Random Walk Works for 2SAT

Theorem 60 *Suppose the random walk algorithm with* $r = 2n^2$ *is applied to any satisfiable* 2SAT *problem with* n *variables. Then a satisfying truth assignment will be discovered with probability at least 0.5.*

- Let $\hat{T}$ be a truth assignment such that $\hat{T} \models \phi$.
- Let $t(i)$ denote the expected number of repetitions of the flipping step until a satisfying truth assignment is found if our starting T differs from $\hat{T}$ in i values.
  - Their Hamming distance is i.

The Proof

- It can be shown that $t(i)$ is finite.
- $t(0) = 0$ because it means that $T = \hat{T}$ and hence $T \models \phi$.
- If $T \neq \hat{T}$ or T is not equal to any other satisfying truth assignment, then we need to flip at least once.
- We flip to pick among the 2 literals of a clause not satisfied by the present T.
- At least one of the 2 literals is true under $\hat{T}$, because $\hat{T}$ satisfies all clauses.
- So we have at least 0.5 chance of moving closer to $\hat{T}$.

The Proof (continued)

- Thus
$$t(i) \leq \frac{t(i-1) + t(i+1)}{2} + 1$$
for $0 < i < n$.
  - Inequality is used because, for example, T may differ from $\hat{T}$ in both literals.
- It must also hold that
$$t(n) \leq t(n-1) + 1$$
because at $i = n$, we can only decrease i.

The Proof (continued)

- As we are only interested in upper bounds, we solve
$$\begin{aligned} x(0) &= 0 \\ x(n) &= x(n-1) + 1 \\ x(i) &= \frac{x(i-1) + x(i+1)}{2} + 1, \quad 0 < i < n \end{aligned}$$
- This is one-dimensional random walk with a reflecting and an absorbing barrier.

The Proof (continued)

- Add the equations up to obtain

$$
\begin{aligned}
& x(1) + x(2) + \cdots + x(n) \\
= \; & \frac{x(0) + x(1) + 2x(2) + \cdots + 2x(n-2) + x(n-1) + x(n)}{2} \\
& + n + x(n-1).
\end{aligned}
$$

- Simplify to yield

$$\frac{x(1) + x(n) - x(n-1)}{2} = n.$$

- As $x(n) - x(n-1) = 1$, we have

$$x(1) = 2n - 1.$$

The Proof (continued)

- Iteratively, we obtain

$$
\begin{aligned}
x(2) &= 4n - 4, \\
&\vdots \\
x(i) &= 2in - i^2.
\end{aligned}
$$

- The worst case happens when $i = n$, in which case

$$x(n) = n^2.$$

The Proof (concluded)

- We therefore reach the conclusion that

$$t(i) \leq x(i) \leq x(n) = n^2.$$

- So the expected number of steps is at most n^2.
- The algorithm picks a running time $2n^2$.
- This amounts to invoking the Markov inequality (p. 399) with $k = 2$, with the consequence of having a probability of 0.5.

Boosting the Performance

- We can pick $r = 2mn^2$ to have an error probability of $\leq (2m)^{-1}$ by Markov's inequality.
- Alternatively, with the same running time, we can run the "$r = 2n^2$" algorithm m times.
- But the error probability is reduced to $\leq 2^{-m}$!
- Again, the gain comes from the fact that Markov's inequality does not take advantage of any specific feature of the random variable.
- The gain also comes from the fact that the two algorithms are different.

How about Random CNF?

- Select m clauses independently and uniformly from the set of all possible disjunctions of k distinct, non-complementary literals with n boolean variables.
- Let $m = cn$.
- The formula is satisfiable with probability approaching 1 as $n \to \infty$ if $c < c_k$ for some $c_k < 2^k \ln 2 - O(1)$.
- The formula is unsatisfiable with probability approaching 1 as $n \to \infty$ if $c > c_k$ for some $c_k > 2^k \ln 2 - O(k)$.
- The above bounds are not tight yet.

Primality Tests

- PRIMES asks if a number N is a prime.
- The classic algorithm tests if $k \mid N$ for $k = 2, 3, \ldots, \sqrt{N}$.
- But it runs in $\Omega(2^{n/2})$ steps, where $n = |\,N\,| = \log_2 N$.

