Theory of Computation Lecture Notes

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Class Information

- Papadimitriou. *Computational Complexity*. 2nd printing. Addison-Wesley. 1995.
  - The best book on the market for graduate students.
  - We more or less follow the topics of the book.
  - More "advanced" materials may be added.
- You may want to review discrete mathematics.

Class Information (concluded)

- More information and future lecture notes (in PDF format) can be found at

  `www.csie.ntu.edu.tw/~lyuu/complexity.html`

- Please ask many questions in class.
  - The best way for me to remember you in a large class.[a]
- Teaching assistants will be announced later.

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[a] "[A] science concentrator [...] said that in his eighth semester of [Harvard] college, there was not a single science professor who could identify him by name." (*New York Times*, September 3, 2003.)

Grading

- No roll calls.
- Homeworks.
- Two to three examinations.
- You must show up for the examinations, in person.
- If you cannot make it to an examination, please email me beforehand (unless there is a legitimate reason).
- Missing the final examination will earn a “fail” grade.

Problems and Algorithms

I have never done anything "useful."
— Godfrey Harold Hardy (1877–1947),
A Mathematician's Apology (1940)

What This Course Is All About

Computability: What can be computed?

- What is computation anyway?
- There are *well-defined* problems that cannot be computed.
- In fact, "most" problems cannot be computed.

What This Course Is All About (continued)

Complexity: What is a computable problem's inherent complexity?

- Some computable problems require at least exponential time and/or space; they are **intractable**.
  - Can't you let Moore's law take care of it?[a]
    - Moore's law says the computing power doubles every 1.5 years.[b]

[a]Contributed by Ms. Amy Liu (`J94922016`) on May 15, 2006.
[b]Moore (1965).

What This Course Is All About (concluded)

- Some practical problems require superpolynomial resources unless certain conjectures are disproved.
- Other resource limits besides time and space?
  - Program size, circuit size (growth), number of random bits, etc.

Tractability and intractability

- Polynomial in terms of the input size n defines tractability.
  - n, $n \log n$, n^2, n^{90}.
  - Time, space, circuit size, number of random bits, etc.
- It results in a fruitful and practical theory of complexity.
- Few practical, tractable problems require a large degree.
- Exponential-time or superpolynomial-time algorithms are usually impractical.
  - $n^{\log n}$, $2^{\sqrt{n}}$,[a] 2^n, $n! \sim \sqrt{2\pi n}\,(n/e)^n$.

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[a] Size of depth-3 circuits to compute the majority function (Wolfovitz (2006)).

Growth of Factorials

n	$n!$	n	$n!$
1	1	9	362,880
2	2	10	3,628,800
3	6	11	39,916,800
4	24	12	479,001,600
5	120	13	6,227,020,800
6	720	14	87,178,291,200
7	5040	15	1,307,674,368,000
8	40320	16	20,922,789,888,000

Turing Machines

Alan Turing (1912–1954)

What Is Computation?

- That can be coded in an **algorithm**.[a]
- An algorithm is a detailed step-by-step method for solving a problem.
  - The Euclidean algorithm for the greatest common divisor is an algorithm.
  - "Let s be the least upper bound of compact set A" is not an algorithm.
  - "Let s be a smallest element of a finite-sized array" can be solved by an algorithm.

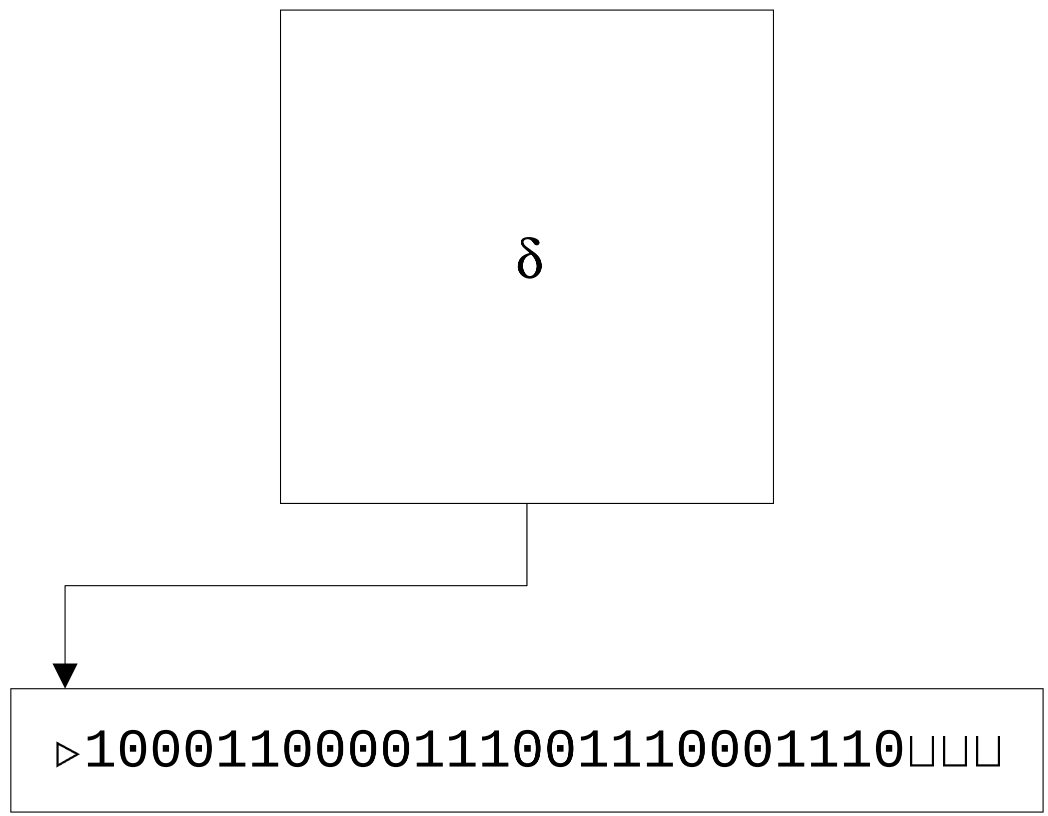
[a]Muhammad ibn Mūsā Al-Khwārizmī (780–850).

