A Few Calculations

- Let $p = 13$.
- From p. 362, we know $\phi(p-1) = 4$.
- Hence $R(12) = 4$.
- And there are 4 primitives roots of p.
- As $\Phi(p-1) = \{1, 5, 7, 11\}$, the primitive roots are g^1, g^5, g^7, g^{11} for any primitive root g.

The Other Direction of Theorem 47 (p. 346)

- We must show p is a prime only if there is a number r (called primitive root) such that
  1. $r^{p-1} = 1 \bmod p$, and
  2. $r^{(p-1)/q} \neq 1 \bmod p$ for all prime divisors q of $p-1$.
- Suppose p is not a prime.
- We proceed to show that no primitive roots exist.
- Suppose $r^{p-1} = 1 \bmod p$ (note $\gcd(r, p) = 1$).
- We will show that the 2nd condition must be violated.

The Proof (concluded)

- $r^{\phi(p)} = 1 \bmod p$ by the Fermat-Euler theorem (p. 362).
- Because p is not a prime, $\phi(p) < p-1$.
- Let k be the smallest integer such that $r^k = 1 \bmod p$.
- As $k \leq \phi(p)$, $k < p-1$.
- Let q be a prime divisor of $(p-1)/k > 1$.
- Then $k \mid (p-1)/q$.
- Therefore, by virtue of the definition of k,
$$r^{(p-1)/q} = 1 \bmod p.$$
- But this violates the 2nd condition.

Function Problems

- Decisions problem are yes/no problems (SAT, TSP (D), etc.).
- **Function problems** require a solution (a satisfying truth assignment, a best TSP tour, etc.).
- Optimization problems are clearly function problems.
- What is the relation between function and decision problems?
- Which one is harder?

Function Problems Cannot Be Easier than Decision Problems

- If we know how to generate a solution, we can solve the corresponding decision problem.
  - If you can find a satisfying truth assignment efficiently, then SAT is in P.
  - If you can find the best TSP tour efficiently, then TSP (D) is in P.
- But decision problems can be as hard as the corresponding function problems.

FSAT

- FSAT is this function problem:
  - Let $\phi(x_1, x_2, \ldots, x_n)$ be a boolean expression.
  - If ϕ is satisfiable, then return a satisfying truth assignment.
  - Otherwise, return "no."
- We next show that if SAT $\in$ P, then FSAT has a polynomial-time algorithm.

An Algorithm for FSAT Using SAT

$t := \epsilon$;
if $\phi \in$ SAT **then**
  for $i = 1, 2, \ldots, n$ **do**
    if $\phi[\, x_i = \texttt{true} \,] \in$ SAT **then**
      $t := t \cup \{\, x_i = \texttt{true} \,\}$;
      $\phi := \phi[\, x_i = \texttt{true} \,]$;
    else
      $t := t \cup \{\, x_i = \texttt{false} \,\}$;
      $\phi := \phi[\, x_i = \texttt{false} \,]$;
    end if
  end for
  return t;
else
  return "no";
end if

Analysis

- There are $\leq n + 1$ calls to the algorithm for SAT.[a]
- Shorter boolean expressions than ϕ are used in each call to the algorithm for SAT.
- So if SAT can be solved in polynomial time, so can FSAT.
- Hence SAT and FSAT are equally hard (or easy).

[a] Contributed by Ms. Eva Ou (R93922132) on November 24, 2004.

TSP and TSP (D) Revisited

- We are given n cities $1, 2, \ldots, n$ and integer distances $d_{ij} = d_{ji}$ between any two cities i and j.
- The TSP asks for a tour with the shortest total distance (not just the shortest total distance, as earlier).
  - The shortest total distance must be at most $2^{|x|}$, where x is the input.
- TSP (D) asks if there is a tour with a total distance at most B.
- We next show that if TSP (D) $\in$ P, then TSP has a polynomial-time algorithm.

