## Randomized Complexity Classes; RP

- Let $N$ be a polynomial-time precise NTM that runs in time $p(n)$ and has 2 nondeterministic choices at each step.
- $N$ is a **polynomial Monte Carlo Turing machine** for a language $L$ if the following conditions hold:
  - If $x \in L$, then at least half of the $2^{p(|x|)}$ computation paths of $N$ on $x$ halt with "yes."
  - If $x \notin L$, then all computation paths halt with "no."
- The class of all languages with polynomial Monte Carlo TMs is denoted **RP** (**randomized polynomial time**).



## Comments on RP

- Nondeterministic steps can be seen as fair coin flips.
- There are no false positive answers.
- The probability of false negatives, $1 - \epsilon$, is at most 0.5.
- Any constant between 0 and 1 can replace 0.5.
  - By repeating the algorithm $k = \lceil -\frac{1}{\log_2 1-\epsilon} \rceil$ times, the probability of false negatives becomes $(1-\epsilon)^k \leq 0.5$.
- In fact, $\epsilon$ can be arbitrarily close to 0 as long as it is of the order $1/p(n)$ for some polynomial $p(n)$.
  - $-\frac{1}{\log_2 1-\epsilon} = O(\frac{1}{\epsilon}) = O(p(n))$.



## Where RP Fits

- $\text{P} \subseteq \text{RP} \subseteq \text{NP}$.
  - A deterministic TM is like a Monte Carlo TM except that all the coin flips are ignored.
  - A Monte Carlo TM is an NTM with extra demands on the number of accepting paths.
- COMPOSITENESS $\in$ RP; PRIMES $\in$ coRP; PRIMES $\in$ RP.[a]
  - In fact, PRIMES $\in$ P.
- $\text{RP} \cup \text{coRP}$ is a "plausible" notion of efficient computation.

[a] Adleman and Huang (1987).



## ZPP[a] (Zero Probabilistic Polynomial)

- The class **ZPP** is defined as $\text{RP} \cap \text{coRP}$.
- A language in ZPP has *two* Monte Carlo algorithms, one with no false positives and the other with no false negatives.
- If we repeatedly run both Monte Carlo algorithms, *eventually* one definite answer will come (unlike RP).
  - A *positive* answer from the one without false positives.
  - A *negative* answer from the one without false negatives.

[a] Gill (1977).

## The ZPP Algorithm (**Las Vegas**)

```
{Suppose L ∈ ZPP.}
{N_1 has no false positives, and N_2 has no false negatives.}
while true do
  if N_1(x) = "yes" then
    return "yes";
  end if
  if N_2(x) = "no" then
    return "no";
  end if
end while
```



## ZPP (concluded)

- The *expected* running time for the correct answer to emerge is polynomial.
  - The probability that a run of the 2 algorithms does not generate a definite answer is 0.5.
  - Let $p(n)$ be the running time of each run.
  - The expected running time for a definite answer is
  $$\sum_{i=1}^{\infty} 0.5^i i p(n) = 2p(n).$$
- Essentially, ZPP is the class of problems that can be solved without errors in expected polynomial time.



## *Et Tu*, RP?

```
{Suppose L ∈ RP.}
{N decides L without false positives.}
while true do
  if N(x) = "yes" then
    return "yes";
  end if
  {But what to do here?}
end while
```

- You eventually get a "yes" if $x \in L$.
- But how to get a "no" when $x \notin L$?
- You have to sacrifice either correctness or bounded running time.



## Large Deviations

- You have a *biased* coin.
- One side has probability $0.5 + \epsilon$ to appear and the other $0.5 - \epsilon$, for some $0 < \epsilon < 1$.
- But you do not know which is which.
- How to decide which side is the more likely—with high confidence?
- Answer: Flip the coin many times and pick the side that appeared the most times.
- Question: Can you quantify the confidence?

## The Chernoff Bound[a]

**Theorem 65 (Chernoff (1952))** *Suppose $x_1, x_2, \ldots, x_n$ are independent random variables taking the values 1 and 0 with probabilities $p$ and $1-p$, respectively. Let $X = \sum_{i=1}^n x_i$. Then for all $0 \leq \theta \leq 1$,*

$$\text{prob}[\, X \geq (1+\theta)\, pn\,] \leq e^{-\theta^2 pn/3}.$$

- The probability that the deviate of a **binomial random variable** from its expected value decreases exponentially with the deviation.
- The Chernoff bound is asymptotically optimal.