Turing Machines[a]

- A Turing machine (TM) is a quadruple $M = (K, \Sigma, \delta, s)$.
- K is a finite set of **states**.
- $s \in K$ is the **initial state**.
- Σ is a finite set of **symbols** (disjoint from K).
  - Σ includes $\sqcup$ (blank) and $\triangleright$ (first symbol).
- $\delta : K \times \Sigma \rightarrow (K \cup \{h, \text{"yes"}, \text{"no"}\}) \times \Sigma \times \{\leftarrow, \rightarrow, -\}$ is a **transition function**.
  - $\leftarrow$ (left), $\rightarrow$ (right), and $-$ (stay) signify cursor movements.

[a] Turing (1936).

A TM Schema



“Physical” Interpretations

- The tape: computer memory and registers.
- δ: program.
- K: instruction numbers.
- s: “`main()`” in C.
- Σ: **alphabet** much like the ASCII code.

More about δ

- The program has the **halting state** (h), the **accepting state** ("yes"), and the **rejecting state** ("no").
- Given current state $q \in K$ and current symbol $\sigma \in \Sigma$,

$$\delta(q, \sigma) = (p, \rho, D).$$

  - It specifies the next state p, the symbol ρ to be written over σ, and the direction D the cursor will move *afterwards*.
- We require $\delta(q, \triangleright) = (p, \triangleright, \rightarrow)$ so that the cursor never falls off the left end of the string.

