Chapter 10

Sensitivity Analysis of Options

Contents

Cleopatra's nose, had it been shorter,
the whole face of the world
would have been changed.
—Blaise Pascal (1623–1662)

Understanding how the value of a security changes relative to changes in a chosen parameter is key to hedging. Duration, for instance, measures the rate of change of the bond value with respect to interest rate changes. This chapter asks similar questions of options. The materials here will be used for hedging later in the book.

10.1 Sensitivity Measures (the Greeks)

In the following, we take

$$x \equiv \frac{\ln(S/X) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}$$

as in the Black-Scholes formula (see Theorem 9.3.3). We will see a common feature that sensitivity tends to be highest in absolute terms for options near the money.

10.1.1 Delta

For a derivative such as option, its **delta** is defined as $\Delta \equiv \partial f/\partial S$, where f is the price of the derivative, and S is that of the underlying asset. The delta of a portfolio of derivatives on the same underlying asset is the sum of the deltas of the individual derivatives.

For a European call on a non-dividend-paying stock, its delta equals

$$\frac{\partial C}{\partial S} = N(x) > 0. \tag{10.1}$$

For a European put on a non-dividend-paying stock, the delta is

$$\frac{\partial P}{\partial S} = N(x) - 1 < 0.$$

The delta of stock is of course one. We remark that the delta used in the binomial option pricing model to replicate options is the discrete analogue of the delta here.

A position with a delta of zero is said to be **delta-neutral**. Since it is immune to *small* price changes, a delta-neutral portfolio is typically constructed for hedging purposes. See Fig. 10.1 for illustration. An obvious example of a delta-neutral portfolio consists of a long position in a call option and a short position in Δ units of the underlying asset. In general, one can hedge one option with another option on the same underlying asset by taking positions in the right proportions: Being short Δ_1/Δ_2 units of option 2 for each unit of option 1 held long is delta-neutral where Δ_i is the delta of option i for $i = 1, 2$.

10.1.2 Theta

The **theta** (or **time decay**) of a portfolio of derivatives is defined as the rate of change of the portfolio's value with respect to time, $\Theta \equiv -\partial \Pi/\partial \tau$, where Π is the value of the portfolio. For a European call on a non-dividend-paying stock,

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N\left(x - \sigma\sqrt{\tau}\right) < 0.$$

Note that $N'(y) = \frac{1}{\sqrt{2\pi}}\,e^{-y^2/2} > 0$ is the density function of the standard normal distribution. The call hence loses value with the passage of time. For a European put,

$$\Theta = -\frac{SN'(x)\,\sigma}{2\sqrt{\tau}} + rXe^{-r\tau}N\left(-x + \sigma\sqrt{\tau}\right),$$

which may be negative or positive. The ambiguity results from two countervailing forces with the passage of time (i.e., as τ decreases): (1) The put is less likely to be exercised because of lower variance, hence more valuable, and (2) the present value of the exercise payment rises, lifting the put value. See Fig. 10.2 for illustration. Consult [422] for thetas of European options on stocks that pay known dividend yields.

Theta tends to be negative because an option usually loses value as the expiration date draws near. In fact, theta is positive only for puts that are deep in the money. The reason is that, as explained on Page 105, a European put can be worth *less than* its intrinsic value but must eventually rise to its intrinsic value at expiration. Theta achieves the maximum magnitude near the strike price for calls.

