Tackling Intractable Problems

- Many important problems are NP-complete or worse.
- **Heuristics** have been developed to attack them.
- They are **approximation algorithms**.
- How good are the approximations?
  - We are looking for theoretically *guaranteed* bounds, not "empirical" bounds.
- Are there NP problems that cannot be approximated well (assuming NP $\neq$ P)?
- Are there NP problems that cannot be approximated at all (assuming NP $\neq$ P)?

Some Definitions

- Given an **optimization problem**, each problem instance x has a set of **feasible solutions** $F(x)$.
- Each feasible solution $s \in F(x)$ has a cost $c(s) \in \mathbb{Z}^+$.
- The **optimum cost** is $\textsc{opt}(x) = \min_{s \in F(x)} c(s)$ for a minimization problem.
- It is $\textsc{opt}(x) = \max_{s \in F(x)} c(s)$ for a maximization problem.

Optimization Problem and Threshold Language

- Given a maximization (minimization) problem, its decision version, the **threshold language**, asks if the optimal cost is at least (at most, resp.) a given threshold.
- If the decision version is hard, the optimization problem cannot be easy.
  - Otherwise, we can solve the optimization problem first and then do a simple test.
- If the optimization problem is hard, its decision version is not expected to be easy.
  - Otherwise, we can often do a binary search to bracket the optimal cost.

Approximation Algorithms

- Let algorithm M on x returns a feasible solution.
- M is an **ϵ-approximation algorithm**, where $\epsilon \geq 0$, if for all x,

$$\frac{|c(M(x)) - \textsc{opt}(x)|}{\max(\textsc{opt}(x), c(M(x)))} \leq \epsilon.$$

  - For a minimization problem,

$$\frac{c(M(x)) - \min_{s \in F(x)} c(s)}{c(M(x))} \leq \epsilon.$$

  - For a maximization problem,

$$\frac{\max_{s \in F(x)} c(s) - c(M(x))}{\max_{s \in F(x)} c(s)} \leq \epsilon.$$

Lower and Upper Bounds

- For a minimization problem,

$$\min_{s \in F(x)} c(s) \leq c(M(x)) \leq \frac{\min_{s \in F(x)} c(s)}{1-\epsilon}.$$

  - So **approximation ratio** $\frac{\min_{s \in F(x)} c(s)}{c(M(x))} \geq 1-\epsilon$.
- For a maximization problem,

$$(1-\epsilon) \times \max_{s \in F(x)} c(s) \leq c(M(x)) \leq \max_{s \in F(x)} c(s).$$

  - So approximation ratio $\frac{c(M(x))}{\max_{s \in F(x)} c(s)} \geq 1-\epsilon$.
- The above are alternative definitions of ϵ-approximation algorithms.

Range Bounds

- ϵ takes values between 0 and 1.
- For maximization problems, an ϵ-approximation algorithm returns solutions within $[(1-\epsilon) \times \text{OPT}, \text{OPT}]$.
- For minimization problems, an ϵ-approximation algorithm returns solutions within $[\text{OPT}, \frac{\text{OPT}}{1-\epsilon}]$.
- For each NP-complete optimization problem, we shall be interested in determining the *smallest* ϵ for which there is a polynomial-time ϵ-approximation algorithm.
- Sometimes ϵ has no minimum value.

Approximation Thresholds

- The **approximation threshold** is the greatest lower bound of all $\epsilon \geq 0$ such that there is a polynomial-time ϵ-approximation algorithm.
- The approximation threshold of an optimization problem can be anywhere between 0 (approximation to any desired degree) and 1 (no approximation is possible).
- If P = NP, then all optimization problems in NP have approximation threshold 0.
- So we assume P $\neq$ NP for the rest of the discussion.

NODE COVER

- NODE COVER seeks the smallest $C \subseteq V$ in graph $G = (V, E)$ such that for each edge in E, at least one of its endpoints is in C.
- A heuristic to obtain a good node cover is to iteratively move a node with the highest degree to the cover.
- This turns out to produce approximation ratio $\frac{c(M(x))}{\text{OPT}(x)} = \Theta(\log n)$.
- It is not an ϵ-approximation algorithm for any $\epsilon < 1$.

