The Quantified Halting Problem

- Let $f(n) \geq n$ be proper.
- Define

$$H_f = \{M; x : M \textit{ accepts} \text{ input } x \text{ after at most } f(|\,x\,|) \text{ steps}\},$$

  where M is deterministic.
- Assume the input is binary.

$H_f \in \mathsf{TIME}(f(n)^3)$

- For each input $M; x$, we simulate M on x with an alarm clock of length $f(|\,x\,|)$.
  - Use the single-string simulator (p. 57), the universal TM (p. 107), and the linear speedup theorem (p. 62).
- From p. 61, the total running time is $O(\ell k^2 f(n)^2)$, where ℓ is the length to encode each symbol or state of M and k is M's number of strings.
- As $\ell = O(\log n)$, the running time is $O(f(n)^3)$, where the constant is independent of M.

$H_f \notin \mathsf{TIME}(f(\lfloor n/2 \rfloor))$

- Suppose there is a TM M_{H_f} deciding H_f in time $f(\lfloor n/2 \rfloor)$.
- Consider machine $D_f(M)$:

$$\textbf{if } M_{H_f}(M; M) = \text{“yes”} \textbf{ then } \text{“no”} \textbf{ else } \text{“yes”}$$

- D_f on input M runs in the same time as M_{H_f} on input $M; M$, i.e., in time $f(\lfloor \frac{2n+1}{2} \rfloor) = f(n)$.
- $D_f(D_f) = \text{“yes”} \Rightarrow D_f; D_f \notin H_f \Rightarrow D_f(D_f) = \text{“no.”}$
- Similarly, $D_f(D_f) = \text{“no”} \Rightarrow D_f(D_f) = \text{“yes.”}$

The Time Hierarchy Theorem

Theorem 16 *If $f(n) \geq n$ is proper, then*

$$\mathrm{TIME}(f(n)) \subsetneq \mathrm{TIME}(f(2n+1)^3).$$

- The quantified halting problem makes it so.

Corollary 17 $\mathrm{P} \subsetneq \mathrm{EXP}$.

- $\mathrm{P} \subseteq \mathrm{TIME}(2^n)$ because $\mathrm{poly}(n) \leq 2^n$ for n large enough.
- But by Theorem 16,

$$\mathrm{TIME}(2^n) \subsetneq \mathrm{TIME}((2^{2n+1})^3) \subseteq \mathrm{TIME}(2^{n^2}) \subseteq \mathrm{EXP}.$$

The Space Hierarchy Theorem

Theorem 18 *If $f(n)$ is proper, then*

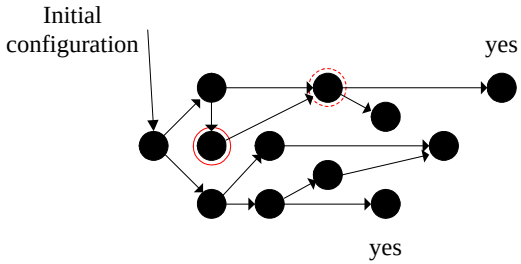
$$\text{SPACE}(f(n)) \subsetneq \text{SPACE}(f(n)\log f(n)).$$

Corollary 19 $\text{L} \subsetneq \text{PSPACE}$.

The Reachability Method

- A computation of a TM can be represented by directional transitions between configurations.
- The reachability method constructs a directed graph with all the TM configurations as its nodes and edges connecting two nodes if one yields the other.
- The start node representing the initial configuration has zero in degree.
- When the TM is nondeterministic, a node may have an out degree greater than one.

Illustration of the Reachability Method



The reachability method may give the edges on the fly without explicitly storing the whole configuration graph.

Relations between Complexity Classes

Theorem 20 *Suppose $f(n)$ is proper. Then*

1. $\text{SPACE}(f(n)) \subseteq \text{NSPACE}(f(n))$, $\text{TIME}(f(n)) \subseteq \text{NTIME}(f(n))$.
2. $\text{NTIME}(f(n)) \subseteq \text{SPACE}(f(n))$.
3. $\text{NSPACE}(f(n)) \subseteq \text{TIME}(k^{\log n + f(n)})$.

- Proof of 2:
  - Explore the computation *tree* of the NTM for "yes."
  - Use the *depth-first* search as f is proper.

