Legendre's Law of Quadratic Reciprocity[a]

- Let p and q be two odd primes.
- The next result says their Legendre symbols are distinct if and only if both numbers are 3 mod 4.

Lemma 65 (Legendre (1785), Gauss)

$$(p|q)(q|p) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}.$$

[a]First stated by Euler in 1751. Legendre (1785) did not give a correct proof. Gauss proved the theorem when he was 19. He gave at least 6 different proofs during his life. The 152nd proof appeared in 1963.

The Proof (continued)

- Sum the elements of R' in the previous proof in mod2.
- On one hand, this is just $\sum_{i=1}^{(p-1)/2} i \bmod 2$.
- On the other hand, the sum equals

$$
\begin{aligned}
& \sum_{i=1}^{(p-1)/2} \left(qi - p \left\lfloor \frac{iq}{p} \right\rfloor \right) + mp \bmod 2 \\
= \quad & \left(q \sum_{i=1}^{(p-1)/2} i - p \sum_{i=1}^{(p-1)/2} \left\lfloor \frac{iq}{p} \right\rfloor \right) + mp \bmod 2.
\end{aligned}
$$

  - Signs are irrelevant under mod2.
  - m is as in Lemma 64 (p. 469).

The Proof (continued)

- Ignore odd multipliers to make the sum equal

$$\left(\sum_{i=1}^{(p-1)/2} i - \sum_{i=1}^{(p-1)/2} \left\lfloor \frac{iq}{p} \right\rfloor \right) + m \bmod 2.$$

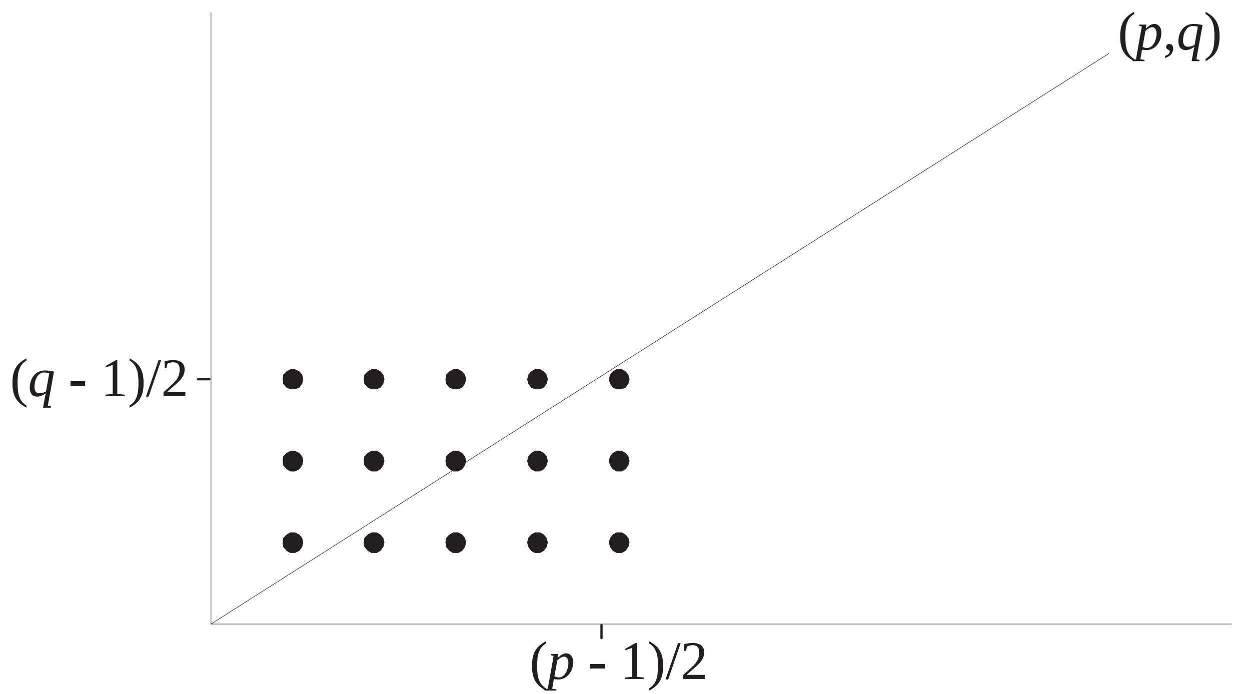
- Equate the above with $\sum_{i=1}^{(p-1)/2} i \bmod 2$ to obtain

$$m = \sum_{i=1}^{(p-1)/2} \left\lfloor \frac{iq}{p} \right\rfloor \bmod 2.$$

The Proof (concluded)