An Algorithm for TSP Using TSP (D)

```
Perform a binary search over interval [0, 2^{|x|}] by calling
  TSP (D) to obtain the shortest distance C;
for i, j = 1, 2, ..., n do
  Call TSP (D) with B = C and d_ij = C + 1;
  if "no" then
    Restore d_ij to old value; {Edge [i, j] is critical.}
  end if
end for
return the tour with edges whose d_ij ≤ C;
```


Analysis

- An edge that is not on *any* optimal tour will be eliminated, with its d_{ij} set to $C + 1$.
- An edge which is not on all remaining optimal tours will also be eliminated.
- So the algorithm ends with n edges which are not eliminated (why?).
- There are $O(|x| + n^2)$ calls to the algorithm for TSP (D).
- So if TSP (D) can be solved in polynomial time, so can TSP.
- Hence TSP (D) and TSP are equally hard (or easy).

Randomized Computation

I know that half my advertising works,
I just don't know which half.
— John Wanamaker

I know that half my advertising is
a waste of money,
I just don't know which half!
— McGraw-Hill ad.

Randomized Algorithms[a]

- Randomized algorithms flip unbiased coins.
- There are important problems for which there are no known efficient *deterministic* algorithms but for which very efficient randomized algorithms exist.
  - Extraction of square roots, for instance.
- There are problems where randomization is *necessary*.
  - Secure protocols.
- Randomized version can be more efficient.
  - Parallel algorithm for maximal independent set.
- Are randomized algorithms algorithms?

[a]Rabin (1976); Solovay and Strassen (1977).

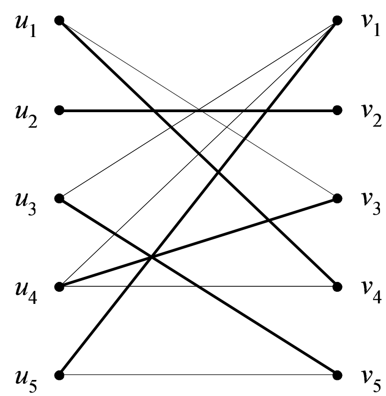
Bipartite Perfect Matching

- We are given a **bipartite graph** $G = (U, V, E)$.
  - $U = \{u_1, u_2, \ldots, u_n\}$.
  - $V = \{v_1, v_2, \ldots, v_n\}$.
  - $E \subseteq U \times V$.
- We are asked if there is a **perfect matching**.
  - A permutation π of $\{1, 2, \ldots, n\}$ such that

$$(u_i, v_{\pi(i)}) \in E$$

for all $u_i \in U$.

A Perfect Matching



Symbolic Determinants

- Given a bipartite graph G, construct the $n \times n$ matrix A^G whose (i, j)th entry A^G_{ij} is a variable x_{ij} if $(u_i, v_j) \in E$ and zero otherwise.
- The **determinant** of A^G is

$$\det(A^G) = \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^{n} A^G_{i,\pi(i)}. \tag{5}$$

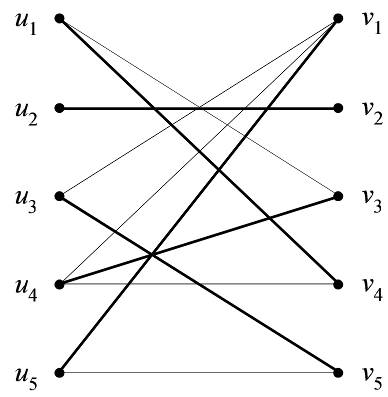
  - π ranges over all permutations of n elements.
  - $\operatorname{sgn}(\pi)$ is 1 if π is the product of an even number of transpositions and -1 otherwise.

Determinant and Bipartite Perfect Matching

- In $\sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^{n} A^G_{i,\pi(i)}$, note the following:
  - Each summand corresponds to a possible prefect matching π.
  - As all variables appear only *once*, all of these summands are different monomials and will not cancel.
- It is essentially an exhaustive enumeration.