The Operations of TMs

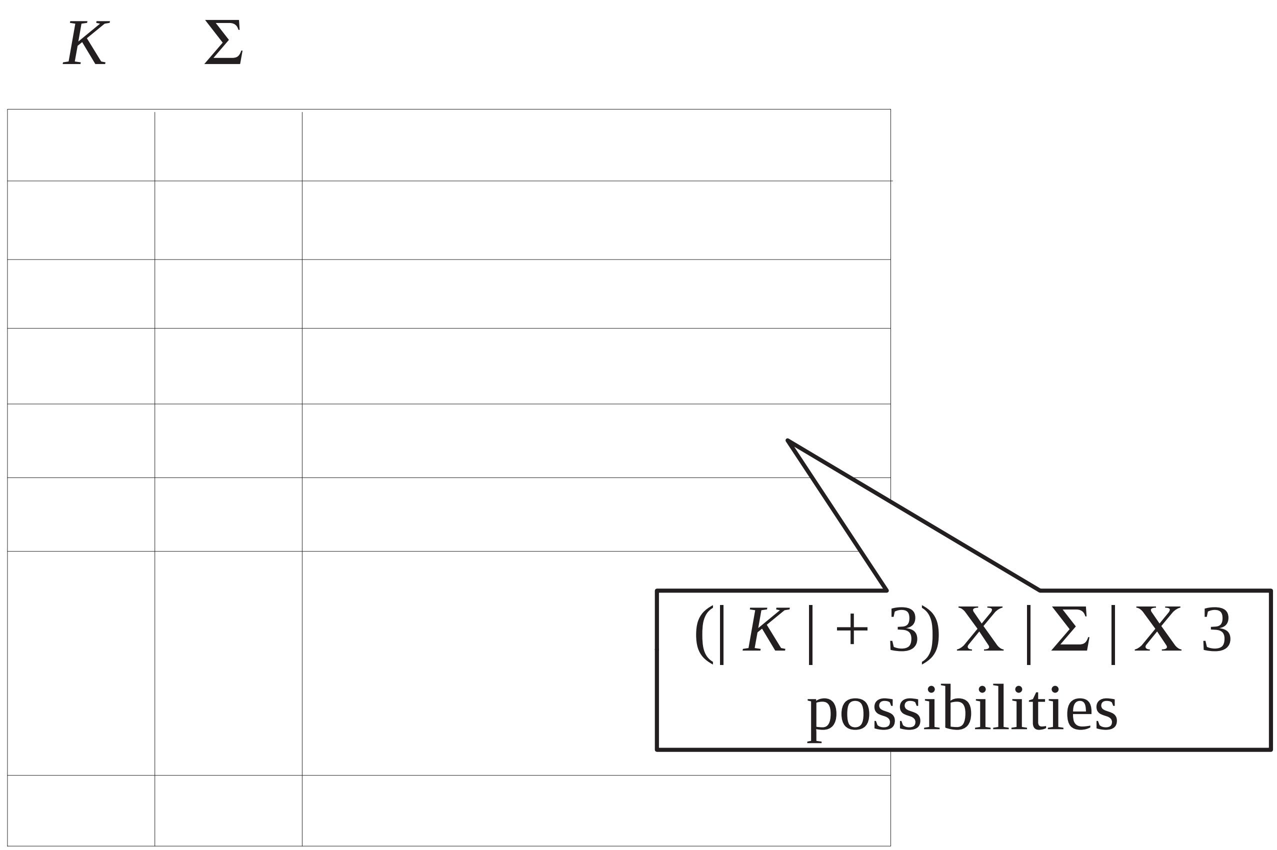
- Initially the state is s.
- The string on the tape is initialized to a $\triangleright$, followed by a *finite-length* string $x \in (\Sigma - \{\sqcup\})^*$.
- x is the **input** of the TM.
  - The input must not contain $\sqcup$s (why?)!
- The cursor is pointing to the first symbol, always a $\triangleright$.
- The TM takes each step according to δ.
- The cursor may overwrite $\sqcup$ to make the string longer during the computation.

Program Count

- A program has a *finite* size.
- Recall that
  $\delta : K \times \Sigma \to (K \cup \{h, \text{“yes”}, \text{“no”}\}) \times \Sigma \times \{\leftarrow, \rightarrow, -\}$.
- So $|K| \times |\Sigma|$ “lines” suffice to specify a program, one line per pair from $K \times \Sigma$ ($|\, x \,|$ denotes the length of x).
- Given K and Σ, there are

  $$((|K| + 3) \times |\Sigma| \times 3)^{|K| \times |\Sigma|}$$

  possible δ's (see next page).
  - This is a constant—albeit large.
- Different δ's may define the same behavior.


K
Σ
$(|K| + 3) \times |\Sigma| \times 3$
possibilities

The Halting of a TM

- A TM M may **halt** in three cases.

  "yes": M **accepts** its input x, and $M(x) =$ "yes".

  "no": M **rejects** its input x, and $M(x) =$ "no".

  h**:** $M(x) = y$, where the string (tape) consists of a $\triangleright$, followed by a finite string y, whose last symbol is not $\sqcup$, followed by a string of $\sqcup$s.

  – y is the **output** of the computation.

  – y may be empty denoted by ϵ.

- If M never halts on x, then write $M(x) = \nearrow$.

Why TMs?

- Because of the simplicity of the TM, the model has the advantage when it comes to complexity issues.
- One can develop a complexity theory based on C++ or Java, say.
- But the added complexity does not yield additional fundamental insights.
- We will describe TMs in pseudocode.

Remarks

- A problem is computable if there is a TM that halts with the correct answer.
  - If a TM (i.e., program) does not always halt, it does not solve the problem, assuming the problem is computable.[a]
  - OS does not halt as it does not solve a well-defined problem (but parts of it do).[b]

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[a] Contributed by Ms. Amy Liu (`J94922016`) on May 15, 2006. Control-C is not a legitimate way to halt a program.

[b] Contributed by Mr. Shuai-Peng Huang (`J94922019`) on May 15, 2006.

Remarks (concluded)

- Any computation model must be physically realizable.
  - A model that requires nearly infinite precision to build is not physically realizable.
  - For example, if the TM required a voltage of exactly 100 to work, it would not be considered a successful model for computation.
- Although a TM requires a tape of infinite length, which is not realizable, it is not a major conceptual problem.[a]
- A tape of infinite length cannot be used to realize infinite precision within a finite time span.[b]

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[a] Thanks to a lively discussion on September 20, 2006.