- $\sum_{i=1}^{(p-1)/2} \lfloor \frac{iq}{p} \rfloor$ is the number of integral points under the line $y = (q/p)\, x$ for $1 \leq x \leq (p-1)/2$.
- Gauss's lemma (p. 469) says $(q|p) = (-1)^m$.
- Repeat the proof with p and q reversed.
- So $(p|q) = (-1)^{m'}$, where m' is the number of integral points *above* the line $y = (q/p)\, x$ for $1 \leq y \leq (q-1)/2$.
- As a result, $(p|q)(q|p) = (-1)^{m+m'}$.
- But $m + m'$ is the total number of integral points in the $\frac{p-1}{2} \times \frac{q-1}{2}$ rectangle, which is $\frac{p-1}{2}\, \frac{q-1}{2}$.

Eisenstein's Rectangle



$p = 11$ and $q = 7$.

The Jacobi Symbol[a]

- The Legendre symbol only works for odd *prime* moduli.
- The **Jacobi symbol** $(a \mid m)$ extends it to cases where m is not prime.
- Let $m = p_1 p_2 \cdots p_k$ be the prime factorization of m.
- When $m > 1$ is odd and $\gcd(a, m) = 1$, then

$$(a|m) = \prod_{i=1}^{k} (a \mid p_i).$$

  - Note that the Jacobi symbol equals ± 1.
- Define $(a \mid 1) = 1$.

[a] Carl Jacobi (1804–1851).

Properties of the Jacobi Symbol

The Jacobi symbol has the following properties, for arguments for which it is defined.

1. $(ab \,|\, m) = (a \,|\, m)(b \,|\, m)$.
2. $(a \,|\, m_1 m_2) = (a \,|\, m_1)(a \,|\, m_2)$.
3. If $a = b \bmod m$, then $(a \,|\, m) = (b \,|\, m)$.
4. $(-1 \,|\, m) = (-1)^{(m-1)/2}$ (by Lemma 64 on p. 469).
5. $(2 \,|\, m) = (-1)^{(m^2-1)/8}$.[a]
6. If a and m are both odd, then $(a \,|\, m)(m \,|\, a) = (-1)^{(a-1)(m-1)/4}$.

---

[a] By Lemma 64 (p. 469) and some parity arguments.

Calculation of $(2200|999)$

Similar to the Euclidean algorithm and does *not* require factorization.

$$\begin{aligned}
&(202|999) = (-1)^{(999^2-1)/8}(101|999)\\
= \;& (-1)^{124750}(101|999) = (101|999)\\
= \;& (-1)^{(100)(998)/4}(999|101) = (-1)^{24950}(999|101)\\
= \;& (999|101) = (90|101) = (-1)^{(101^2-1)/8}(45|101)\\
= \;& (-1)^{1275}(45|101) = -(45|101)\\
= \;& -(-1)^{(44)(100)/4}(101|45) = -(101|45) = -(11|45)\\
= \;& -(-1)^{(10)(44)/4}(45|11) = -(45|11)\\
= \;& -(1|11) = -(11|1) = -1.
\end{aligned}$$

A Result Generalizing Proposition 10.3 in the Textbook

Theorem 66 *The group of set $\Phi(n)$ under multiplication* mod n *has a primitive root if and only if n is either 1, 2, 4, p^k, or $2p^k$ for some nonnegative integer k and and odd prime p.*

This result is essential in the proof of the next lemma.

The Jacobi Symbol and Primality Test[a]

Lemma 67 *If* $(M|N) = M^{(N-1)/2} \bmod N$ *for all* $M \in \Phi(N)$*, then* N *is prime. (Assume* N *is odd.)*

- Assume $N = mp$, where p is an odd prime, $\gcd(m, p) = 1$, and $m > 1$ (not necessarily prime).
- Let $r \in \Phi(p)$ such that $(r \,|\, p) = -1$.
- The Chinese remainder theorem says that there is an $M \in \Phi(N)$ such that

$$\begin{aligned} M &= r \bmod p, \\ M &= 1 \bmod m. \end{aligned}$$

[a] Mr. Clement Hsiao (`R88526067`) pointed out that the textbook's proof for Lemma 11.8 is incorrect while he was a senior in January 1999.