Proposition 56 (Edmonds (1967)) *G has a perfect matching if and only if* $\det(A^G)$ *is not identically zero.*

A Perfect Matching in a Bipartite Graph



The Perfect Matching in the Determinant

- The matrix is

$$A^G = \begin{bmatrix} 0 & 0 & x_{13} & \boxed{x_{14}} & 0 \\ 0 & \boxed{x_{22}} & 0 & 0 & 0 \\ x_{31} & 0 & 0 & 0 & \boxed{x_{35}} \\ x_{41} & 0 & \boxed{x_{43}} & x_{44} & 0 \\ \boxed{x_{51}} & 0 & 0 & 0 & x_{55} \end{bmatrix}.$$

- $\det(A^G) = -x_{14}x_{22}x_{35}x_{43}x_{51} + x_{13}x_{22}x_{35}x_{44}x_{51} + x_{14}x_{22}x_{31}x_{43}x_{55} - x_{13}x_{22}x_{31}x_{44}x_{55}$, each denoting a perfect matching.

How To Test If a Polynomial Is Identically Zero?

- $\det(A^G)$ is a polynomial in n^2 variables.
- There are exponentially many terms in $\det(A^G)$.
- Expanding the determinant polynomial is not feasible.
  - Too many terms.
- Observation: If $\det(A^G)$ is *identically zero*, then it remains zero if we substitute *arbitrary* integers for the variables $x_{11}, \ldots, x_{nn}$.
- What is the likelihood of obtaining a zero when $\det(A^G)$ is *not* identically zero?

Number of Roots of a Polynomial

Lemma 57 (Schwartz (1980)) *Let $p(x_1, x_2, \ldots, x_m) \not\equiv 0$ be a polynomial in m variables each of degree at most d. Let $M \in \mathbb{Z}^+$. Then the number of m-tuples*

$$(x_1, x_2, \ldots, x_m) \in \{0, 1, \ldots, M-1\}^m$$

such that $p(x_1, x_2, \ldots, x_m) = 0$ is

$$\leq mdM^{m-1}.$$

- By induction on m (consult the textbook).

Density Attack

- The density of roots in the domain is at most

$$\frac{mdM^{m-1}}{M^m} = \frac{md}{M}.$$

- So suppose $p(x_1, x_2, \ldots, x_m) \not\equiv 0$.
- Then a random

$$(x_1, x_2, \ldots, x_n) \in \{\, 0, 1, \ldots, M-1 \,\}^n$$

has a probability of $\leq md/M$ of being a root of p.

Density Attack (concluded)

Here is a sampling algorithm to test if $p(x_1, x_2, \ldots, x_m) \not\equiv 0$.

```
Choose i_1, ..., i_m from {0, 1, ..., M - 1} randomly;
if p(i_1, i_2, ..., i_m) ≠ 0 then
  return "p is not identically zero";
else
  return "p is identically zero";
end if
```

A Randomized Bipartite Perfect Matching Algorithm[a]

We now return to the original problem of bipartite perfect matching.

```
Choose n^2 integers i_11, ..., i_nn from {0, 1, ..., b − 1} randomly;
Calculate det(A^G(i_11, ..., i_nn)) by Gaussian elimination;
if det(A^G(i_11, ..., i_nn)) ≠ 0 then
   return "G has a perfect matching";
else
   return "G has no perfect matchings";
end if
```

[a] Lovász (1979).

Analysis

- Pick $b = 2n^2$.
- If G has no perfect matchings, the algorithm will always be correct.
- Suppose G has a perfect matching.
  - The algorithm will answer incorrectly with probability at most $n^2 d/b = 0.5$ because $d = 1$.
  - Run the algorithm *independently* k times and output "G has no perfect matchings" if they all say no.
  - The error probability is now reduced to at most 2^{-k}.
- Is there an $(i_{11}, \ldots, i_{nn})$ that will always give correct answers for all bipartite graphs of $2n$ nodes?[a]

[a] Thanks to a lively class discussion on November 24, 2004.

Perfect Matching for General Graphs

- Page 382 is about bipartite perfect matching
- Now we are given a graph $G = (V, E)$.
  - $V = \{v_1, v_2, \ldots, v_{2n}\}$.
- We are asked if there is a perfect matching.
  - A permutation π of $\{1, 2, \ldots, 2n\}$ such that

$$(v_i, v_{\pi(i)}) \in E$$

for all $v_i \in V$.