The Density Attack for PRIMES

```
Pick k ∈ {2, ..., N − 1} randomly; {Assume N > 2.}
if k | N then
  return "N is composite";
else
  return "N is a prime";
end if
```


Analysis[a]

- Suppose $N = PQ$, a product of 2 primes.
- The probability of success is

$$< 1 - \frac{\phi(N)}{N} = 1 - \frac{(P-1)(Q-1)}{PQ} = \frac{P+Q-1}{PQ}.$$

- In the case where $P \approx Q$, this probability becomes

$$< \frac{1}{P} + \frac{1}{Q} \approx \frac{2}{\sqrt{N}}.$$

- This probability is exponentially small.

[a] See also p. 358.

The Fermat Test for Primality

Fermat's "little" theorem on p. 360 suggests the following primality test for any given number p:

```
Pick a number a randomly from {1, 2, ..., N − 1};
if a^(N−1) ≠ 1 mod N then
    return "N is composite";
else
    return "N is probably a prime";
end if
```


The Fermat Test for Primality (concluded)

- Unfortunately, there are composite numbers called **Carmichael numbers** that will pass the Fermat test for *all* $a \in \{1, 2, \ldots, N-1\}$.
- There are infinitely many Carmichael numbers.[a]

[a] Alford, Granville, and Pomerance (1992).

Square Roots Modulo a Prime

- Equation $x^2 = a \bmod p$ has at most two (distinct) roots by Lemma 55 (p. 365).
  - The roots are called **square roots**.
  - Numbers a with square roots and $\gcd(a, p) = 1$ are called **quadratic residues**.
    - They are $1^2 \bmod p, 2^2 \bmod p, \ldots, (p-1)^2 \bmod p$.
- We shall show that a number either has two roots or has none, and testing which one is true is trivial.
- There are no known efficient *deterministic* algorithms to find the roots.

Euler's Test

Lemma 61 (Euler) *Let p be an odd prime and $a \neq 0 \bmod p$.*

1. *If $a^{(p-1)/2} = 1 \bmod p$, then $x^2 = a \bmod p$ has two roots.*
2. *If $a^{(p-1)/2} \neq 1 \bmod p$, then $a^{(p-1)/2} = -1 \bmod p$ and $x^2 = a \bmod p$ has no roots.*

- Let r be a primitive root of p.
- By Fermat's "little" theorem, $r^{(p-1)/2}$ is a square root of 1, so $r^{(p-1)/2} = \pm 1 \bmod p$.
- But as r is a primitive root, $r^{(p-1)/2} \neq 1 \bmod p$.
- Hence $r^{(p-1)/2} = -1 \bmod p$.

The Proof (continued)

- Suppose $a = r^{2j}$ for some $1 \leq j \leq (p-1)/2$.
- Then $a^{(p-1)/2} = r^{j(p-1)} = 1 \bmod p$ and its two *distinct* roots are $r^j, -r^j (= r^{j+(p-1)/2})$.
  - If $r^j = -r^j \bmod p$, then $2r^j = 0 \bmod p$, which implies $r^j = 0 \bmod p$, a contradiction.
- As $1 \leq j \leq (p-1)/2$, there are $(p-1)/2$ such a's.

The Proof (concluded)

- Each such a has 2 distinct square roots.
- The square roots of all the a's are distinct.
  - The square roots of different a's must be different.
- Hence the set of *square roots* is $\{1, 2, \ldots, p-1\}$.
  - That is, $\bigcup_{1 \leq a \leq p-1} \{x : x^2 = a \bmod p\} = \{1, 2, \ldots, p-1\}$.
- If $a = r^{2j+1}$, then it has no roots because all the square roots have been taken.
- $a^{(p-1)/2} = [\, r^{(p-1)/2} \,]^{2j+1} = (-1)^{2j+1} = -1 \bmod p$.

The Legendre Symbol[a] and Quadratic Residuacity Test

- By Lemma 61 (p. 420) $a^{(p-1)/2} \bmod p = \pm 1$ for $a \neq 0 \bmod p$.
- For odd prime p, define the **Legendre symbol** $(a \,|\, p)$ as

$$(a \,|\, p) = \begin{cases} 0 & \text{if } p \,|\, a, \\ 1 & \text{if } a \text{ is a quadratic residue modulo } p, \\ -1 & \text{if } a \text{ is a } \textbf{quadratic nonresidue} \text{ modulo } p. \end{cases}$$

- Euler's test implies $a^{(p-1)/2} = (a \,|\, p) \bmod p$ for any odd prime p and any integer a.
- Note that $(ab|p) = (a|p)(b|p)$.

[a] Andrien-Marie Legendre (1752–1833).