The Proof (continued)

- By the hypothesis,

$$M^{(N-1)/2} = (M \mid N) = (M \mid p)(M \mid m) = -1 \bmod N.$$

- Hence

$$M^{(N-1)/2} = -1 \bmod m.$$

- But because $M = 1 \bmod m$,

$$M^{(N-1)/2} = 1 \bmod m,$$

  a contradiction.

The Proof (continued)

- Second, assume that $N = p^a$, where p is an odd prime and $a \geq 2$.
- By Theorem 66 (p. 481), there exists a primitive root r modulo p^a.
- From the assumption,

$$M^{N-1} = \left[M^{(N-1)/2} \right]^2 = (M|N)^2 = 1 \bmod N$$

for all $M \in \Phi(N)$.

The Proof (continued)

- As $r \in \Phi(N)$ (prove it), we have

$$r^{N-1} = 1 \bmod N.$$

- As r's exponent modulo $N = p^a$ is $\phi(N) = p^{a-1}(p-1)$,

$$p^{a-1}(p-1) \mid N-1,$$

  which implies that $p \mid N-1$.

- But this is impossible given that $p \mid N$.

The Proof (continued)

- Third, assume that $N = mp^a$, where p is an odd prime, $\gcd(m, p) = 1$, $m > 1$ (not necessarily prime), and a is even.
- The proof mimics that of the second case.
- By Theorem 66 (p. 481), there exists a primitive root r modulo p^a.
- From the assumption,

$$M^{N-1} = \left[M^{(N-1)/2} \right]^2 = (M|N)^2 = 1 \bmod N$$

for all $M \in \Phi(N)$.

The Proof (continued)

- In particular,
$$M^{N-1} = 1 \bmod p^a \tag{7}$$
for all $M \in \Phi(N)$.

- The Chinese remainder theorem says that there is an $M \in \Phi(N)$ such that
$$\begin{aligned} M &= r \bmod p^a, \\ M &= 1 \bmod m. \end{aligned}$$

- Because $M = r \bmod p^a$ and Eq. (7),
$$r^{N-1} = 1 \bmod p^a.$$

The Proof (concluded)

- As r's exponent modulo $N = p^a$ is $\phi(N) = p^{a-1}(p-1)$,

$$p^{a-1}(p-1) \mid N-1,$$

which implies that $p \mid N-1$.

- But this is impossible given that $p \mid N$.

The Number of Witnesses to Compositeness

Theorem 68 (Solovay and Strassen (1977)) *If N is an odd composite, then $(M|N) \neq M^{(N-1)/2} \bmod N$ for at least half of $M \in \Phi(N)$.*

- By Lemma 67 (p. 482) there is at least one $a \in \Phi(N)$ such that $(a|N) \neq a^{(N-1)/2} \bmod N$.
- Let $B = \{b_1, b_2, \ldots, b_k\} \subseteq \Phi(N)$ be the set of *all* distinct residues such that $(b_i|N) = b_i^{(N-1)/2} \bmod N$.
- Let $aB = \{ab_i \bmod N : i = 1, 2, \ldots, k\}$.

The Proof (concluded)

- $|aB| = k$.
  - $ab_i = ab_j \bmod N$ implies $N | a(b_i - b_j)$, which is impossible because $\gcd(a, N) = 1$ and $N > |b_i - b_j|$.
- $aB \cap B = \emptyset$ because