A 0.5-Approximation Algorithm

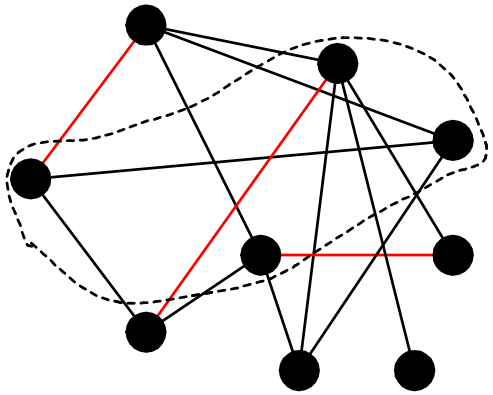
1: $C := \emptyset$;
2: **while** $E \neq \emptyset$ **do**
3: Delete an arbitrary edge $[u, v]$ from E;
4: Delete edges incident with u and v from E;
5: Add u and v to C; {Add 2 nodes to C each time.}
6: **end while**
7: **return** C;

Analysis

- C contains $|C|/2$ edges.
- No two edges of C share a node.
- *Any* node cover must contain at least one node from each of these edges.
- This means that $\textsc{opt}(G) \geq |C|/2$.
- So

$$\frac{\textsc{opt}(G)}{|C|} \geq 1/2.$$

- The approximation threshold is ≤ 0.5.



Maximum Satisfiability

- Given a set of clauses, MAXSAT seeks the truth assignment that satisfies the most.
- MAX2SAT is already NP-complete (p. 263).
- Consider the more general k-MAXGSAT for constant k.
  - Given a set of boolean expressions $\Phi = \{\phi_1, \phi_2, \ldots, \phi_m\}$ in n variables.
  - Each ϕ_i is a *general* expression involving k variables.
  - k-MAXGSAT seeks the truth assignment that satisfies the most expressions.

A Probabilistic Interpretation of an Algorithm

- Each ϕ_i involves exactly k variables and is satisfied by t_i of the 2^k truth assignments.
- A random truth assignment $\in \{0,1\}^n$ satisfies ϕ_i with probability $p(\phi_i) = t_i/2^k$.
  - $p(\phi_i)$ is easy to calculate for a $k = O(\log n)$.
- Hence a random truth assignment satisfies an expected number

$$p(\Phi) = \sum_{i=1}^{m} p(\phi_i)$$

of expressions ϕ_i.

The Search Procedure

- Clearly

$$p(\Phi) = \frac{1}{2}\{\, p(\Phi[x_1 = \texttt{true}\,]) + p(\Phi[x_1 = \texttt{false}\,])\,\}.$$

- Select the $t_1 \in \{\texttt{true}, \texttt{false}\}$ such that $p(\Phi[x_1 = t_1\,])$ is the larger one.
- Note that $p(\Phi[x_1 = t_1\,]) \geq p(\Phi)$.
- Repeat with expression $\Phi[x_1 = t_1\,]$ until all variables x_i have been given truth values t_i and all ϕ_i either true or false.

The Search Procedure (concluded)

- By our hill-climbing procedure,

$$\begin{aligned} & p(\Phi[x_1 = t_1, x_2 = t_2, \ldots, x_n = t_n\,]) \\ \geq\ & \cdots \\ \geq\ & p(\Phi[x_1 = t_1, x_2 = t_2\,]) \\ \geq\ & p(\Phi[x_1 = t_1\,]) \\ \geq\ & p(\Phi). \end{aligned}$$

- So at least $p(\Phi)$ expressions are satisfied by truth assignment $(t_1, t_2, \ldots, t_n)$.
- The algorithm is deterministic.