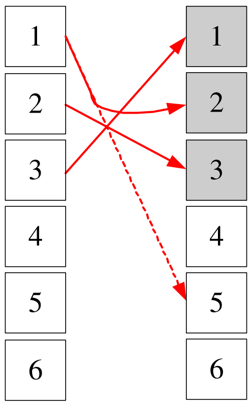
Gauss's Lemma

Lemma 62 (Gauss) *Let p and q be two odd primes. Then $(q|p) = (-1)^m$, where m is the number of residues in $R = \{iq \bmod p : 1 \leq i \leq (p-1)/2\}$ that are greater than $(p-1)/2$.*

- All residues in R are distinct.
  - If $iq = jq \bmod p$, then $p|(j-i)\,q$ or $p|q$.
- No two elements of R add up to p.
  - If $iq + jq = 0 \bmod p$, then $p|(i+j)\,q$ or $p|q$.

The Proof (continued)

- Consider the set R' of residues that result from R if we replace each of the m elements $a \in R$ such that $a > (p-1)/2$ by $p-a$.
- All residues in R' are now at most $(p-1)/2$.
- In fact, $R' = \{1, 2, \ldots, (p-1)/2\}$ (see illustration next page).
  - Otherwise, two elements of R would add up to p.



$p = 7$ and $q = 5$.

The Proof (concluded)

- Alternatively, $R' = \{\pm iq \bmod p : 1 \leq i \leq (p-1)/2\}$, where exactly m of the elements have the minus sign.
- Take the product of all elements in the two representations of R'.
- So $[(p-1)/2]! = (-1)^m q^{(p-1)/2} [(p-1)/2]! \bmod p$.
- Because $\gcd([(p-1)/2]!, p) = 1$, the lemma follows.

Legendre's Law of Quadratic Reciprocity[a]

- Let p and q be two odd primes.
- The next result says their Legendre symbols are distinct if and only if both numbers are 3 mod 4.

Lemma 63 (Legendre (1785), Gauss)

$$(p|q)(q|p) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}.$$

[a] First stated by Euler in 1751. Legendre (1785) did not give a correct proof. Gauss proved the theorem when he was 19. He gave at least 6 different proofs during his life. The 152nd proof appeared in 1963.

The Proof (continued)

- Sum the elements of R' in the previous proof in mod2.
- On one hand, this is just $\sum_{i=1}^{(p-1)/2} i \bmod 2$.
- On the other hand, the sum equals

$$\sum_{i=1}^{(p-1)/2} \left(qi - p \left\lfloor \frac{iq}{p} \right\rfloor \right) + mp \bmod 2$$

$$= \left(q \sum_{i=1}^{(p-1)/2} i - p \sum_{i=1}^{(p-1)/2} \left\lfloor \frac{iq}{p} \right\rfloor \right) + mp \bmod 2.$$

  - Signs are irrelevant under mod2.
  - m is as in Lemma 62 (p. 424).

The Proof (continued)

- Ignore odd multipliers to make the sum equal

$$\left(\sum_{i=1}^{(p-1)/2} i - \sum_{i=1}^{(p-1)/2} \left\lfloor \frac{iq}{p} \right\rfloor \right) + m \bmod 2.$$

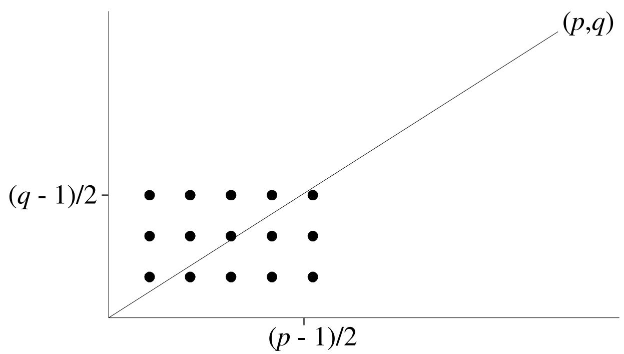
- Equate the above with $\sum_{i=1}^{(p-1)/2} i \bmod 2$ to obtain

$$m = \sum_{i=1}^{(p-1)/2} \left\lfloor \frac{iq}{p} \right\rfloor \bmod 2.$$

The Proof (concluded)

- $\sum_{i=1}^{(p-1)/2} \lfloor \frac{iq}{p} \rfloor$ is the number of integral points under the line $y = (q/p)\,x$ for $1 \leq x \leq (p-1)/2$.
- Gauss's lemma (p. 424) says $(q|p) = (-1)^m$.
- Repeat the proof with p and q reversed.
- We obtain $(p|q)$ is -1 raised to the number of integral points *above* the line $y = (q/p)\,x$ for $1 \leq y \leq (q-1)/2$.
- So $(p|q)(q|p)$ is -1 raised to the total number of integral points in the $\frac{p-1}{2} \times \frac{q-1}{2}$ rectangle, which is $\frac{p-1}{2}\,\frac{q-1}{2}$.

