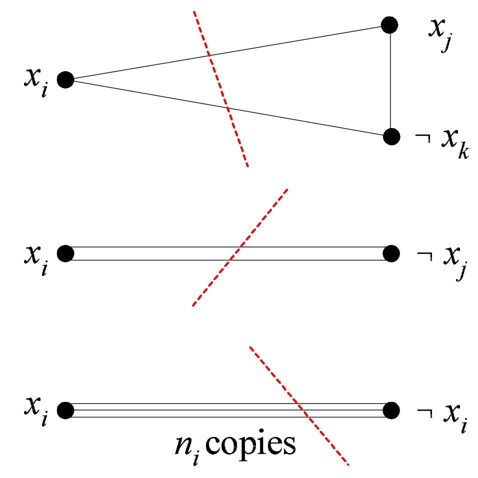
MAX CUT Is NP-Complete[a]

- We will reduce NAESAT to MAX CUT.
- Given an instance ϕ of 3SAT with m clauses, we shall construct a graph $G = (V, E)$ and a goal K such that:
  - There is a cut of size at least K if and only if ϕ is NAE-satisfiable.
- Our graph will have multiple edges between two nodes.
  - Each such edge contributes one to the cut if its nodes are separated.

[a] Garey, Johnson, and Stockmeyer (1976).

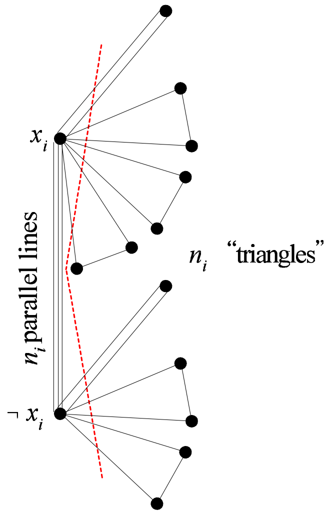
The Proof

- Suppose ϕ's m clauses are $C_1, C_2, \ldots, C_m$.
- The boolean variables are $x_1, x_2, \ldots, x_n$.
- G has $2n$ nodes: $x_1, x_2, \ldots, x_n, \neg x_1, \neg x_2, \ldots, \neg x_n$.
- Each clause with 3 distinct literals makes a triangle in G.
- For each clause with two identical literals, there are two parallel edges between the two distinct literals.
- No need to consider clauses with one literal (why?).
- For each variable x_i, add n_i copies of the edge $[x_i, \neg x_i]$, where n_i is the number of occurrences of x_i and $\neg x_i$ in ϕ.



The Proof (continued)

- Set $K = 5m$.
- Suppose there is a cut $(S, V - S)$ of size $5m$ or more.
- A clause (a triangle or two parallel edges) contributes at most 2 to a cut no matter how you split it.
- Suppose both x_i and $\neg x_i$ are on the same side of the cut.
- Then they *together* contribute at most $2n_i$ edges to the cut as they appear in at most n_i different clauses.

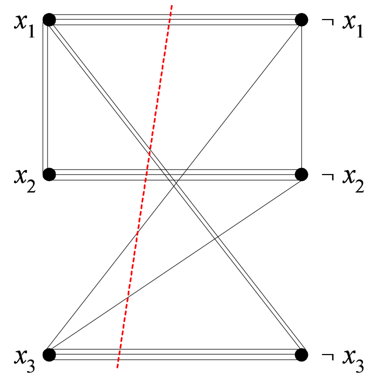


The Proof (continued)

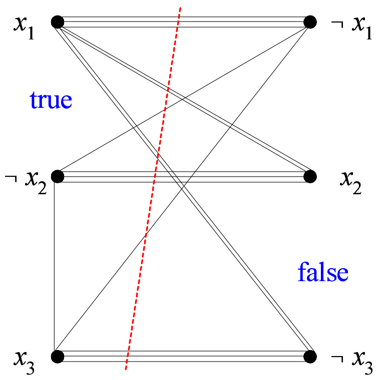
- Changing the side of a literal contributing at most n_i to the cut does not decrease the size of the cut.
- Hence we assume variables are separated from their negations.
- The total number of edges in the cut that join opposite literals is $\sum_i n_i = 3m$.
  - The total number of literals is $3m$.

The Proof (concluded)

- The *remaining* $2m$ edges in the cut must come from the m triangles or parallel edges that correspond to the clauses.
- As each can contribute at most 2 to the cut, all are split.
- A split clause means at least one of its literals is true and at least one false.
- The other direction is left as an exercise.



- $(x_1 \vee x_2 \vee x_2) \wedge (x_1 \vee \neg x_3 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$.
- The cut size is $13 < 5 \times 3 = 15$.



- $(x_1 \lor x_2 \lor x_2) \land (x_1 \lor \neg x_3 \lor \neg x_3) \land (\neg x_1 \lor \neg x_2 \lor x_3)$.
- The cut size is now 15.

MAX BISECTION

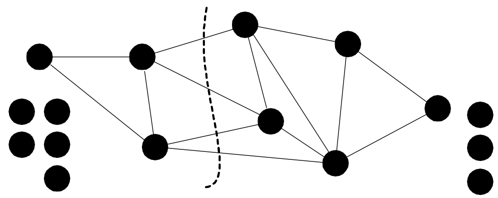
- MAX CUT becomes MAX BISECTION if we require that $|S| = |V - S|$.
- It has many applications, especially in VLSI layout.
- Sometimes imposing additional restrictions makes a problem easier.
  - SAT to 2SAT.
- Other times, it makes the problem as hard or harder.
  - MIN CUT to BISECTION WIDTH.
  - LINEAR PROGRAMMING to INTEGER PROGRAMMING.