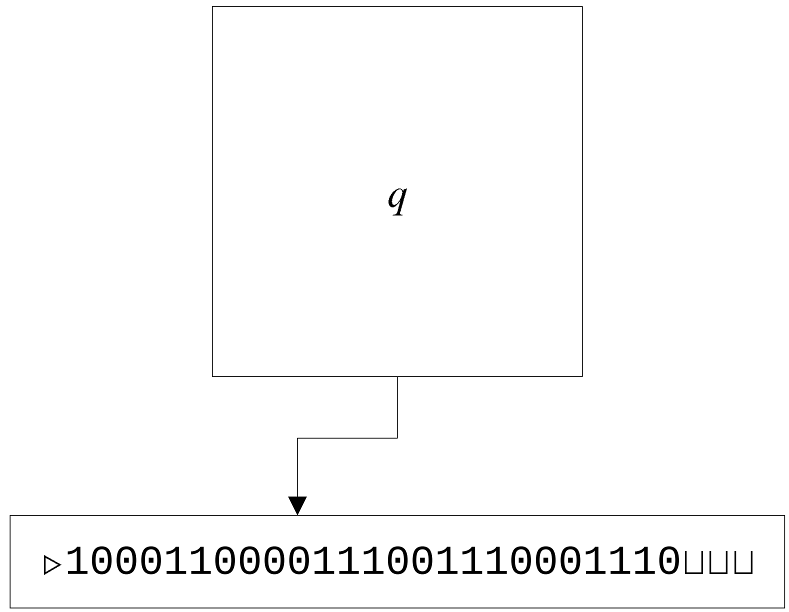
[b] Thanks to a lively discussion on September 20, 2006.

The Concept of Configuration

- A **configuration** is a complete description of the current state of the computation.
- The specification of a configuration is sufficient for the computation to continue as if it had not been stopped.
  - What does your PC save before it sleeps?
  - Enough for it to resume work later.
- Similar to the concept of **Markov process** in stochastic processes or dynamic systems.

Configurations (concluded)

- A configuration is a triple (q, w, u):
  - $q \in K$.
  - $w \in \Sigma^*$ is the string to the left of the cursor (inclusive).
  - $u \in \Sigma^*$ is the string to the right of the cursor.
- Note that (w, u) describes both the string and the cursor position.



- $w = \triangleright$1000110000.
- $u =$ 111001110001110.

Yielding

- Fix a TM M.
- Configuration (q, w, u) **yields** configuration (q', w', u') in one step,

$$(q, w, u) \xrightarrow{M} (q', w', u'),$$

  if a step of M from configuration (q, w, u) results in configuration (q', w', u').
- $(q, w, u) \xrightarrow{M^k} (q', w', u')$: Configuration (q, w, u) yields configuration (q', w', u') in $k \in \mathbb{N}$ steps.
- $(q, w, u) \xrightarrow{M^*} (q', w', u')$: Configuration (q, w, u) yields configuration (q', w', u').

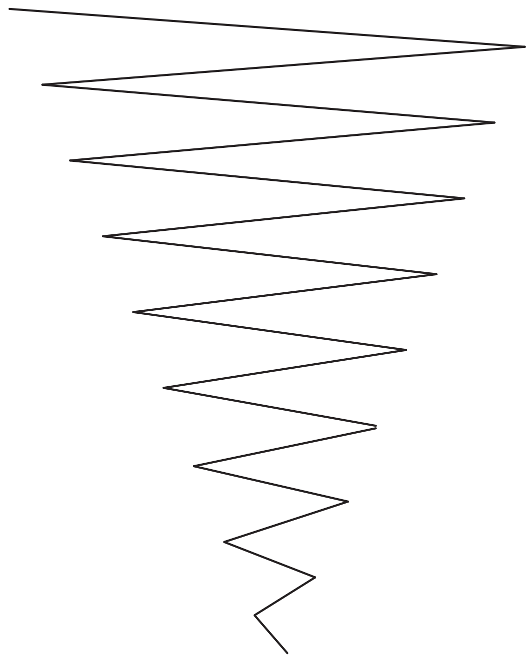
Example: How to Insert a Symbol

- We want to compute $f(x) = ax$.
  - The TM moves the last symbol of x to the right by one position.
  - It then moves the next to last symbol to the right, and so on.
  - The TM finally writes a in the first position.
- The total number of steps is $O(n)$, where n is the length of x.

Palindromes

- A string is a **palindrome** if it reads the same forwards and backwards (e.g., `001100`).
- A TM program can be written to recognize palindromes:
  - It matches the first character with the last character.
  - It matches the second character with the next to last character, etc. (see next page).
  - "yes" for palindromes and "no" for nonpalindromes.
- This program takes $O(n^2)$ steps.
- Can we do better?