$$(ab_i)^{(N-1)/2} = a^{(N-1)/2} b_i^{(N-1)/2} \neq (a|N)(b_i|N) = (ab_i|N).$$

- Combining the above two results, we know

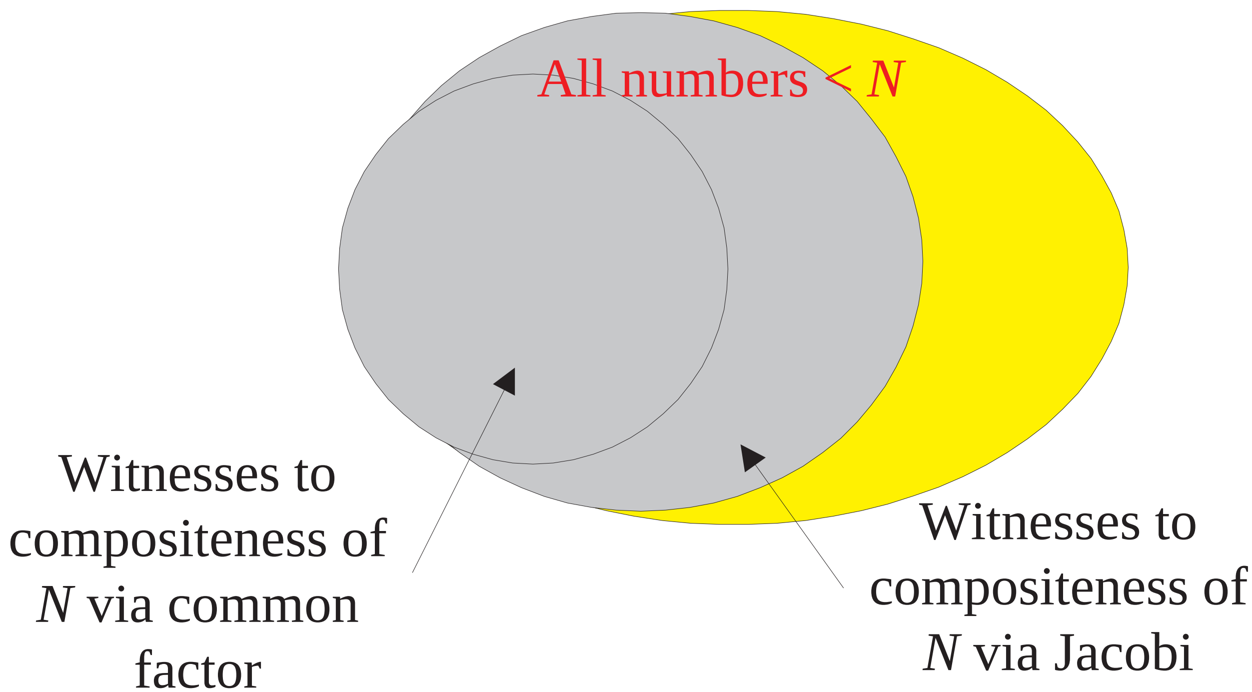
$$\frac{|\,B\,|}{\phi(N)} \leq 0.5.$$

if N is even but $N \neq 2$ **then**
    return "N is composite";
else if $N = 2$ **then**
    return "N is a prime";
end if
Pick $M \in \{2, 3, \ldots, N-1\}$ randomly;
if $\gcd(M, N) > 1$ **then**
    return "N is a composite";
else
    if $(M|N) \neq M^{(N-1)/2} \bmod N$ **then**
        return "N is composite";
    else
        return "N is a prime";
    end if
end if

Analysis

- The algorithm certainly runs in polynomial time.
- There are no false positives (for COMPOSITENESS).
  - When the algorithm says the number is composite, it is always correct.
- The probability of a false negative is at most one half.
  - If the input is composite, then the probability that the algorithm says the number is a prime is ≤ 0.5.
- The error probability can be reduced but not eliminated.

The Improved Density Attack for COMPOSITENESS



Randomized Complexity Classes; RP

- Let N be a polynomial-time precise NTM that runs in time $p(n)$ and has 2 nondeterministic choices at each step.
- N is a **polynomial Monte Carlo Turing machine** for a language L if the following conditions hold:
  - If $x \in L$, then at least half of the $2^{p(n)}$ computation paths of N on x halt with "yes" where $n = |\, x\,|$.
  - If $x \notin L$, then all computation paths halt with "no."
- The class of all languages with polynomial Monte Carlo TMs is denoted **RP** (**randomized polynomial time**).[a]

[a] Adleman and Manders (1977).

Comments on RP

- Nondeterministic steps can be seen as fair coin flips.
- There are no false positive answers.
- The probability of false negatives, $1 - \epsilon$, is at most 0.5.
- But any constant between 0 and 1 can replace 0.5.
  - By repeating the algorithm $k = \lceil -\frac{1}{\log_2 1-\epsilon} \rceil$ times, the probability of false negatives becomes $(1-\epsilon)^k \leq 0.5$.
- In fact, ϵ can be arbitrarily close to 0 as long as it is of the order $1/p(n)$ for some polynomial $p(n)$.
  - $-\frac{1}{\log_2 1-\epsilon} = O(\frac{1}{\epsilon}) = O(p(n))$.

Where RP Fits

- $P \subseteq RP \subseteq NP$.
  - A deterministic TM is like a Monte Carlo TM except that all the coin flips are ignored.
  - A Monte Carlo TM is an NTM with extra demands on the number of accepting paths.
- COMPOSITENESS $\in$ RP; PRIMES $\in$ coRP; PRIMES $\in$ RP.[a]
  - In fact, PRIMES $\in$ P.[b]
- $RP \cup coRP$ is another "plausible" notion of efficient computation.