[a]Herman Chernoff (1923–).




## The Proof

- Let $t$ be any positive real number.
- Then

$$\text{prob}[\, X \geq (1+\theta)\, pn\,] = \text{prob}[\, e^{tX} \geq e^{t(1+\theta)\, pn}\,].$$

- Markov's inequality (p. 372) generalized to real-valued random variables says that

$$\text{prob}\left[\, e^{tX} \geq kE[\, e^{tX}\,]\,\right] \leq 1/k.$$

- With $k = e^{t(1+\theta)\, pn}/E[\, e^{tX}\,]$, we have

$$\text{prob}[\, X \geq (1+\theta)\, pn\,] \leq e^{-t(1+\theta)\, pn} E[\, e^{tX}\,].$$




## The Proof (continued)

- Because $X = \sum_{i=1}^n x_i$ and $x_i$'s are independent,

$$E[\, e^{tX}\,] = (E[\, e^{tx_1}\,])^n = [\, 1 + p(e^t - 1)\,]^n.$$

- Substituting, we obtain

$$\begin{aligned} \text{prob}[\, X \geq (1+\theta)\, pn\,] &\leq e^{-t(1+\theta)\, pn}[\, 1 + p(e^t - 1)\,]^n \\ &\leq e^{-t(1+\theta)\, pn} e^{pn(e^t - 1)} \end{aligned}$$

as $(1+a)^n \leq e^{an}$ for all $a > 0$.




## The Proof (concluded)

- With the choice of $t = \ln(1+\theta)$, the above becomes

$$\text{prob}[\, X \geq (1+\theta)\, pn\,] \leq e^{pn[\, \theta - (1+\theta)\ln(1+\theta)\,]}.$$

- The exponent expands to $-\frac{\theta^2}{2} + \frac{\theta^3}{6} - \frac{\theta^4}{12} + \cdots$ for $0 \leq \theta \leq 1$, which is less than

$$-\frac{\theta^2}{2} + \frac{\theta^3}{6} \leq \theta^2 \left( -\frac{1}{2} + \frac{\theta}{6} \right) \leq \theta^2 \left( -\frac{1}{2} + \frac{1}{6} \right) = -\frac{\theta^2}{3}.$$

## Power of the Majority Rule

From prob$[\, X \leq (1-\theta)\, pn \,] \leq e^{-\frac{\theta^2}{2} pn}$ (prove it):

**Corollary 66** *If $p = (1/2) + \epsilon$ for some $0 \leq \epsilon \leq 1/2$, then*

$$\text{prob}\left[\sum_{i=1}^{n} x_i \leq n/2\right] \leq e^{-\epsilon^2 n/2}.$$

- The textbook's corollary to Lemma 11.9 seems incorrect.
- Our original problem (p. 406) hence demands $\approx 1.4k/\epsilon^2$ independent coin flips to guarantee making an error with probability at most $2^{-k}$ with the majority rule.



## BPP[a] (Bounded Probabilistic Polynomial)

- The class **BPP** contains all languages for which there is a precise polynomial-time NTM $N$ such that:
  - If $x \in L$, then at least $3/4$ of the computation paths of $N$ on $x$ lead to "yes."
  - If $x \notin L$, then at least $3/4$ of the computation paths of $N$ on $x$ lead to "no."
- $N$ accepts or rejects by a *clear* majority.

[a] Gill (1977).



## Magic $3/4$?

- The number $3/4$ bounds the probability of a right answer away from $1/2$.
- Any constant *strictly* between $1/2$ and 1 can be used without affecting the class BPP.
- In fact, 0.5 plus any inverse polynomial between $1/2$ and 1,

$$0.5 + 1/p(n),$$

can be used.



## The Majority Vote Algorithm

Suppose $L$ is decided by $N$ by majority $(1/2) + \epsilon$.

**for** $i = 1, 2, \ldots, 2k+1$ **do**
  Run $N$ on input $x$;
**end for**
**if** "yes" is the majority answer **then**
  "yes";
**else**
  "no";
**end if**

## Analysis

- The running time remains polynomial, being $2k+1$ times $N$'s running time.
- By Corollary 66 (p. 411), the probability of a false answer is at most $e^{-\epsilon^2 k}$.
- By taking $k = \lceil 2/\epsilon^2 \rceil$, the error probability is at most $1/4$.
- As with the RP case, $\epsilon$ can be any inverse polynomial, because $k$ remains polynomial in $n$.