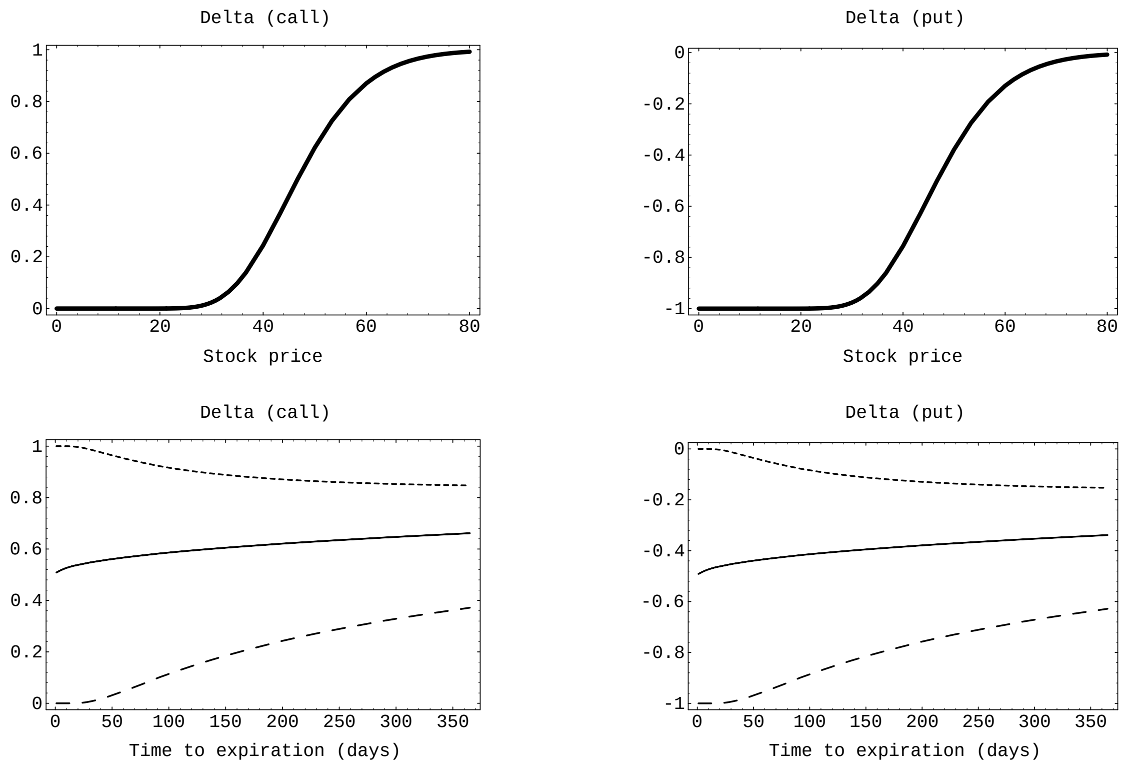

Figure 10.1: OPTION DELTA. The default parameters are $S = 50$, $X = 50$, $\tau = 201$ (days), $\sigma = 0.3$, and $r = 8\%$. The dotted lines use $S = 60$ (in-the-money call or out-of-the-money put), the solid lines use $S = 50$ (at-the-money option), and the dashed lines use $S = 40$ (out-of-the-money call or in-the-money put).

10.1.3 Gamma

The **gamma** of a portfolio of derivatives is the rate of change of the portfolio's delta with respect to the price of the underlying asset, $\Gamma \equiv \partial^2 \Pi / \partial S^2$. Gamma measures how sensitive delta is to changes in the price of the underlying asset. The implication is that a delta-neutral portfolio needs to be rebalanced more often if its gamma is high. Gamma therefore measures the degree of risk exposure a hedged position will develop if the hedge is not adjusted. Delta and gamma have obvious counterparts in bonds: duration and convexity. The gamma of a European call or put on a non-dividend-paying stock is $\frac{N'(x)}{S\sigma\sqrt{\tau}} > 0$. See Fig. 10.3 for illustration.

The change in the value of a portfolio of derivatives can be expanded as

$$d\Pi = \frac{\partial \Pi}{\partial S}\, dS + \frac{\partial \Pi}{\partial \tau}\, d\tau + \frac{1}{2}\, \frac{\partial^2 \Pi}{\partial S^2}\, (dS)^2 + \frac{1}{2}\, \frac{\partial^2 \Pi}{\partial \tau^2}\, (d\tau)^2 + \frac{\partial^2 \Pi}{\partial S \partial \tau}\, dS\, d\tau + \cdots$$

if the volatility of the underlying asset is a constant. The above can be simplied to

$$d\Pi \approx \frac{\partial \Pi}{\partial \tau}\, d\tau + \frac{1}{2}\, \frac{\partial^2 \Pi}{\partial S^2}\, (dS)^2 = -\Theta\, d\tau + \frac{1}{2}\, \Gamma\, (dS)^2$$

if the portfolio is also delta-neutral.