---

[a] Adleman and Huang (1987).
[b] Agrawal, Kayal, and Saxena (2002).

ZPP[a] (Zero Probabilistic Polynomial)

- The class **ZPP** is defined as $\text{RP} \cap \text{coRP}$.
- A language in ZPP has *two* Monte Carlo algorithms, one with no false positives and the other with no false negatives.
- If we repeatedly run both Monte Carlo algorithms, *eventually* one definite answer will come (unlike RP).
  - A *positive* answer from the one without false positives.
  - A *negative* answer from the one without false negatives.

[a] Gill (1977).

The ZPP Algorithm (**Las Vegas**)

{Suppose $L \in \mathrm{ZPP}$.}
{N_1 has no false positives, and N_2 has no false negatives.}
while `true` **do**
  if $N_1(x) =$ "yes" **then**
    return "yes";
  end if
  if $N_2(x) =$ "no" **then**
    return "no";
  end if
end while

ZPP (concluded)

- The *expected* running time for the correct answer to emerge is polynomial.
  - The probability that a run of the 2 algorithms does not generate a definite answer is 0.5.
  - Let $p(n)$ be the running time of each run.
  - The expected running time for a definite answer is
$$\sum_{i=1}^{\infty} 0.5^i i p(n) = 2p(n).$$
- Essentially, ZPP is the class of problems that can be solved without errors in expected polynomial time.

Et Tu, RP?

```
{Suppose L ∈ RP.}
{N decides L without false positives.}
while true do
   if N(x) = "yes" then
      return "yes";
   end if
   {But what to do here?}
end while
```

- You eventually get a "yes" if $x \in L$.
- But how to get a "no" when $x \notin L$?
- You have to sacrifice either correctness or bounded running time.

Large Deviations

- Suppose you have a *biased* coin.
- One side has probability $0.5 + \epsilon$ to appear and the other $0.5 - \epsilon$, for some $0 < \epsilon < 0.5$.
- But you do not know which is which.
- How to decide which side is the more likely—with high confidence?
- Answer: Flip the coin many times and pick the side that appeared the most times.
- Question: Can you quantify the confidence?

The Chernoff Bound[a]

Theorem 69 (Chernoff (1952)) *Suppose* $x_1, x_2, \ldots, x_n$ *are independent random variables taking the values 1 and 0 with probabilities* p *and* $1-p$*, respectively. Let* $X = \sum_{i=1}^{n} x_i$*. Then for all* $0 \leq \theta \leq 1$*,*

$$\text{prob}[\, X \geq (1+\theta)\, pn\,] \leq e^{-\theta^2 pn/3}.$$

- The probability that the deviate of a **binomial random variable** from its expected value $E[\,X\,] = E[\,\sum_{i=1}^{n} x_i\,] = pn$ decreases exponentially with the deviation.
- The Chernoff bound is asymptotically optimal.

---

[a] Herman Chernoff (1923–).

The Proof

- Let t be any positive real number.
- Then

$$\text{prob}[\, X \geq (1+\theta)\, pn\,] = \text{prob}[\, e^{tX} \geq e^{t(1+\theta)\, pn}\,].$$

- Markov's inequality (p. 443) generalized to real-valued random variables says that

$$\text{prob}\left[\, e^{tX} \geq kE[\, e^{tX}\,]\right] \leq 1/k.$$

- With $k = e^{t(1+\theta)\, pn}/E[\, e^{tX}\,]$, we have

$$\text{prob}[\, X \geq (1+\theta)\, pn\,] \leq e^{-t(1+\theta)\, pn} E[\, e^{tX}\,].$$

The Proof (continued)

- Because $X = \sum_{i=1}^{n} x_i$ and x_i's are independent,

$$E[\, e^{tX} \,] = (E[\, e^{tx_1} \,])^n = [\, 1 + p(e^t - 1) \,]^n.$$

- Substituting, we obtain

$$\begin{aligned} \text{prob}[\, X \geq (1+\theta)\, pn \,] &\leq e^{-t(1+\theta)\, pn} [\, 1 + p(e^t - 1) \,]^n \\ &\leq e^{-t(1+\theta)\, pn} e^{pn(e^t - 1)} \end{aligned}$$

as $(1+a)^n \leq e^{an}$ for all $a > 0$.