The Tutte Matrix[a]

- Given a graph $G = (V, E)$, construct the $2n \times 2n$ **Tutte** matrix T^G such that

$$T^G_{ij} = \begin{cases} x_{ij} & \text{if } (v_i, v_j) \in E \text{ and } i < j, \\ -x_{ij} & \text{if } (v_i, v_j) \in E \text{ and } i > j, \\ 0 & \text{othersie.} \end{cases}$$

- The Tutte matrix is a skew-symmetric symbolic matrix.
- Similar to Proposition 56 (p. 385):

Proposition 58 *G has a perfect matching if and only if* $\det(T^G)$ *is not identically zero.*

[a] William Thomas Tutte (1917–2002).

Monte Carlo Algorithms[a]

- The randomized bipartite perfect matching algorithm is called a **Monte Carlo algorithm** in the sense that
  - If the algorithm finds that a matching exists, it is always correct (no **false positives**).
  - If the algorithm answers in the negative, then it may make an error (**false negative**).
- The algorithm makes a false negative with probability ≤ 0.5.
- This probability is *not* over the space of all graphs or determinants, but *over* the algorithm's own coin flips.
  - It holds for *any* bipartite graph.

[a] Metropolis and Ulam (1949).

The Markov Inequality[a]

Lemma 59 *Let x be a random variable taking nonnegative integer values. Then for any $k > 0$,*

$$\text{prob}[x \geq kE[x]] \leq 1/k.$$

- Let p_i denote the probability that $x = i$.

$$\begin{aligned} E[x] &= \sum_i ip_i \\ &= \sum_{i<kE[x]} ip_i + \sum_{i \geq kE[x]} ip_i \\ &\geq kE[x] \times \text{prob}[x \geq kE[x]]. \end{aligned}$$

[a] Andrei Andreyevich Markov (1856–1922).

An Application of Markov's Inequality

- Algorithm C runs in expected time $T(n)$ and always gives the right answer.
- Consider an algorithm that runs C for time $kT(n)$ and rejects the input if C does not stop within the time bound.
- By Markov's inequality, this new algorithm runs in time $kT(n)$ and gives the wrong answer with probability $\leq 1/k$.
- By running this algorithm m times, we reduce the error probability to $\leq k^{-m}$.

An Application of Markov's Inequality (concluded)

- Suppose, instead, we run the algorithm for the same running time $mkT(n)$ once and rejects the input if it does not stop within the time bound.
- By Markov's inequality, this new algorithm gives the wrong answer with probability $\leq 1/(mk)$.
- This is a far cry from the previous algorithm's error probability of $\leq k^{-m}$.
- The loss comes from the fact that Markov's inequality does not take advantage of any specific feature of the random variable.

FSAT for k-SAT Formulas (p. 373)

- Let $\phi(x_1, x_2, \ldots, x_n)$ be a k-SAT formula.
- If ϕ is satisfiable, then return a satisfying truth assignment.
- Otherwise, return "no."
- We next propose a randomized algorithm for this problem.

A Random Walk Algorithm for ϕ in CNF Form

```
Start with an arbitrary truth assignment T;
for i = 1, 2, ..., r do
  if T ⊨ φ then
    return "φ is satisfiable with T";
  else
    Let c be an unsatisfiable clause in φ under T; {All
    of its literals are false under T.}
    Pick any x of these literals at random;
    Modify T to make x true;
  end if
end for
return "φ is unsatisfiable";
```


3SAT vs. 2SAT Again

- Note that if ϕ is unsatisfiable, the algorithm will not refute it.
- The random walk algorithm needs expected exponential time for 3SAT.
  - In fact, it runs in expected $O((1.333 \cdots + \epsilon)^n)$ time with $r = 3n$, much better than $O(2^n)$.[a]
- We will show immediately that it works well for 2SAT.
- The state of the art is expected $O(1.324^n)$ time for 3SAT and expected $O(1.474^n)$ time for 4SAT.[b]

[a] Schöning (1999).

[b] Kwama and Tamaki (2004).