Proof of Theorem 20(2)

- (continued)
  - Specifically, generate a $f(n)$-bit sequence denoting the nondeterministic choices over $f(n)$ steps.
  - Simulate the NTM based on the choices.
  - Recycle the space and then repeat the above steps until a "yes" is encountered or the tree is exhausted.
  - Each path simulation consumes at most $O(f(n))$ space because it takes $O(f(n))$ time.
  - The total space is $O(f(n))$ as space is recycled.

Proof of Theorem 20(3)

- Let k-string NTM

$$M = (K, \Sigma, \Delta, s)$$

  with input and output decide $L \in \text{NSPACE}(f(n))$.
- Use the reachability method on the configuration graph of M on input x of length n.
- A configuration is a $(2k+1)$-tuple

$$(q, w_1, u_1, w_2, u_2, \ldots, w_k, u_k).$$

Proof of Theorem 20(3) (continued)

- We only care about

$$(q, i, w_2, u_2, \ldots, w_{k-1}, u_{k-1}),$$

  where i is an integer between 0 and n for the position of the first cursor.
- The number of configurations is therefore at most

$$|K| \times (n+1) \times |\Sigma|^{(2k-4)f(n)} = O(c_1^{\log n + f(n)}) \quad (3)$$

  for some c_1, which depends on M.
- Add edges to the configuration graph based on the transition function.

Proof of Theorem 20(3) (concluded)

- $x \in L \Leftrightarrow$ there is a path in the configuration graph from the initial configuration to a configuration of the form ("yes", $i, \ldots$) [there may be many of them].
- The problem is therefore that of REACHABILITY on a graph with $O(c_1^{\log n + f(n)})$ nodes.
- It is in $\text{TIME}(c^{\log n + f(n)})$ for some c because REACHABILITY is in $\text{TIME}(n^k)$ for some k and

$$\left(c_1^{\log n + f(n)}\right)^k = (c_1^k)^{\log n + f(n)}.$$

A Corollary of the Reachability Method

Corollary 21 *For any NTM M in NSPACE($f(n)$), where $f(n) = \Omega(\log n)$, there is a TM in SPACE($f(n)$) that writes out the configuration graph of $M(x)$, given input x.*

- From the proof of Theorem 20 (p. 169), especially Eq. (3) on p. 172, the number of configurations is $O(c^{f(n)})$ for some constant c.
- Use two counters each with space $O(f(n))$ to enumerate all possible pairs of configurations, (C_1, C_2).
- Write (C_1, C_2) to the output string if C_1 yields C_2.

The Grand Chain of Inclusions

$$\mathrm{L} \subseteq \mathrm{NL} \subseteq \mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE} \subseteq \mathrm{EXP}.$$

- It is known that PSPACE $\subsetneq$ EXP.
- By Corollary 19 (p. 166), we know L $\subsetneq$ PSPACE.
- The chain must break somewhere between L and PSPACE.
- We suspect all four inclusions are proper.
- But there are no proofs yet.

Nondeterministic Space and Deterministic Space

- By Theorem 5 (p. 88),

$$\mathrm{NTIME}(f(n)) \subseteq \mathrm{TIME}(c^{f(n)}),$$

  an exponential gap.
- There is no proof that the exponential gap is inherent, however.
- How about NSPACE vs. SPACE?
- Surprisingly, the relation is only quadratic, a polynomial, by Savitch's theorem.

Savitch's Theorem

Theorem 22 (Savitch, 1970)

$$\textsc{reachability} \in \mathrm{SPACE}(\log^2 n).$$

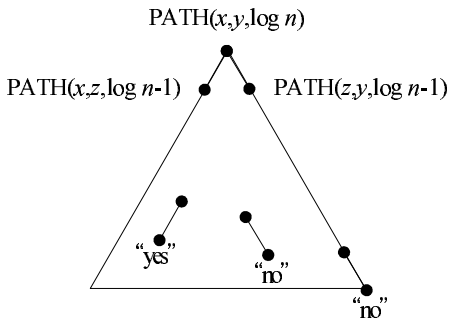
- Let G be a graph with n nodes.
- For $i \geq 0$, let

$$\mathrm{PATH}(x, y, i)$$

  mean there is a path from node x to node y of length at most 2^i.
- There is a path from x to y if and only if $\mathrm{PATH}(x, y, \lceil \log n \rceil)$ holds.

