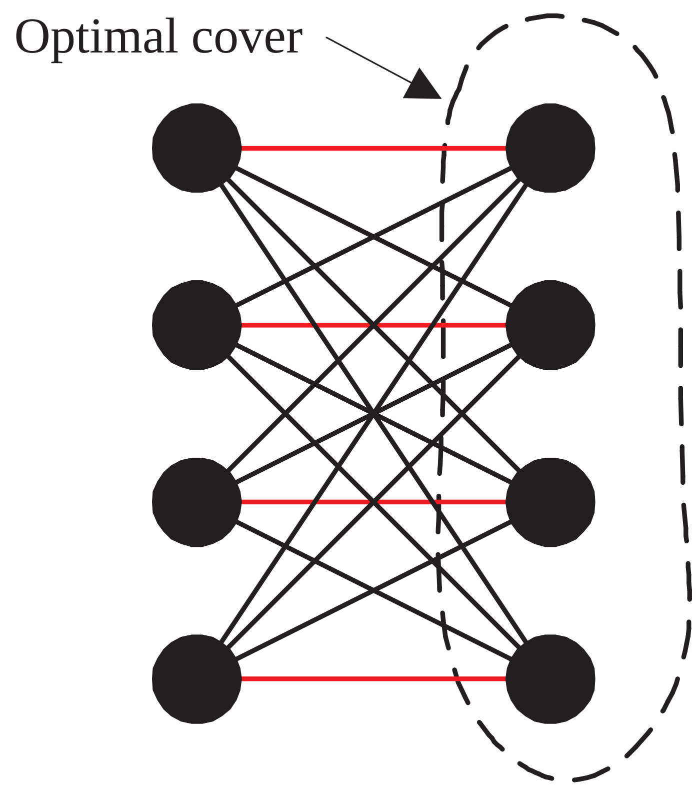
---

[a] In fact, C is a *maximal* matching.

[b] 0.5 is also the lower bound for any "greedy" algorithms (see Davis and Impagliazzo (2004)).



The 0.5 Bound Is Tight for the Algorithm[a]



[a]Contributed by Mr. Jenq-Chung Li (R92922087) on December 20, 2003.