The Proof (concluded)

- With the choice of $t = \ln(1+\theta)$, the above becomes

$$\text{prob}[\, X \geq (1+\theta)\, pn\,] \leq e^{pn[\,\theta-(1+\theta)\ln(1+\theta)\,]}.$$

- The exponent expands to $-\frac{\theta^2}{2}+\frac{\theta^3}{6}-\frac{\theta^4}{12}+\cdots$ for $0 \leq \theta \leq 1$, which is less than

$$-\frac{\theta^2}{2}+\frac{\theta^3}{6} \leq \theta^2\left(-\frac{1}{2}+\frac{\theta}{6}\right) \leq \theta^2\left(-\frac{1}{2}+\frac{1}{6}\right) = -\frac{\theta^2}{3}.$$

Power of the Majority Rule

From prob$[\, X \leq (1-\theta)\, pn \,] \leq e^{-\frac{\theta^2}{2} pn}$ (prove it):

Corollary 70 *If* $p = (1/2) + \epsilon$ *for some* $0 \leq \epsilon \leq 1/2$, *then*

$$\text{prob}\left[\sum_{i=1}^{n} x_i \leq n/2\right] \leq e^{-\epsilon^2 n/2}.$$

- The textbook's corollary to Lemma 11.9 seems incorrect.
- Our original problem (p. 501) hence demands $\approx 1.4k/\epsilon^2$ independent coin flips to guarantee making an error with probability at most 2^{-k} with the majority rule.

BPP[a] (Bounded Probabilistic Polynomial)

- The class **BPP** contains all languages for which there is a precise polynomial-time NTM N such that:
  - If $x \in L$, then at least $3/4$ of the computation paths of N on x lead to "yes."
  - If $x \notin L$, then at least $3/4$ of the computation paths of N on x lead to "no."
- N accepts or rejects by a *clear* majority.

---

[a] Gill (1977).

Magic $3/4$?

- The number $3/4$ bounds the probability of a right answer away from $1/2$.
- Any constant *strictly* between $1/2$ and 1 can be used without affecting the class BPP.
- In fact, 0.5 plus any inverse polynomial between $1/2$ and 1,
$$0.5 + \frac{1}{p(n)},$$
can be used.

The Majority Vote Algorithm

Suppose L is decided by N by majority $(1/2) + \epsilon$.

for $i = 1, 2, \ldots, 2k + 1$ **do**
    Run N on input x;
end for
if "yes" is the majority answer **then**
    "yes";
else
    "no";
end if

Analysis

- The running time remains polynomial, being $2k+1$ times N's running time.
- By Corollary 70 (p. 506), the probability of a false answer is at most $e^{-\epsilon^2 k}$.
- By taking $k = \lceil 2/\epsilon^2 \rceil$, the error probability is at most $1/4$.
- As with the RP case, ϵ can be any inverse polynomial, because k remains polynomial in n.

Probability Amplification for BPP

- Let m be the number of random bits used by a BPP algorithm.
  - By definition, m is polynomial in n.
- With $k = \Theta(\log m)$ in the majority vote algorithm, we can lower the error probability to, say, $\leq (3m)^{-1}$.