MAX BISECTION Is NP-Complete

- We shall reduce the *more general* MAX CUT to MAX BISECTION.
- Add $|V|$ **isolated nodes** to G to yield G'.
- G' has $2 \times |V|$ nodes.
- As the new nodes have no edges, moving them around contributes nothing to the cut.

The Proof (concluded)

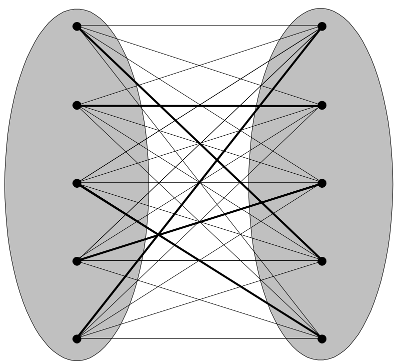
- Every cut $(S, V - S)$ of $G = (V, E)$ can be made into a bisection by appropriately allocating the new nodes between S and $V - S$.
- Hence each cut of G can be made a cut of G' of the same size, and vice versa.



BISECTION WIDTH

- BISECTION WIDTH is like MAX BISECTION except that it asks if there is a bisection of size *at most* K (sort of MIN BISECTION).
- Unlike MIN CUT, BISECTION WIDTH remains NP-complete.
  - A graph $G = (V, E)$, where $|V| = 2n$, has a bisection of size K if and only if the complement of G has a bisection of size $n^2 - K$.
  - So G has a bisection of size $\geq K$ if and only if its complement has a bisection of size $\leq n^2 - K$.

Illustration



HAMILTONIAN PATH Is NP-Complete[a]

Theorem 41 *Given an* undirected *graph, the question whether it has a Hamiltonian path is NP-complete.*

- Reduce 3SAT to HAMILTONIAN PATH.
- We skip the messy proof in the text.

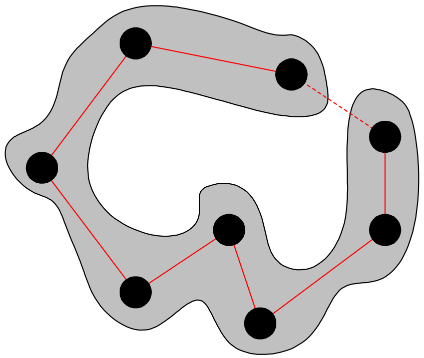
[a]Karp (1972).

TSP (D) Is NP-Complete

Corollary 42 TSP (D) *is NP-complete.*

- Consider a graph G with n nodes.
- Define $d_{ij} = 1$ if $[\,i, j\,] \in G$ and $d_{ij} = 2$ if $[\,i, j\,] \notin G$.
- Set the budget $B = n + 1$.
- If G has no Hamiltonian paths, then every tour on the new graph must contain at least two edges with weight 2.
- The total cost is then at least $(n - 2) + 2 \cdot 2 = n + 2$.
- There is a tour of length B or less if and only if G has a Hamiltonian path.

Hamiltonian Path and TSP Tour



Graph Coloring

- k-COLORING asks if the nodes of a graph can be colored with $\leq k$ colors such that no two adjacent nodes have the same color.
- 2-COLORING is in P (why?).
- But 3-COLORING is NP-complete.

3-COLORING Is NP-Complete[a]

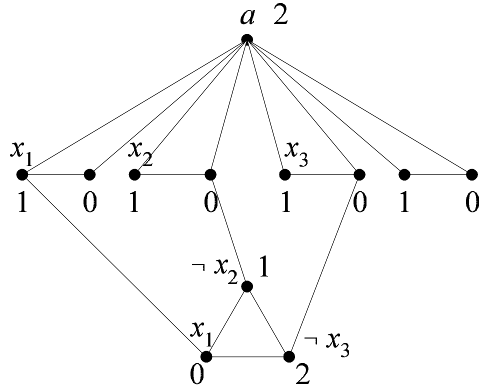
- We will reduce NAESAT to 3-COLORING.
- We are given a set of clauses $C_1, C_2, \ldots, C_m$ each with 3 literals.
- The boolean variables are $x_1, x_2, \ldots, x_n$.
- We shall construct a graph G such that it can be colored with colors $\{0, 1, 2\}$ if and only if all the clauses can be NAE-satisfied.

[a]Karp (1972).

The Proof (continued)

- Every variable x_i is involved in a triangle $[\,a, x_i, \neg x_i\,]$ with a common node a.
- Each clause $C_i = (c_{i1} \lor c_{i2} \lor c_{i3})$ is also represented by a triangle
$$[\,c_{i1}, c_{i2}, c_{i3}\,].$$
- There is an edge between c_{ij} and the node that represents the jth literal of C_i.

Construction for $\cdots \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \wedge \cdots$



The Proof (continued)

Suppose the graph is 3-colorable.

- Assume without loss of generality that node a takes the color 2, x_i takes the color 1, and $\neg x_i$ takes the color 0.
- A triangle must use all 3 colors.
- The clause triangle cannot be linked to nodes with all 1s or all 0s; otherwise, it cannot be colored with 3 colors.
- Treat 1 as true and 0 as false (it is consistent).
- Treat 2 as either true or false; it does not matter.
- As each clause triangle contains one color 1 and one color 0, the clauses are NAE-satisfied.

The Proof (concluded)

Suppose the clauses are NAE-satisfiable.

- Color node a with color 2.
- Color the nodes representing literals by their truth values (color 0 for false and color 1 for true).
- For each clause triangle:
  - Pick any two literals with opposite truth values and color the corresponding nodes with 0 if the literal is true and 1 if it is false.
  - Color the remaining node with color 2.