Approximation Analysis

- The optimum is at most the number of satisfiable ϕ_i—i.e., those with $p(\phi_i) > 0$.
- Hence the ratio of algorithm's output vs. the optimum is

$$\geq \frac{p(\Phi)}{\sum_{p(\phi_i)>0} 1} = \frac{\sum_i p(\phi_i)}{\sum_{p(\phi_i)>0} 1} \geq \min_{p(\phi_i)>0} p(\phi_i).$$

- The heuristic is a polynomial-time ϵ-approximation algorithm with $\epsilon = 1 - \min_{p(\phi_i)>0} p(\phi_i)$.
- Because $p(\phi_i) \geq 2^{-k}$, the heuristic is a polynomial-time ϵ-approximation algorithm with $\epsilon = 1 - 2^{-k}$.

Back to MAXSAT

- In MAXSAT, the ϕ_i's are clauses.
- Hence $p(\phi_i) \geq 1/2$, which happens when ϕ_i contains a single literal.
- And the heuristic becomes a polynomial-time ϵ-approximation algorithm with $\epsilon = 1/2$.[a]
- If the clauses have k distinct literals, $p(\phi_i) = 1 - 2^{-k}$.
- And the heuristic becomes a polynomial-time ϵ-approximation algorithm with $\epsilon = 2^{-k}$.
  - This is the best possible for $k \geq 3$ unless P = NP.

[a] Johnson (1974).

MAX CUT Revisited

- The NP-complete MAX CUT seeks to partition the nodes of graph $G = (V, E)$ into $(S, V - S)$ so that there are as many edges as possible between S and $V - S$ (p. 284).
- **Local search** starts from a feasible solution and performs "local" improvements until none are possible.

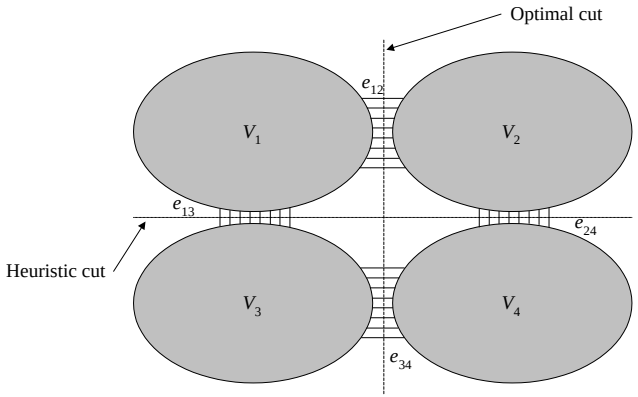
A 0.5-Approximation Algorithm for MAX CUT

1: $S := \emptyset$;
2: **while** $\exists v \in V$ whose switching sides results in a larger cut **do**
3: $\quad S := S \cup \{v\}$;
4: **end while**
5: **return** S;

- A 0.12-approximation algorithm exists.[a]
- 0.059-approximation algorithms do not exist unless NP = ZPP.

[a] Goemans and Williamson (1995).

Analysis



Analysis (continued)

- Partition $V = V_1 \cup V_2 \cup V_3 \cup V_4$, where our algorithm returns $(V_1 \cup V_2, V_3 \cup V_4)$ and the optimum cut is $(V_1 \cup V_3, V_2 \cup V_4)$.
- Let e_{ij} be the number of edges between V_i and V_j.
- Because no migration of nodes can improve the algorithm's cut, for each node in V_1, its edges to $V_1 \cup V_2$ are outnumbered by those to $V_3 \cup V_4$.
- Considering all nodes in V_1 together, we have $2e_{11} + e_{12} \leq e_{13} + e_{14}$, which implies

$$e_{12} \leq e_{13} + e_{14}.$$

Analysis (concluded)

- Similarly,

$$\begin{aligned} e_{12} &\leq e_{23} + e_{24} \\ e_{34} &\leq e_{23} + e_{13} \\ e_{34} &\leq e_{14} + e_{24} \end{aligned}$$

- Adding all four inequalities, dividing both sides by 2, and adding the inequality $e_{14} + e_{23} \leq e_{14} + e_{23} + e_{13} + e_{24}$, we obtain

$$e_{12} + e_{34} + e_{14} + e_{23} \leq 2(e_{13} + e_{14} + e_{23} + e_{24}).$$

- The above says our solution is at least half the optimum.