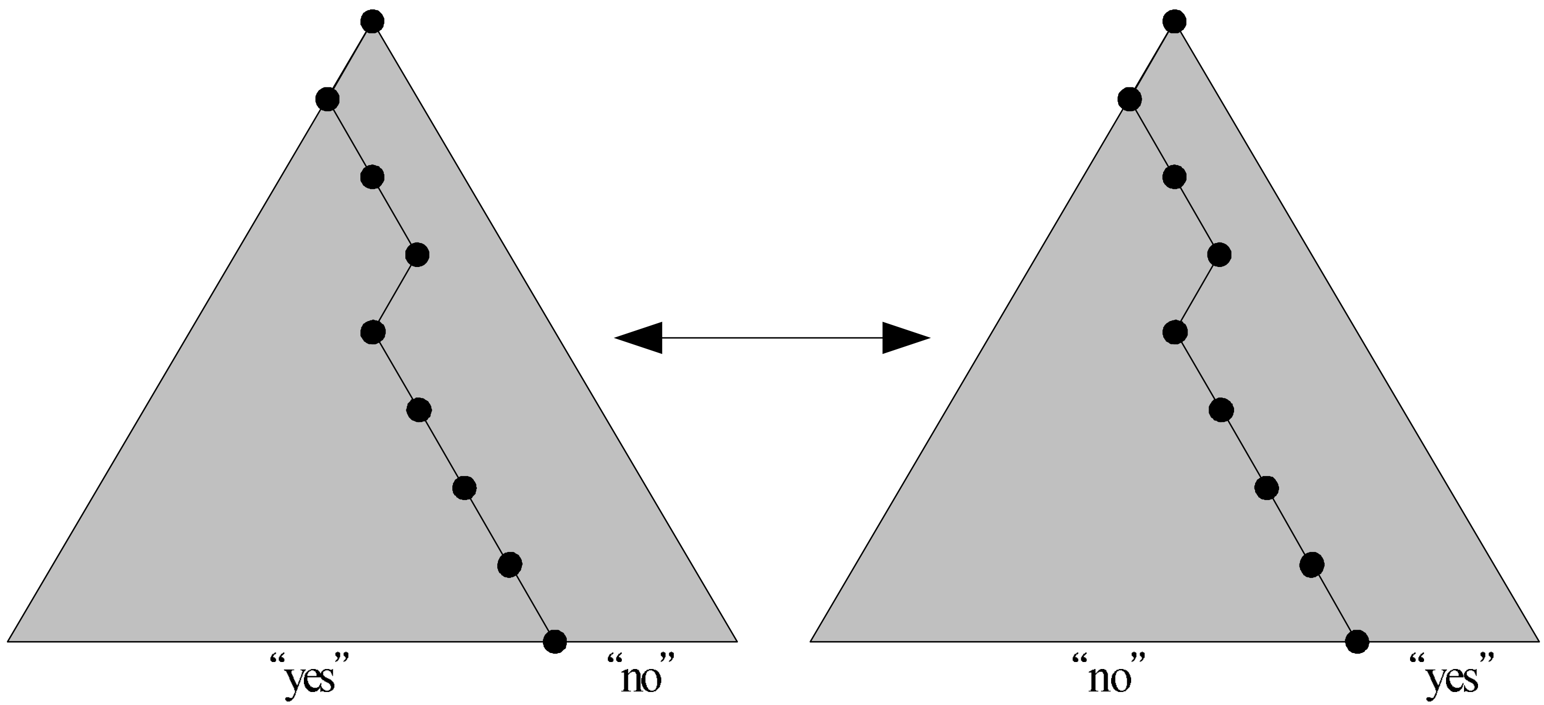
Aspects of BPP

- BPP is the most comprehensive yet plausible notion of efficient computation.
  - If a problem is in BPP, we take it to mean that the problem can be solved efficiently.
  - In this aspect, BPP has effectively replaced P.
- $(\mathrm{RP} \cup \mathrm{coRP}) \subseteq (\mathrm{NP} \cup \mathrm{coNP})$.
- $(\mathrm{RP} \cup \mathrm{coRP}) \subseteq \mathrm{BPP}$.
- Whether $\mathrm{BPP} \subseteq (\mathrm{NP} \cup \mathrm{coNP})$ is unknown.
- But it is unlikely that $\mathrm{NP} \subseteq \mathrm{BPP}$ (p. 527).