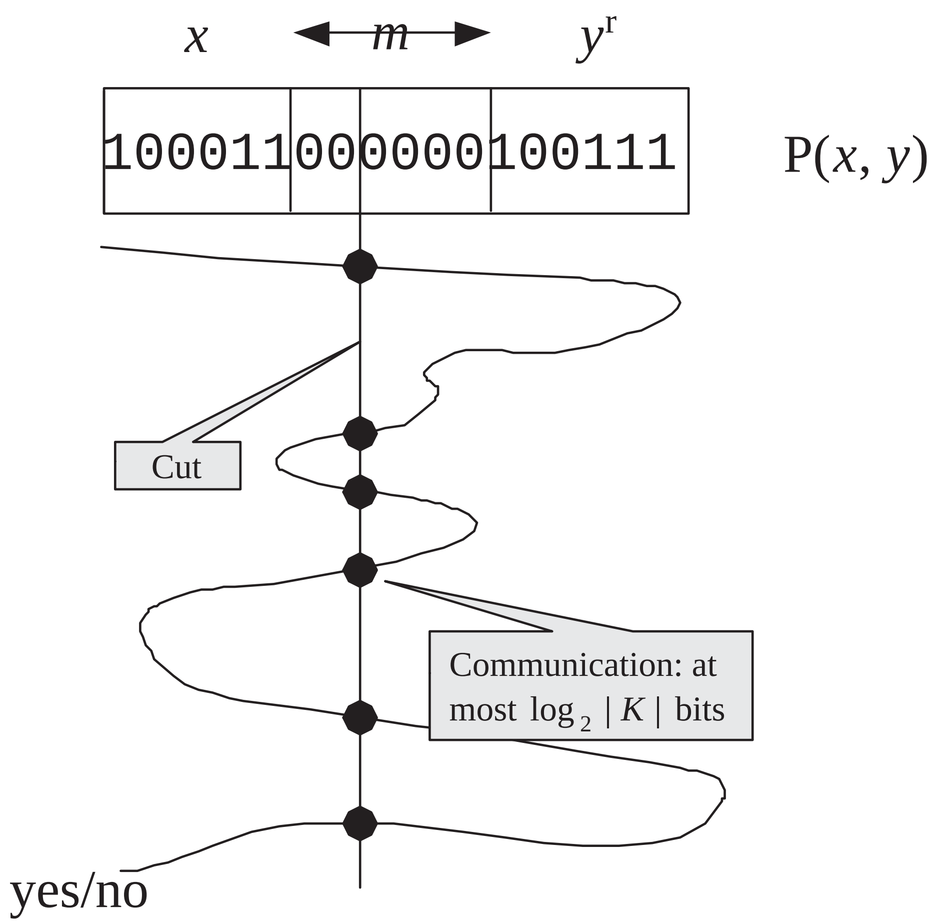
100011000000100111



A Matching Lower Bound for PALINDROME

Theorem 1 (Hennie (1965)) PALINDROME *on single-string TMs takes* $\Omega(n^2)$ *steps in the worst case.*

