# Theory of Computation

## Solutions to Homework 1

**Problem 1.** Briefly describe a Turing machine that accepts a string $x \in \{0, 1\}^*$ if and only if $x$ contains at least one 0. You do not need to specify the exact states and state transitions of the Turing machine. Just sketch the idea.

*Solution.* The Turing machine scans the input from left to right and accepts once it reads a 0. □

**Problem 2.** Let $M$ be a one-string Turing machine and denote by $T(M, \epsilon)$ the number of configurations that $M$ goes through on the empty input $\epsilon$. Denote by $(q^{(n)}, w^{(n)}, u^{(n)})$ the $n$-th configuration of $M$ on $\epsilon$, $1 \leq n \leq T(M, \epsilon)$, where we adhere to the representation of configurations in the slides. For $1 \leq n \leq T(M, \epsilon)$ and $i \geq 1$, write $A^{(n)}[i]$ for the $i$-th symbol, counting from left to right, of the concatenation of $w^{(n)}, u^{(n)}$ and an infinite string of $\sqcup$s. That is, $A^{(n)}[i]$ is the $i$-th symbol of $M$'s string at the $n$-th configuration starting from the input $\epsilon$. Briefly argue why $A^{(n)}[k+1]$ is uniquely determined given $q^{(n-1)}$, $A^{(n-1)}[k]$, $A^{(n-1)}[k+1]$, $A^{(n-1)}[k+2]$ and the length of $w^{(n-1)}$, for each $2 \leq n \leq T(M, \epsilon)$ and $k \geq 1$.

*Proof.* Clearly, $A^{(n)}[k+1]$ is uniquely determined given the $(n-1)$-th configuration of $M$ on $\epsilon$. As a Turing machine moves its cursor at most once in each step, $A^{(n)}[k+1]$ must be uniquely determined given $q^{(n-1)}$, $A^{(n-1)}[k]$, $A^{(n-1)}[k+1]$, $A^{(n-1)}[k+2]$ and the length of $w^{(n-1)}$. □

**Comment 1.** *A similar observation leads us somewhat close to proving the Cook-Levin theorem. In this problem, however, $A^{(n)}[k+1]$ can also be uniquely determined given $q^{(n-1)}$, $A^{(n-1)}[k+1]$ and the length of $w^{(n-1)}$ because a Turing machine alters only the character under the cursor at any time step.*