The Simple Idea for Computing PATH(x, y, i)

- For $i > 0$, PATH(x, y, i) if and only if there exists a z such that PATH$(x, z, i-1)$ and PATH$(z, y, i-1)$.
- For PATH$(x, y, 0)$, check the input graph or if $x = y$.
- We compute PATH$(x, y, \lceil \log n \rceil)$ with a depth-first search on a tree with nodes (x, y, i)s.
- Like stacks in recursive calls, we keep only the current path of (x, y, i)s.
- The space requirement is proportional to the depth of the tree, $\lceil \log n \rceil$.



- Depth is $\lceil \log n \rceil$, and each node (x, y, i) needs space $O(\log n)$.
- The total space is $O(\log^2 n)$.

The Algorithm for PATH(x, y, i)

```
1: if i = 0 then
2:    if x = y or (x, y) ∈ G then
3:       return true;
4:    else
5:       return false;
6:    end if
7: else
8:    for z = 1, 2, ..., n do
9:       if PATH(x, z, i − 1) and PATH(z, y, i − 1) then
10:         return true;
11:      end if
12:   end for
13:   return false;
14: end if
```


The Relation between Nondeterministic Space and Deterministic Space Only Quadratic

Corollary 23 *Let $f(n) \geq \log n$ be proper. Then*

$$\text{NSPACE}(f(n)) \subseteq \text{SPACE}(f^2(n)).$$

- Apply Savitch's theorem to the configuration graph of the NTM on the input.
- From p. 172, the configuration graph has $O(c^{f(n)})$ nodes; hence each node takes space $O(f(n))$.
- But the graph is implicit—we check for connectedness only when $i = 0$, by examining the input string.

Implications of Savitch's Theorem

- PSPACE = NSPACE.
- Nondeterminism is less powerful with respect to space.
- It may be very powerful with respect to time as it is not known if P = NP.

Nondeterministic Space Is Closed under Complement

- Closure under complement is trivially true for deterministic complexity classes (p. 160).
- On p. 186, we shall prove

$$\text{coNSPACE}(f(n)) = \text{NSPACE}(f(n)).$$

- So

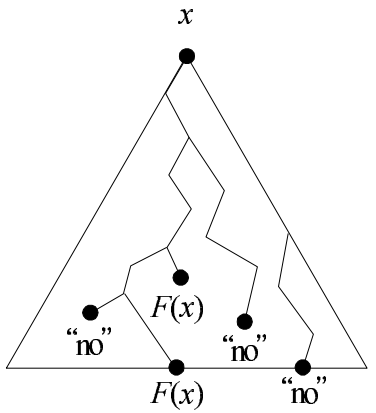
$$\begin{aligned} \text{coNL} &= \text{NL}, \\ \text{coPSPACE} &= \text{NPSPACE}. \end{aligned}$$

- But there are still no hints of coNP = NP.

Functions and Nondeterministic TMs

- An NTM computes function F if the following hold:
  - On input x, each computation path either outputs the correct answer $F(x)$ or ends up in state "no."
  - At least one computation path ends up with $F(x)$.
- So all successful paths agree on their output.
- Existence of output indicates successful computation.
- As before, the machine observes a space bound $f(n)$ if at halting all strings (except for the input and output ones) are of length at most $f(|\,x\,|)$.

How an NTM Computes a Function



The Immerman-Szelepscényi Theorem

Theorem 24 (Szelepscényi, 1987, Immerman, 1988) *Given a graph G and a node x, the number of nodes reachable from x in G can be computed by an NTM within space $O(\log n)$.*

- The algorithm has four nested loops.
- Let n be the number of nodes.
- $S(k)$ denotes the set of nodes in G that can be reached from x by paths of length at most k.
- So $|S(n-1)|$ is the desired answer.

The Algorithm: Top 2 Levels

```
 1: |S(0)| := 1;
 2: for k = 1, 2, ..., n − 1 do
 3:    {Compute |S(k)| from |S(k − 1)| saved in previous loop.}
 4:    ℓ := 0;
 5:    for u = 1, 2, ..., n do
 6:       if u ∈ S(k) then
 7:          ℓ := ℓ + 1;
 8:       end if
 9:    end for
10:    |S(k)| := ℓ;
11: end for
12: return |S(n − 1)|;
```

(Need $|S(k-1)|$, but not earlier ones.)