## Probability Amplification for BPP

- Let $m$ be the number of random bits used by a BPP algorithm.
  - By definition, $m$ is polynomial in $n$.
- With $k = \Theta(\log m)$ in the majority vote algorithm, we can lower the error probability to $\leq (3m)^{-1}$.




## Aspects of BPP

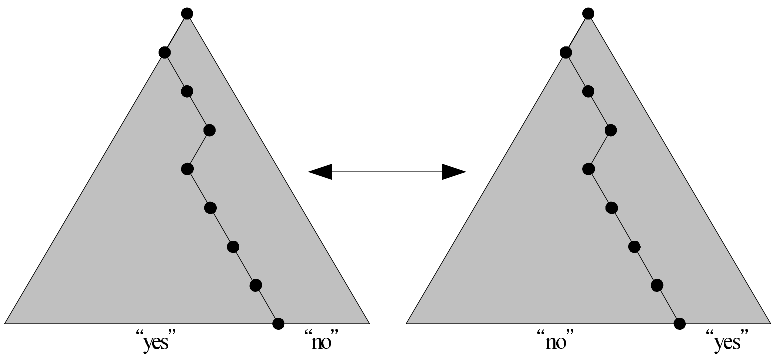
- BPP is the most comprehensive yet plausible notion of efficient computation.
  - If a problem is in BPP, we take it to mean that the problem can be solved efficiently.
  - In this aspect, BPP has effectively replaced P.
- $(\text{RP} \cup \text{coRP}) \subseteq (\text{NP} \cup \text{coNP})$.
- $(\text{RP} \cup \text{coRP}) \subseteq \text{BPP}$.
- Whether $\text{BPP} \subseteq (\text{NP} \cup \text{coNP})$ is unknown.
- But it is unlikely that $\text{NP} \subseteq \text{BPP}$ (p. 641).




## coBPP

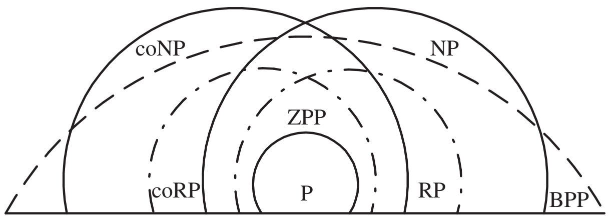
- The definition of BPP is symmetric: acceptance by clear majority and rejection by clear majority.
- An algorithm for $L \in \text{BPP}$ becomes one for $\bar{L} \in \text{coBPP}$ by reversing the answer.
- Hence $\text{BPP} = \text{coBPP}$.
- This approach does not work for RP.
- It did not work for NP either.

## BPP and coBPP






## “The Good, the Bad, and the Ugly”






## Circuit Complexity

- Circuit complexity is based on boolean circuits instead of Turing machines.
- A boolean circuit with $n$ inputs computes a boolean function of $n$ variables.
- By identify `true` with 1 and `false` with 0, a boolean circuit with $n$ inputs accepts certain strings in $\{ 0, 1 \}^n$.
- To relate circuits with arbitrary languages, we need one circuit for each possible input length $n$.



## Formal Definitions

- The **size** of a circuit is the number of *gates* in it.
- A **family of circuits** is an infinite sequence $\mathcal{C} = (C_0, C_1, \ldots)$ of boolean circuits, where $C_n$ has $n$ boolean inputs.
- $L \subseteq \{0, 1\}^*$ has **polynomial circuits** if there is a family of circuits $\mathcal{C}$ such that:
  – The size of $C_n$ is at most $p(n)$ for some fixed polynomial $p$.
  – For input $x \in \{0, 1\}^*$, $C_{|\,x\,|}$ outputs 1 if and only if $x \in L$.
    * $C_n$ accepts $L \cap \{0, 1\}^n$.

## Exponential Circuits Contain All Languages

- Theorem 16 (p. 157) implies that there are languages that cannot be solved by circuits of size $2^n/(2n)$.
- But exponential circuits can solve all problems.

**Proposition 67** *All decision problems (decidable or otherwise) can be solved by a circuit of size* $2^{n+2}$.

- We will show that for any language $L \subseteq \{0,1\}^*$, $L \cap \{0,1\}^n$ can be decided by a circuit of size $2^{n+2}$.