The Proof: Setup

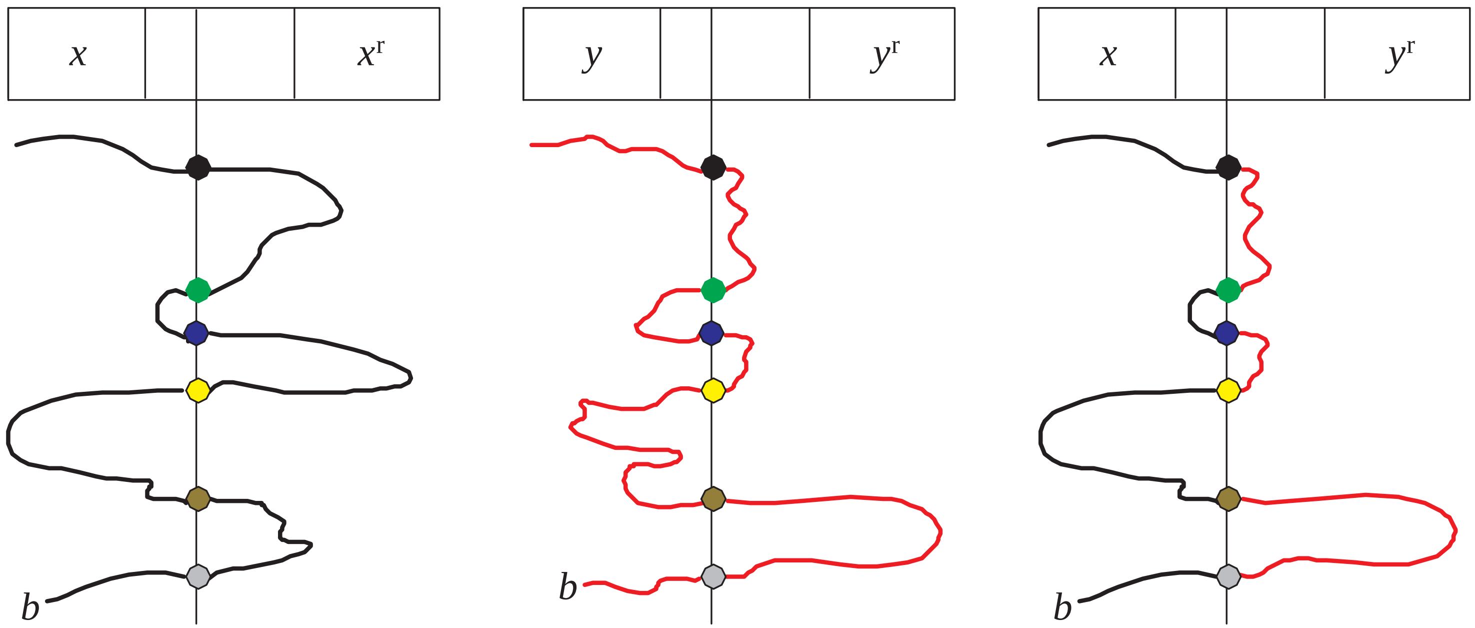


The Proof: Communications

- Our input is more restricted; hence any lower bound holds for the original problem.
- Each communication between the two halves across the cut is a state from K, hence of size $O(1)$.
- C(x, y): the sequence of communications for palindrome problem P(x, y) *across* the cut.
  - This crossing sequence is a sequence of states from K.

The Proof: Communications (concluded)

- $\mathrm{C}(x, x) \neq \mathrm{C}(y, y)$ when $x \neq y$.
  - Suppose otherwise, $C(x, x) = C(y, y)$.
  - Then $C(y, y) = C(x, y)$ by the cut-and-paste argument (see next page).
  - Hence $\mathrm{P}(x, y)$ has the same answer as $\mathrm{P}(y, y)$!
- So $\mathrm{C}(x, x)$ is distinct for each x.



x
x^{r}
b
y
y^{r}
b
x
y^{r}
b

The Proof: Amount of Communications

- Assume $|\,x\,| = |\,y\,| = m = n/3$.
- $|\,\mathrm{C}(x,x)\,|$ is the number of times the cut is crossed.
- We first seek a lower bound on the total number of communications:
$$\sum_{x \in \{0,1\}^m} |\,\mathrm{C}(x,x)\,|.$$
- Define
$$\kappa \equiv (m+1)\log_{|\,K\,|} 2 - \log_{|\,K\,|} m - 1 + \log_{|\,K\,|}(|\,K\,| - 1).$$

The Proof: Amount of Communications (continued)

- There are $\leq |\, K \,|^i$ distinct $\mathrm{C}(x, x)$s with $|\, \mathrm{C}(x, x) \,| = i$.
- Hence there are at most

$$\sum_{i=0}^{\kappa} |\, K \,|^i = \frac{|\, K \,|^{\kappa+1} - 1}{|\, K \,| - 1} \leq \frac{|\, K \,|^{\kappa+1}}{|\, K \,| - 1} = \frac{2^{m+1}}{m}$$

  distinct $\mathrm{C}(x, x)$s with $|\, \mathrm{C}(x, x) \,| \leq \kappa$.
- The rest must have $|\, \mathrm{C}(x, x) \,| > \kappa$.
- Because $\mathrm{C}(x, x)$ is distinct for each x (p. 36), there are at least $2^m - \frac{2^{m+1}}{m}$ of them with $|\, \mathrm{C}(x, x) \,| > \kappa$.

The Proof: Amount of Communications (concluded)

- Thus

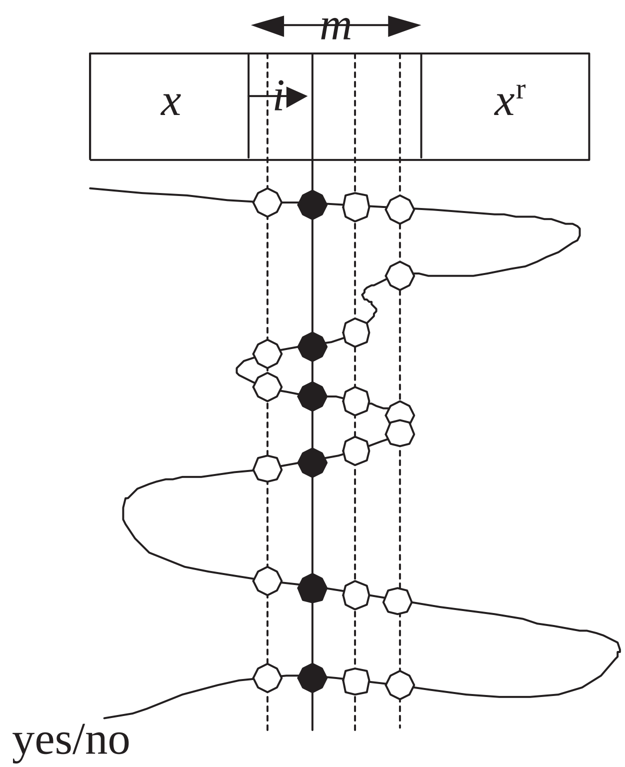
$$\begin{aligned}
\sum_{x \in \{0,1\}^m} |\, \mathrm{C}(x,x) \,| &\geq \sum_{x \in \{0,1\}^m, |\, \mathrm{C}(x,x) \,| > \kappa} |\, \mathrm{C}(x,x) \,| \\
&> \left(2^m - \frac{2^{m+1}}{m} \right) \kappa \\
&= \kappa 2^m \frac{m-2}{m}.
\end{aligned}$$