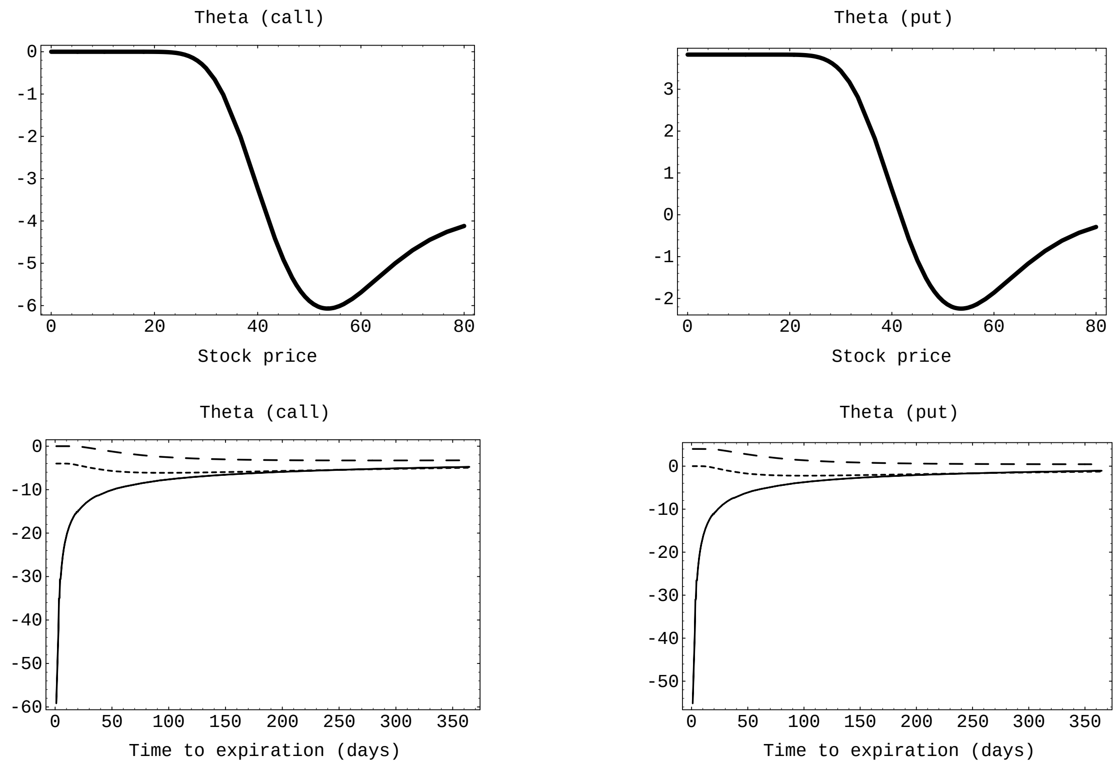

Figure 10.2: OPTION THETA. The default parameters are $S = 50$, $X = 50$, $\tau = 201$ (days), $\sigma = 0.3$, and $r = 8\%$. The dotted lines use $S = 60$ (in-the-money call or out-of-the-money put), the solid lines use $S = 50$ (at-the-money option), and the dashed lines use $S = 40$ (out-of-the-money call or in-the-money put).

10.1.4 Vega

Volatility often changes over time. The **vega** (sometimes called **lambda**, **kappa**, or **sigma**) of a portfolio of derivatives is the rate of change of the portfolio's value with respect to the volatility of the underlying asset, $\Lambda \equiv \partial \Pi / \partial \sigma$. (Vega is of course not Greek.) A security with a high vega means it is very sensitive to small changes in volatility. The vega of a European call or put on a non-dividend-paying stock is[1] $S\sqrt{\tau}\, N'(x) > 0$. (This inequality incidentally solves Exercise 9.4.1.) See Fig. 10.4 for illustration. A positive vega is consistent with the intuition that higher volatility increases option value.

10.1.5 Rho

The **rho** of a portfolio of derivatives is the rate of change in the value of the portfolio with respect to interest rates, $\rho \equiv \partial \Pi / \partial r$. The rho of a European call on a non-dividend-paying stock is $X\tau e^{-r\tau} N\left(x - \sigma\sqrt{\tau}\right) > 0$. The rho of a European put on a non-dividend-paying stock is $-X\tau e^{-r\tau} N\left(-x + \sigma\sqrt{\tau}\right) < 0$.

[1]For users of *Mathematica*'s Finance Pack: Its `Lambda` function is incorrectly implemented and should be multiplied by τ.