Approximability, Unapproximability, and Between

- KNAPSACK, NODE COVER, MAXSAT, and MAX CUT have approximation thresholds less than 1.
  - KNAPSACK has a threshold of 0.
  - But NODE COVER and MAXSAT have a threshold larger than 0.
- The situation is maximally pessimistic for TSP: It cannot be approximated unless P = NP.
  - The approximation threshold of TSP is 1.
    - The threshold is 1/3 if the TSP satisfies the triangular inequality.
  - The same holds for INDEPENDENT SET.

Unapproximability of TSP[a]

Theorem 72 *The approximation threshold of* TSP *is 1 unless P = NP.*

- Suppose there is a polynomial-time ϵ-approximation algorithm for TSP for some $\epsilon < 1$.
- We shall construct a polynomial-time algorithm for the NP-complete HAMILTONIAN CYCLE.
- Given any graph $G = (V, E)$, construct a TSP with $|V|$ cities with distances

$$d_{ij} = \begin{cases} 1, & \text{if } \{ i, j \} \in E \\ \frac{|V|}{1-\epsilon}, & \text{otherwise} \end{cases}$$

[a] Sahni and Gonzales (1976).

The Proof (concluded)

- Run the alleged approximation algorithm on this TSP.
- Suppose a tour of cost $|V|$ is returned.
  - This tour must be a Hamiltonian cycle.
- Suppose a tour with at least one edge of length $\frac{|V|}{1-\epsilon}$ is returned.
  - The total length of this tour is $> \frac{|V|}{1-\epsilon}$.
  - Because the algorithm is ϵ-approximate, the optimum is at least $1-\epsilon$ times the returned tour's length.
  - The optimum tour has a cost exceeding $|V|$.
  - Hence G has no Hamiltonian cycles.

KNAPSACK Has an Approximation Threshold of Zero[a]

Theorem 73 *For any ϵ, there is a polynomial-time ϵ-approximation algorithm for* KNAPSACK.

- We have n weights $w_1, w_2, \ldots, w_n$, a weight limit W, and n values $v_1, v_2, \ldots, v_n$.
- We must find an $S \subseteq \{1, 2, \ldots, n\}$ such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i$ is the largest possible.
- Let

$$V = \max\{v_1, v_2, \ldots, v_n\}.$$

[a] Ibarra and Kim (1975).

The Proof (continued)

- For $0 \leq i \leq n$ and $0 \leq v \leq nV$, define $W(i, v)$ to be the minimum weight attainable by selecting some among the i first items, so that their value is exactly v.
- Start with $W(0, v) = \infty$ for all v.
- Then

$$W(i+1, v) = \min\{W(i, v), W(i, v - v_{i+1}) + w_{i+1}\}.$$

- Finally, pick the largest v such that $W(n, v) \leq W$.
- The running time is $O(n^2 V)$, not polynomial time.
- Key idea: Limit the number of precision bits.

The Proof (continued)

- Given the instance $x = (w_1, \ldots, w_n, W, v_1, \ldots, v_n)$, we define the approximate instance

$$x' = (w_1, \ldots, w_n, W, v'_1, \ldots, v'_n),$$

where

$$v'_i = 2^b \left\lfloor \frac{v_i}{2^b} \right\rfloor.$$

- Solving x' takes time $O(n^2 V / 2^b)$.
- The solution S' is close to the optimum solution S:

$$\sum_{i \in S} v_i \geq \sum_{i \in S'} v_i \geq \sum_{i \in S'} v'_i \geq \sum_{i \in S} v'_i \geq \sum_{i \in S} (v_i - 2^b) \geq \sum_{i \in S} v_i - n2^b.$$

The Proof (concluded)

- Hence

$$\sum_{i \in S'} v_i \geq \sum_{i \in S} v_i - n2^b.$$

- Because V is a lower bound on OPT (if, without loss of generality, $w_i \leq W$), the relative deviation from the optimum is at most $n2^b/V$.
- By truncating the last $b = \lfloor \log_2 \frac{\epsilon V}{n} \rfloor$ bits of the values, the algorithm becomes ϵ-approximate.
- The running time is then $O(n^2 V/b) = O(n^3/\epsilon)$, a polynomial in n and ϵ.