- As $\kappa = \Theta(m)$, the total number of communications is

$$\sum_{x \in \{0,1\}^m} |\, \mathrm{C}(x,x) \,| = \Omega(m 2^m). \tag{1}$$

The Proof (continued)

We now lower-bound the worst-case number of communication points in the middle section.



The Proof (continued)

- $\mathrm{C}_i(x, x)$ denotes the sequence of communications for $\mathrm{P}(x, x)$ given the cut at position i.
- Then $\sum_{i=1}^{m} |\, \mathrm{C}_i(x, x) \,|$ is the number of steps spent in the middle section for $P(x, x)$.
- Let $T(n) = \max_{x \in \{0,1\}^m} \sum_{i=1}^{m} |\, \mathrm{C}_i(x, x) \,|$.
  - $T(n)$ is the worst-case running time spent in the middle section when dealing with any $P(x, x)$ with $|\, x \,| = m$.
- Note that $T(n) \geq \sum_{i=1}^{m} |\, \mathrm{C}_i(x, x) \,|$ for any $x \in \{0, 1\}^m$.

The Proof (continued)

- Now,

$$
\begin{aligned}
& 2^m T(n) \\
= & \sum_{x \in \{0,1\}^m} T(n) \\
\geq & \sum_{x \in \{0,1\}^m} \sum_{i=1}^{m} |\, \mathrm{C}_i(x, x) \,| \\
= & \sum_{i=1}^{m} \sum_{x \in \{0,1\}^m} |\, \mathrm{C}_i(x, x) \,|.
\end{aligned}
$$

The Proof (concluded)

- By the pigeonhole principle,[a] there exists an $1 \leq i^* \leq m$,

$$\sum_{x \in \{0,1\}^m} |\, \mathrm{C}_{i^*}(x, x) \,| \leq \frac{2^m T(n)}{m}.$$

- Eq. (1) on p. 40 says that

$$\sum_{x \in \{0,1\}^m} |\, \mathrm{C}_{i^*}(x, x) \,| = \Omega(m 2^m).$$

- Hence

$$T(n) = \Omega(m^2) = \Omega(n^2).$$

[a]Dirichlet (1805–1859).

Comments on Lower-Bound Proofs

- They are usually difficult.
  - Worthy of a Ph.D. degree.
- A lower bound that matches a known upper bound given by an algorithm shows that the algorithm is optimal.
  - The simple $O(n^2)$ algorithm for PALINDROME is optimal.
- This happens rarely and is model dependent.
  - Searching, sorting, PALINDROME, matrix-vector multiplication, etc.

Decidability and Recursive Languages

- Let $L \subseteq (\Sigma - \{\sqcup\})^*$ be a **language**, i.e., a set of strings of symbols with a finite length.
  - For example, $\{0, 01, 10, 210, 1010, \ldots\}$.
- Let M be a TM such that for any string x:
  - If $x \in L$, then $M(x) =$ "yes."
  - If $x \notin L$, then $M(x) =$ "no."
- We say M **decides** L.
- If L is decided by some TM, then L is **recursive**.
  - Palindromes over $\{0, 1\}^*$ are recursive.

Acceptability and Recursively Enumerable Languages

- Let $L \subseteq (\Sigma - \{\sqcup\})^*$ be a language.
- Let M be a TM such that for any string x:
  - If $x \in L$, then $M(x) =$ "yes."
  - If $x \notin L$, then $M(x) = \nearrow$.
- We say M **accepts** L.

Acceptability and Recursively Enumerable Languages (concluded)

- If L is accepted by some TM, then L is a **recursively enumerable language**.[a]
  - A recursively enumerable language can be generated by a TM, thus the name.
  - That is, there is an algorithm such that for every $x \in L$, it will be printed out eventually.

[a] Post (1944).