## The Proof (concluded)

- Define boolean function $f : \{0,1\}^n \to \{0,1\}$, where

$$f(x_1 x_2 \cdots x_n) = \begin{cases} 1 & x_1 x_2 \cdots x_n \in L, \\ 0 & x_1 x_2 \cdots x_n \notin L. \end{cases}$$

- $f(x_1 x_2 \cdots x_n) = (x_1 \wedge f(1 x_2 \cdots x_n)) \vee (\neg x_1 \wedge f(0 x_2 \cdots x_n))$.
- The circuit size $s(n)$ for $f(x_1 x_2 \cdots x_n)$ hence satisfies

$$s(n) = 3 + 2s(n-1)$$

  with $s(1) = 1$.
- Solve it to obtain $s(n) = 2^{n+1} + 2^{n-1} - 4$.




## The Circuit Complexity of P

**Proposition 68** *All languages in P have polynomial circuits.*

- Let $L \in \text{P}$ be decided by a TM in time $p(n)$.
- By Corollary 31 (p. 240), there is a circuit with $O(p(n)^2)$ gates that accepts $L \cap \{0,1\}^n$.
- The size of the circuit depends only on $L$ and the length of the input.
- The size of the circuit is polynomial in $n$.




## Languages That Polynomial Circuits Accept

- Do polynomial circuits accept only languages in P?
- There are *undecidable* languages that have polynomial circuits.
  - Let $L \subseteq \{0,1\}^*$ be an undecidable language.
  - Let $U = \{1^n : \text{the binary expansion of } n \text{ is in } L\}$.
  - $U$ must be undecidable.
  - $U \cap \{1\}^n$ can be accepted by $C_n$ that is trivially false if $1^n \notin U$ and trivially true if $1^n \in U$.
  - The family of circuits $(C_0, C_1, \ldots)$ is polynomial in size.

## A Patch

- Despite the simplicity of a circuit, the previous discussions imply the following:
  - Circuits are *not* a realistic model of computation.
  - Polynomial circuits are *not* a plausible notion of efficient computation.
- What gives?
- The *effective and efficient constructibility* of

$$C_0, C_1, \ldots .$$



## Uniformity

- A family $(C_0, C_1, \ldots)$ of circuits is **uniform** if there is a $\log n$-space bounded TM which on input $1^n$ outputs $C_n$.
  - Circuits now cannot accept undecidable languages (why?).
  - The circuit family on p. 426 is not constructible by a *single* Turing machine (algorithm).
- A language has **uniformly polynomial circuits** if there is a *uniform* family of polynomial circuits that decide it.



## Uniformly Polynomial Circuits and P

**Theorem 69** *$L \in P$ if and only if $L$ has uniformly polynomial circuits.*

- One direction was proved in Proposition 68 (p. 425).
- Now suppose $L$ has uniformly polynomial circuits.
- Decide $x \in L$ in polynomial time as follows:
  - Let $n = |\, x \,|$.
  - Build $C_n$ in $\log n$ space, hence polynomial time.
  - Evaluate the circuit with input $x$ in polynomial time.
- Therefore $L \in$ P.



## Relation to P vs. NP

- Theorem 69 implies that P $\neq$ NP if and only if NP-complete problems have no *uniformly* polynomial circuits.
- A stronger conjecture: NP-complete problems have no polynomial circuits, *uniformly or not*.
- The above is currently the preferred approach to proving the P $\neq$ NP conjecture—without success so far.
  - Theorem 16 (p. 157) states that there are boolean functions requiring $2^n/(2n)$ gates to compute.
  - In fact, almost all boolean functions do.

## BPP's Circuit Complexity

**Theorem 70 (Adleman (1978))** *All languages in BPP have polynomial circuits.*

- Our proof will be *nonconstructive* in that only the existence of the desired circuits is shown.
  - Something exists if its probability of existence is nonzero.
- How to efficiently generate circuit $C_n$ given $1^n$ is not known.
- If the construction of $C_n$ is efficient, then P = BPP, an unlikely result.



## The Proof

- Let $L \in$ BPP be decided by a precise NTM $N$ by clear majority.
- We shall prove that $L$ has polynomial circuits $C_0, C_1, \ldots$.
- Suppose $N$ runs in time $p(n)$, where $p(n)$ is a polynomial.
- Let $A_n = \{a_1, a_2, \ldots, a_m\}$, where $a_i \in \{0,1\}^{p(n)}$.
- Let $m = 12(n+1)$.
- Each $a_i \in A_n$ represents a sequence of nondeterministic choices—i.e., a computation path—for $N$.