Eisenstein's Rectangle



$p = 11$ and $q = 7$.

Remarks[a]

- $\lfloor \frac{iq}{p} \rfloor = (q-1)/2$ when $i = (p-1)/2$ and $p > q$ (as on p. 432).
  - Note that $\frac{\lfloor (p-1)/2 \rfloor q}{p} = q(\frac{1}{2} - \frac{1}{2p})$.
  - Hence
$$\frac{\lfloor (p-1)/2 \rfloor q}{p} < q\left(\frac{1}{2} + \frac{1}{2q}\right) = (q+1)/2,$$
$$\frac{\lfloor (p-1)/2 \rfloor q}{p} > q\left(\frac{1}{2} - \frac{1}{2q}\right) = (q-1)/2.$$
- Similarly, $\lceil \frac{iq}{p} \rceil = (q-1)/2$ when $i = (p-1)/2$ and $p < q$.

[a]Observation and proof by Mr. Wei-Cheng Cheng (R93922108) on December 1, 2004.

The Jacobi Symbol[a]

- The Legendre symbol only works for odd *prime* moduli.
- The **Jacobi symbol** $(a \mid m)$ extends it to cases where m is not prime.
- Let $m = p_1 p_2 \cdots p_k$ be the prime factorization of m.
- When $m > 1$ is odd and $\gcd(a, m) = 1$, then
$$(a|m) = \prod_{i=1}^{k} (a \mid p_i).$$
- Define $(a \mid 1) = 1$.

[a]Carl Jacobi (1804–1851).

Properties of the Jacobi Symbol

The Jacobi symbol has the following properties, for arguments for which it is defined.

1. $(ab \mid m) = (a \mid m)(b \mid m)$.
2. $(a \mid m_1 m_2) = (a \mid m_1)(a \mid m_2)$.
3. If $a = b \bmod m$, then $(a \mid m) = (b \mid m)$.
4. $(-1 \mid m) = (-1)^{(m-1)/2}$ (by Lemma 62 on p. 424).
5. $(2 \mid m) = (-1)^{(m^2-1)/8}$ (by Lemma 62 on p. 424).
6. If a and m are both odd, then $(a \mid m)(m \mid a) = (-1)^{(a-1)(m-1)/4}$.

Calculation of $(2200|999)$

Similar to the Euclidean algorithm and does *not* require factorization.

$$\begin{aligned}
& (202|999) = (-1)^{(999^2-1)/8}(101|999) \\
= \; & (-1)^{124750}(101|999) = (101|999) \\
= \; & (-1)^{(100)(998)/4}(999|101) = (-1)^{24950}(999|101) \\
= \; & (999|101) = (90|101) = (-1)^{(101^2-1)/8}(45|101) \\
= \; & (-1)^{1275}(45|101) = -(45|101) \\
= \; & -(-1)^{(44)(100)/4}(101|45) = -(101|45) = -(11|45) \\
= \; & -(-1)^{(10)(44)/4}(45|11) = -(45|11) \\
= \; & -(1|11) = -(11|1) = -1.
\end{aligned}$$

A Result Generalizing Proposition 10.3 in the Textbook

Theorem 64 *The group of set* $\Phi(n)$ *under multiplication* mod n *has a primitive root if and only if* n *is either 1, 2, 4,* p^k*, or* $2p^k$ *for some nonnegative integer* k *and and odd prime* p*.*

This result is essential in the proof of the next lemma.

The Jacobi Symbol and Primality Test[a]

Lemma 65 *If* $(M|N) = M^{(N-1)/2} \bmod N$ *for all* $M \in \Phi(N)$*, then* N *is prime. (Assume* N *is odd.)*

- Assume $N = mp$, where p is an odd prime, $\gcd(m, p) = 1$, and $m > 1$ (not necessarily prime).
- Let $r \in \Phi(p)$ such that $(r \,|\, p) = -1$.
- The Chinese remainder theorem says that there is an $M \in \Phi(N)$ such that

$$\begin{aligned} M &= r \bmod p, \\ M &= 1 \bmod m. \end{aligned}$$

[a]Clement Hsiao (R88067) pointed out that the textbook's proof in Lemma 11.8 is incorrect while he was a senior in January 1999.