The Third Loop, for $u \in S(k)$

```
 1: m := 0; {Count members of S(k − 1) encountered.}
 2: reply := false;
 3: for v = 1, 2, ..., n do
 4:    if v ∈ S(k − 1) then
 5:       m := m + 1;
 6:       if G(v, u) then
 7:          reply := true;
 8:       end if
 9:    end if
10: end for
11: if m < |S(k − 1)| then
12:    "no"; {Cannot be sure of the validity of reply.}
13: end if
14: return reply;
```


The Fourth Loop, for $v \in S(k-1)$

```
 1: s := x;
 2: for i = 1, 2, ..., k − 1 do
 3:    Guess a node t ∈ {1, 2, ..., n}; {Nondeterminism.}
 4:    if (s, t) ∉ G then
 5:       "no";
 6:    end if
 7:    s := t;
 8: end for
 9: if t = v then
10:    return true;
11: else
12:    "no";
13: end if
```

Wrap Up the Proof

- Space is needed for k, $|S(k-1)|$, ℓ, u, m, v, s, i, t.
- The nondeterministic algorithm needs space $O(\log n)$.

Closure under Complement of Nondeterministic Space

Corollary 25 *If $f \geq \log n$ is proper, then*

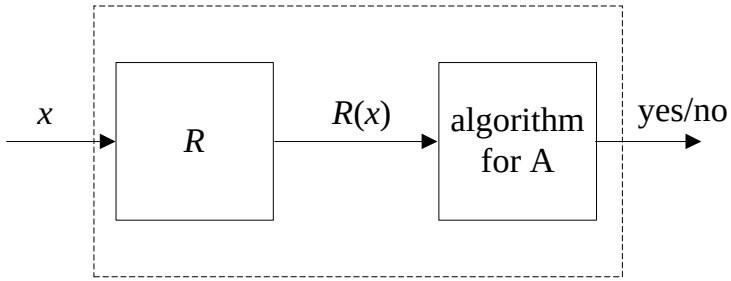
$$\text{NSPACE}(f(n)) = \text{coNSPACE}(f(n)).$$

- Run the above algorithm on the configuration graph of the NTM M deciding $L \in \text{NSPACE}(f(n))$ on input x.
- We accept only if no accepting configurations have been encountered and if $|S(n-1)|$ is computed.
  - The existence of $|S(n-1)|$ means that every reachable configuration has been visited.

Degrees of Difficulty

- When is a problem more difficult than another?
- B **reduces to** A if there is a transformation R which for every input x of B yields an equivalent input $R(x)$ of A.
  - The answer to x for B is the same as the answer to $R(x)$ for A.
  - There must be restrictions on the complexity of computing R.
  - Otherwise, $R(x)$ might as well solve B.
- Problem A is at least as hard as problem B if B reduces to A.

Reduction



Solving problem B by calling the algorithm for problem *once* and *without* further processing its answer.

Reduction between Languages

- Language L_1 is **reducible to** L_2 if there is a function R computable by a deterministic TM in space $O(\log n)$.
- Furthermore, for all inputs x, $x \in L_1$ if and only if $R(x) \in L_2$.
- R is called a (**Karp**) **reduction** from L_1 to L_2.
- Note that by Theorem 20 (p. 169), R runs in polynomial time.

A Paradox?

- Degree of difficulty is not defined in terms of *absolute* complexity.
- A language A $\in$ TIME(n^{99}) may be "easier" than a language B $\in$ TIME(n^3).
- This happens when A is reducible to B.
  - In this situation, it is necessary that $|\, R(x)\,| = \Omega(n^{33})$ or that R runs in time $\Omega(n^{99})$ so that A $\notin$ TIME(n^k) for some $k < 99$.

Reduction of HAMILTONIAN PATH to SAT

- Given a graph G, we shall construct a CNF $R(G)$ such that $R(G)$ is satisfiable if and only if G has a Hamiltonian path.
- Suppose G has n nodes: $1, 2, \ldots, n$.
- $R(G)$ has n^2 boolean variables x_{ij}, $1 \le i, j \le n$.
- x_{ij} means "node j is the ith node in the Hamiltonian path."

The Clauses of $R(G)$

1. Each node j must appear in the path.
   - $x_{1j} \vee x_{2j} \vee \cdots \vee x_{nj}$ for each j.
2. No node j appears twice in the path.
   - $\neg x_{ij} \vee \neg x_{kj}$ for all i, j, k with $i \neq k$.
3. Every position i on the path must be occupied.
   - $x_{i1} \vee x_{i2} \vee \cdots \vee x_{in}$ for each i.
4. No two nodes j and k occupy the same position in the path.
   - $\neg x_{ij} \vee \neg x_{ik}$ for all i, j, k with $j \neq k$.
5. Nonadjacent nodes i and j cannot be adjacent in the path.
   - $\neg x_{ki} \vee \neg x_{k+1,j}$ for all $(i, j) \notin G$ and $k = 1, 2, \ldots, n-1$.