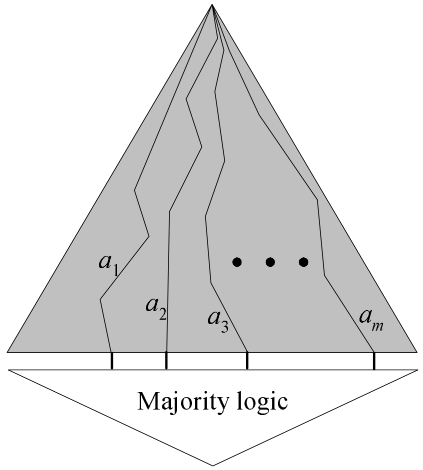


## The Proof (continued)

- Let $x$ be an input with $|x| = n$.
- Circuit $C_n$ simulates $N$ on $x$ with each sequence of choices in $A_n$ and then takes the majority of the $m$ outcomes.
- Because $N$ with $a_i$ is a polynomial-time TM, it can be simulated by polynomial circuits of size $O(p(n)^2)$.
  - See the proof of Proposition 68 (p. 425).
- The size of $C_n$ is therefore $O([\, mp(n)\,]^2) = O(n^2 p(n)^2)$, a polynomial.
- We next prove the existence of $A_n$ making $C_n$ correct.



## The Circuit



## The Proof (continued)

- Call $a_i$ **bad** if it leads $N$ to a false positive or a false negative answer.
- Select $A_n$ *uniformly randomly*.
- For each $x \in \{0,1\}^n$, $1/4$ of the computations of $N$ are erroneous.
- Because the sequences in $A_n$ are chosen randomly and independently, the expected number of bad $a_i$'s is $m/4$.
- By the Chernoff bound (p. 407), the probability that the number of bad $a_i$'s is $m/2$ or more is at most
$$e^{-m/12} < 2^{-(n+1)}.$$

## The Proof (concluded)

- The error probability is $< 2^{-(n+1)}$ for each $x \in \{0,1\}^n$.
- The probability that there is an $x$ such that $A_n$ results in an incorrect answer is $< 2^n 2^{-(n+1)} = 2^{-1}$.
  - $\text{prob}[\,A \cup B \cup \cdots\,] \leq \text{prob}[\,A\,] + \text{prob}[\,B\,] + \cdots$.
- So with probability one half, a random $A_n$ produces a correct $C_n$ for *all* inputs of length $n$.
- Because this probability exceeds 0, an $A_n$ that makes majority vote work for all inputs of length $n$ exists.
- Hence a correct $C_n$ exists.

## Cryptography[a]

- **Alice** (A) wants to send a message to **Bob** (B) over a channel monitored by **Eve** (eavesdropper).
- The protocol should be such that the message is known only to Alice and Bob.
- The art and science of keeping messages secure is **cryptography**.

$$\text{Alice} \xrightarrow{\text{Eve}} \text{Bob}$$

[a] "Whoever wishes to keep a secret must hide the fact that he possesses one." — Johann Wolfgang von Goethe (1749–1832).

## Encryption and Decryption

- Alice and Bob agree on two algorithms $E$ and $D$—the **encryption** and the **decryption algorithms**.
- Both $E$ and $D$ are known to the public in the analysis.
- Alice runs $E$ and wants to send a message $x$ to Bob.
- Bob operates $D$.
- Privacy is assured in terms of two numbers $e, d$, the **encryption** and **decryption keys**.
- Alice sends $y = E(e, x)$ to Bob, who then performs $D(d, y) = x$ to recover $x$.
- $x$ is called **plaintext**, and $y$ is called **ciphertext**.

## Some Requirements

- $D$ should be an inverse of $E$ given $e$ and $d$.
- $D$ and $E$ must both run in (probabilistic) polynomial time.
- Eve should not be able to recover $y$ from $x$ without knowing $d$.
  - As $D$ is public, $d$ must be kept secret.
  - $e$ may or may not be a secret.




## Degrees of Security

- **Perfect secrecy**: After a ciphertext is intercepted by the enemy, the a posteriori probabilities of the plaintext that this ciphertext represents are identical to the a priori probabilities of the same plaintext before the interception.
- Such systems are said to be **informationally secure**.
- A system is **computationally secure** if breaking it is theoretically possible, just computationally infeasible.