The Proof (continued)

- By the hypothesis,

$$M^{(N-1)/2} = (M \,|\, N) = (M \,|\, p)(M \,|\, m) = -1 \bmod N.$$

- Hence

$$M^{(N-1)/2} = -1 \bmod m.$$

- But because $M = 1 \bmod m$,

$$M^{(N-1)/2} = 1 \bmod m,$$

a contradiction.

The Proof (continued)

- Second, assume that $N = p^a$, where p is an odd prime and $a \geq 2$.
- By Theorem 64 (p. 437), there exists a primitive root r modulo p^a.
- From the assumption,

$$M^{N-1} = \left[M^{(N-1)/2} \right]^2 = (M|N)^2 = 1 \bmod N$$

for all $M \in \Phi(N)$.

The Proof (continued)

- As $r \in \Phi(N)$ (prove it), we have

$$r^{N-1} = 1 \bmod N.$$

- As r's exponent modulo $N = p^a$ is $\phi(N) = p^{a-1}(p-1)$,

$$p^{a-1}(p-1) \mid N-1,$$

  which implies that $p \mid N-1$.

- But this is impossible given that $p \mid N$.

The Proof (continued)

- Third, assume that $N = mp^a$, where p is an odd prime, $\gcd(m, p) = 1$, $m > 1$ (not necessarily prime), and a is even.
- The proof mimics that of the second case.
- By Theorem 64 (p. 437), there exists a primitive root r modulo p^a.
- From the assumption,

$$M^{N-1} = \left[M^{(N-1)/2} \right]^2 = (M|N)^2 = 1 \bmod N$$

  for all $M \in \Phi(N)$.

The Proof (continued)

- In particular,

$$M^{N-1} = 1 \bmod p^a \qquad (6)$$

  for all $M \in \Phi(N)$.

- The Chinese remainder theorem says that there is an $M \in \Phi(N)$ such that

$$\begin{aligned} M &= r \bmod p^a, \\ M &= 1 \bmod m. \end{aligned}$$

- Because $M = r \bmod p^a$ and Eq. (6),

$$r^{N-1} = 1 \bmod p^a.$$

The Proof (concluded)

- As r's exponent modulo $N = p^a$ is $\phi(N) = p^{a-1}(p-1)$,

$$p^{a-1}(p-1) \mid N-1,$$

  which implies that $p \mid N-1$.

- But this is impossible given that $p \mid N$.

The Number of Witnesses to Compositeness

Theorem 66 (Solovay and Strassen (1977)) *If N is an odd composite, then $(M|N) \neq M^{(N-1)/2} \bmod N$ for at least half of $M \in \Phi(N)$.*

- By Lemma 65 (p. 438) there is at least one $a \in \Phi(N)$ such that $(a|N) \neq a^{(N-1)/2} \bmod N$.
- Let $B = \{b_1, b_2, \ldots, b_k\} \subseteq \Phi(N)$ be the set of all distinct residues such that $(b_i|N) = b_i^{(N-1)/2} \bmod N$.
- Let $aB = \{ab_i \bmod N : i = 1, 2, \ldots, k\}$.

The Proof (concluded)

- $|aB| = k$.
  - $ab_i = ab_j \bmod N$ implies $N|a(b_i - b_j)$, which is impossible because $\gcd(a, N) = 1$ and $N > |b_i - b_j|$.
- $aB \cap B = \emptyset$ because

$$(ab_i)^{(N-1)/2} = a^{(N-1)/2} b_i^{(N-1)/2} \neq (a|N)(b_i|N) = (ab_i|N).$$

- Combining the above two results, we know

$$\frac{|B|}{\phi(N)} \leq 0.5.$$

if N is even but $N \neq 2$ **then**
  return "N is composite";
else if $N = 2$ **then**
  return "N is a prime";
end if
Pick $M \in \{2, 3, \ldots, N-1\}$ randomly;
if $\gcd(M, N) > 1$ **then**
  return "N is a composite";
else
  if $(M|N) \neq M^{(N-1)/2} \bmod N$ **then**
    return "N is composite";
  else
    return "N is a prime";
  end if
end if

Analysis

- The algorithm certainly runs in polynomial time.
- There are no false positives (for COMPOSITENESS).
  - When the algorithm says the number is composite, it is always correct.
- The probability of a false negative is at most one half.
  - When the algorithm says the number is a prime, it may err.
  - If the input is composite, then the probability that the algorithm errs is one half.
- The error probability can be reduced but not eliminated.

The Improved Density Attack for COMPOSITENESS