The Proof

- $R(G)$ can be computed efficiently.
- Suppose $T \models R(G)$.
- Clauses of 1 and 2 imply that for each j, there is a unique i such that $T \models x_{ij}$.
- Clauses of 3 and 4 imply that for each i, there is a unique j such that $T \models x_{ij}$.
- So there is a permutation π of the nodes such that $\pi(i) = j$ if and only if $T \models x_{ij}$.
- Clauses of 5 guarantees that $(\pi(1), \pi(2), \ldots, \pi(n))$ is a Hamiltonian path.

The Proof (concluded)

- Conversely, suppose G has a Hamiltonian path

$$(\pi(1), \pi(2), \ldots, \pi(n)),$$

  where π is a permutation.
- Clearly, the truth assignment

$$T(x_{ij}) = \texttt{true} \text{ if and only if } \pi(i) = j$$

  satisfies all clauses of $R(G)$.

Reduction of REACHABILITY to CIRCUIT VALUE

- Note that both problems are in P.
- Given a graph $G = (V, E)$, we shall construct a *variable-free* circuit $R(G)$.
- The output of $R(G)$ is true if and only if there is a path from node 1 to node n in G.
- Idea: the Floyd-Warshall algorithm.

The Gates

- The gates are
  - g_{ijk} with $1 \leq i, j \leq n$ and $0 \leq k \leq n$.
  - h_{ijk} with $1 \leq i, j, k \leq n$.
- g_{ijk}: There is a path from node i to node j without passing through a node bigger than k.
- h_{ijk}: There is a path from node i to node j passing through k but not any node bigger than k.
- Input gate $g_{ij0} = \texttt{true}$ if and only if $i = j$ or $(i, j) \in E$.

The Construction

- h_{ijk} is an AND gate with predecessors $g_{i,k,k-1}$ and $g_{k,j,k-1}$, where $k = 1, 2, \ldots, n$.
- g_{ijk} is an OR gate with predecessors $g_{i,j,k-1}$ and $h_{i,j,k}$, where $k = 1, 2, \ldots, n$.
- g_{1nn} is the output gate.
- Interestingly, $R(G)$ uses no $\neg$ gates: It is a **monotone circuit**.
- The depth of $R(G)$ is $O(n)$, which is not optimal.

Reduction of CIRCUIT SAT to SAT

- Given a circuit C, we shall construct a boolean expression $R(C)$ such that $R(C)$ is satisfiable if and only if C is satisfiable.
  - $R(C)$ will turn out to be a CNF.
- The variables of $R(C)$ are those of C plus g for each gate g of C.
- Each gate of C will be turned into equivalent clauses of $R(C)$.
- Recall that clauses are $\wedge$ed together.

The Clauses of $R(C)$

g **is a variable gate** x**:** Add clauses $(\neg g \vee x)$ and $(g \vee \neg x)$.

- Meaning: $g \Leftrightarrow x$.

g **is a `true` gate:** Add clause (g).

- Meaning: g must be true to make $R(C)$ true.

g **is a `false` gate:** Add clause $(\neg g)$.

- Meaning: g must be false to make $R(C)$ true.

g **is a $\neg$ gate with predecessor gate** h**:** Add clauses $(\neg g \vee \neg h)$ and $(g \vee h)$.

- Meaning: $g \Leftrightarrow \neg h$.

The Clauses of $R(C)$ (concluded)

g **is a $\vee$ gate with predecessor gates** h **and** h'**:** Add clauses $(\neg h \vee g)$, $(\neg h' \vee g)$, and $(h \vee h' \vee \neg g)$.

- Meaning: $g \Leftrightarrow (h \vee h')$.

g **is a $\wedge$ gate with predecessor gates** h **and** h'**:** Add clauses $(\neg g \vee h)$, $(\neg g \vee h')$, and $(\neg h \vee \neg h' \vee g)$.

- Meaning: $g \Leftrightarrow (h \wedge h')$.

g **is the output gate:** Add clause (g).

- Meaning: g must be true to make $R(C)$ true.