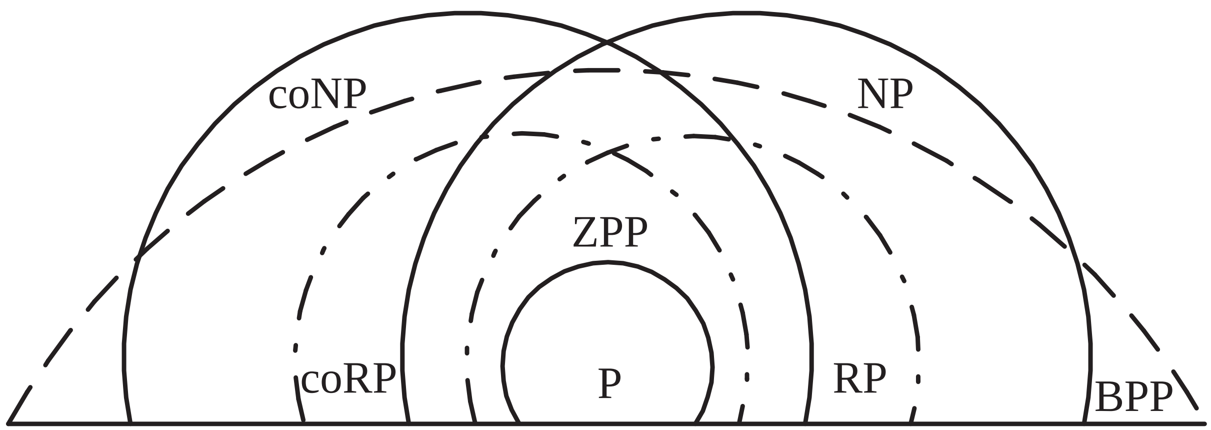
coBPP

- The definition of BPP is symmetric: acceptance by clear majority and rejection by clear majority.
- An algorithm for $L \in \text{BPP}$ becomes one for $\bar{L}$ by reversing the answer.
- So $\bar{L} \in \text{BPP}$ and $\text{BPP} \subseteq \text{coBPP}$.
- Similarly $\text{coBPP} \subseteq \text{BPP}$.
- Hence $\text{BPP} = \text{coBPP}$.
- This approach does not work for RP.
- It did not work for NP either.

BPP and coBPP



"The Good, the Bad, and the Ugly"



Circuit Complexity

- Circuit complexity is based on boolean circuits instead of Turing machines.
- A boolean circuit with n inputs computes a boolean function of n variables.
- By identify `true` with 1 and `false` with 0, a boolean circuit with n inputs accepts certain strings in $\{0, 1\}^n$.
- To relate circuits with arbitrary languages, we need one circuit for each possible input length n.

Formal Definitions

- The **size** of a circuit is the number of *gates* in it.
- A **family of circuits** is an infinite sequence $\mathcal{C} = (C_0, C_1, \ldots)$ of boolean circuits, where C_n has n boolean inputs.
- $L \subseteq \{0,1\}^*$ has **polynomial circuits** if there is a family of circuits $\mathcal{C}$ such that:
  - The size of C_n is at most $p(n)$ for some fixed polynomial p.
  - For input $x \in \{0,1\}^*$, $C_{|\,x\,|}$ outputs 1 if and only if $x \in L$.
    * C_n accepts $L \cap \{0,1\}^n$.

Exponential Circuits Contain All Languages

- Theorem 15 (p. 168) implies that there are languages that cannot be solved by circuits of size $2^n/(2n)$.
- But exponential circuits can solve all problems.

Proposition 71 *All decision problems (decidable or otherwise) can be solved by a circuit of size* 2^{n+2}.

- We will show that for any language $L \subseteq \{0,1\}^*$, $L \cap \{0,1\}^n$ can be decided by a circuit of size 2^{n+2}.

The Proof (concluded)

- Define boolean function $f : \{0,1\}^n \to \{0,1\}$, where

$$f(x_1 x_2 \cdots x_n) = \begin{cases} 1 & x_1 x_2 \cdots x_n \in L, \\ 0 & x_1 x_2 \cdots x_n \notin L. \end{cases}$$

- $f(x_1 x_2 \cdots x_n) = (x_1 \wedge f(1 x_2 \cdots x_n)) \vee (\neg x_1 \wedge f(0 x_2 \cdots x_n))$.
- The circuit size $s(n)$ for $f(x_1 x_2 \cdots x_n)$ hence satisfies

$$s(n) = 4 + 2s(n-1)$$

  with $s(1) = 1$.
- Solve it to obtain $s(n) = 5 \times 2^{n-1} - 4 \leq 2^{n+2}$.

Comments

- Proposition 71 (p. 518) does not contradict anything we knew so far about computation theory.
- Yes, there are only a finite number of circuits with size 2^{n+2}.
- Yes, there are only 2^n possible inputs of length n.
- Yes, those circuits can solve all problems of length n.
- But is there an algorithm to tell which circuit is the correct one?

The Circuit Complexity of P

Proposition 72 *All languages in P have polynomial circuits.*

- Let $L \in \mathrm{P}$ be decided by a TM in time $p(n)$.
- By Corollary 28 (p. 260), there is a circuit with $O(p(n)^2)$ gates that accepts $L \cap \{0,1\}^n$.
- The size of the circuit depends only on L and the length of the input.
- The size of the circuit is polynomial in n.

Languages That Polynomial Circuits Accept

- Do polynomial circuits accept only languages in P?
- There are *undecidable* languages that have polynomial circuits.
  - Let $L \subseteq \{0,1\}^*$ be an undecidable language.
  - Let $U = \{1^n : \text{the binary expansion of } n \text{ is in } L\}$.[a]
  - U is also undecidable.
  - $U \cap \{1\}^n$ can be accepted by C_n that is trivially true if $1^n \in U$ and trivially false if $1^n \notin U$.
  - The family of circuits $(C_0, C_1, \ldots)$ is polynomial in size.

[a] Assume n's leading bit is always 1 without loss of generality.