Emil Post (1897–1954)

Recursive and Recursively Enumerable Languages

Proposition 2 *If L is recursive, then it is recursively enumerable.*

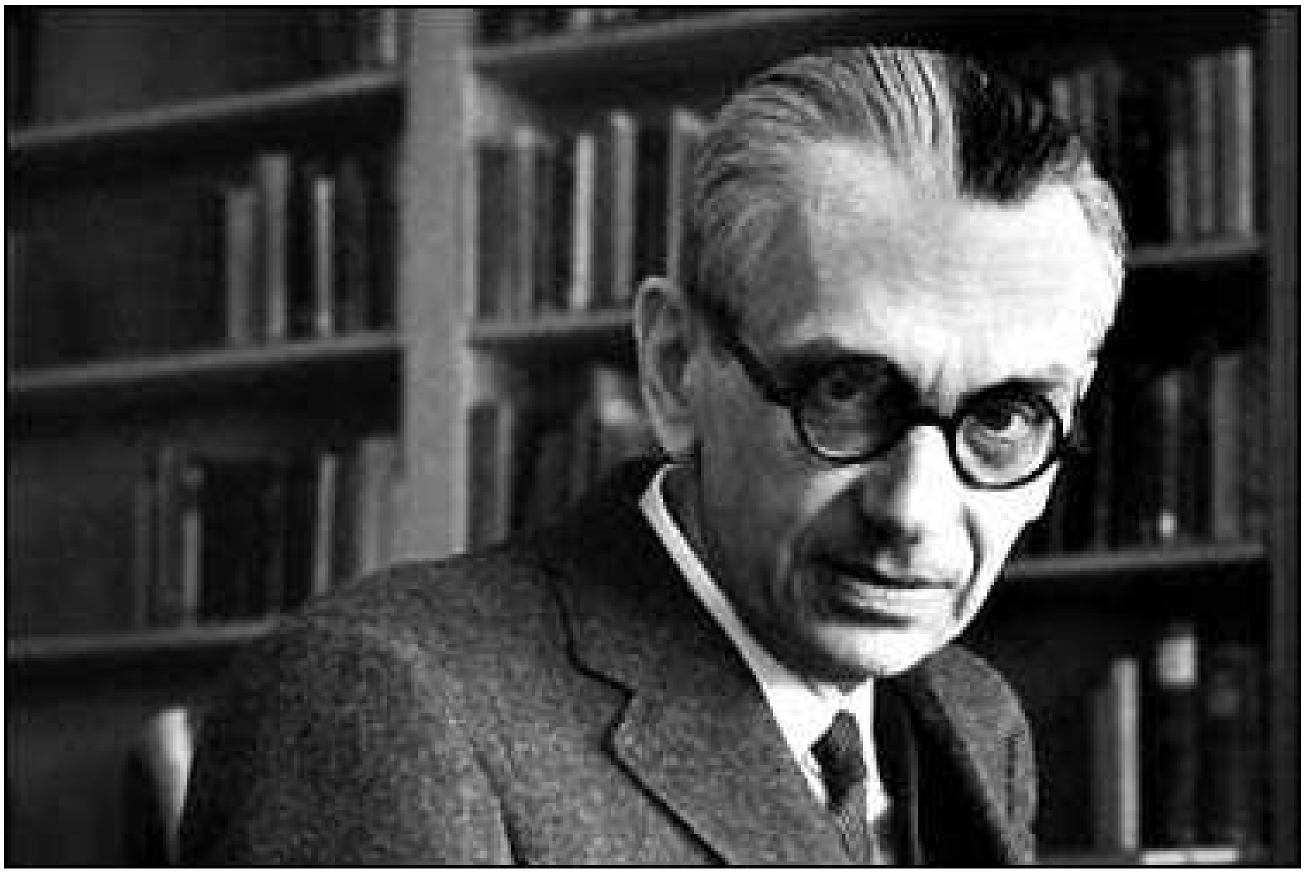
- We need to design a TM that accepts L.
- Let TM M decide L.
- We next modify M's program to obtain M' that accepts L.
- M' is identical to M except that when M is about to halt with a "no" state, M' goes into an infinite loop.
- M' accepts L.

Turing-Computable Functions

- Let $f : (\Sigma - \{\sqcup\})^* \to \Sigma^*$.
  - Optimization problems, root finding problems, etc.
- Let M be a TM with alphabet Σ.
- M **computes** f if for any string $x \in (\Sigma - \{\sqcup\})^*$, $M(x) = f(x)$.
- We call f a **recursive function**[a] if such an M exists.

[a] Kurt Gödel (1931).

Kurt Gödel (1906–1978)



Church's Thesis or the Church-Turing Thesis

- What is computable is Turing-computable; TMs are algorithms (Kleene 1953).
- Many other computation models have been proposed.
  - Recursive function (Gödel), λ calculus (Church), formal language (Post), assembly language-like RAM (Shepherdson & Sturgis), boolean circuits (Shannon), extensions of the Turing machine (more strings, two-dimensional strings, and so on), etc.
- All have been proved to be equivalent.
- No "intuitively computable" problems have been shown not to be Turing-computable (yet).

Alonso Church (1903–1995)

Stephen Kleene (1909–1994)