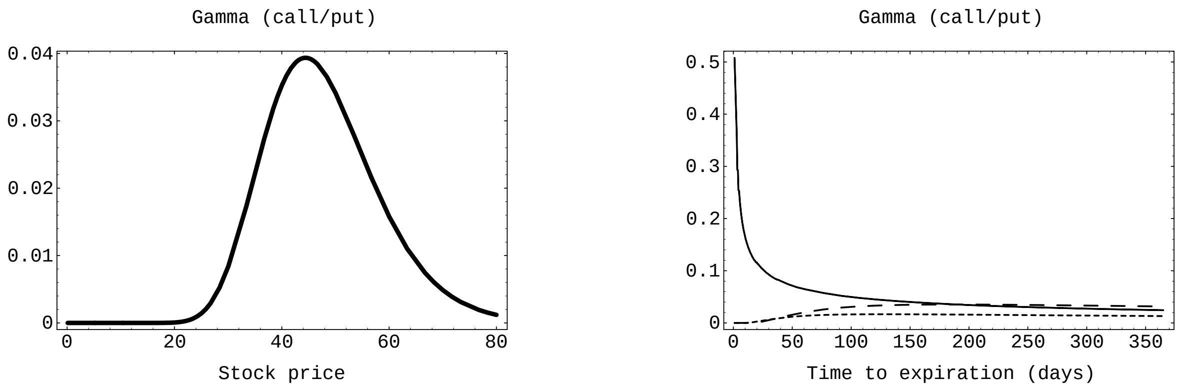

Figure 10.3: Option gamma. The default parameters are $S = 50$, $X = 50$, $\tau = 201$ (days), $\sigma = 0.3$, and $r = 8\%$. The dotted lines use $S = 60$ (in-the-money call or out-of-the-money put), the solid lines use $S = 50$ (at-the-money option), and the dashed lines use $S = 40$ (out-of-the-money call or in-the-money put).

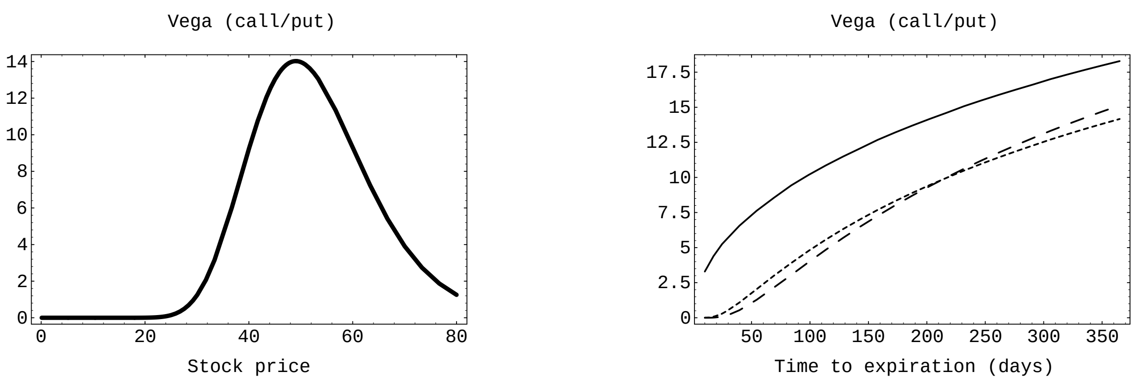

Figure 10.4: Option vega. The default parameters are $S = 50$, $X = 50$, $\tau = 201$ (days), $\sigma = 0.3$, and $r = 8\%$. The dotted lines use $S = 60$ (in-the-money call or out-of-the-money put), the solid lines use $S = 50$ (at-the-money option), and the dashed lines use $S = 40$ (out-of-the-money call or in-the-money put).

10.2 Numerical Techniques

Sensitivity measures of options for which closed-form formulae do not exist have to be computed numerically. The issues of efficiency and accuracy can be treacherous here. Take delta as an example. It is defined as $\Delta f / \Delta S$, where ΔS is a small change in the stock price and Δf is the resulting change in the option price. A naive method computes $f(S - \Delta S)$ and $f(S + \Delta S)$ before finally settling for

$$\frac{f(S + \Delta S) - f(S - \Delta S)}{2\,\Delta S}$$

as an approximation for delta. The computation time for this numerical differentiation scheme roughly doubles that for evaluating the option itself. Worse, numerical differentiation may give wildly inaccurate results.

A preferred approach is to take advantage of the intermediate results of the binomial

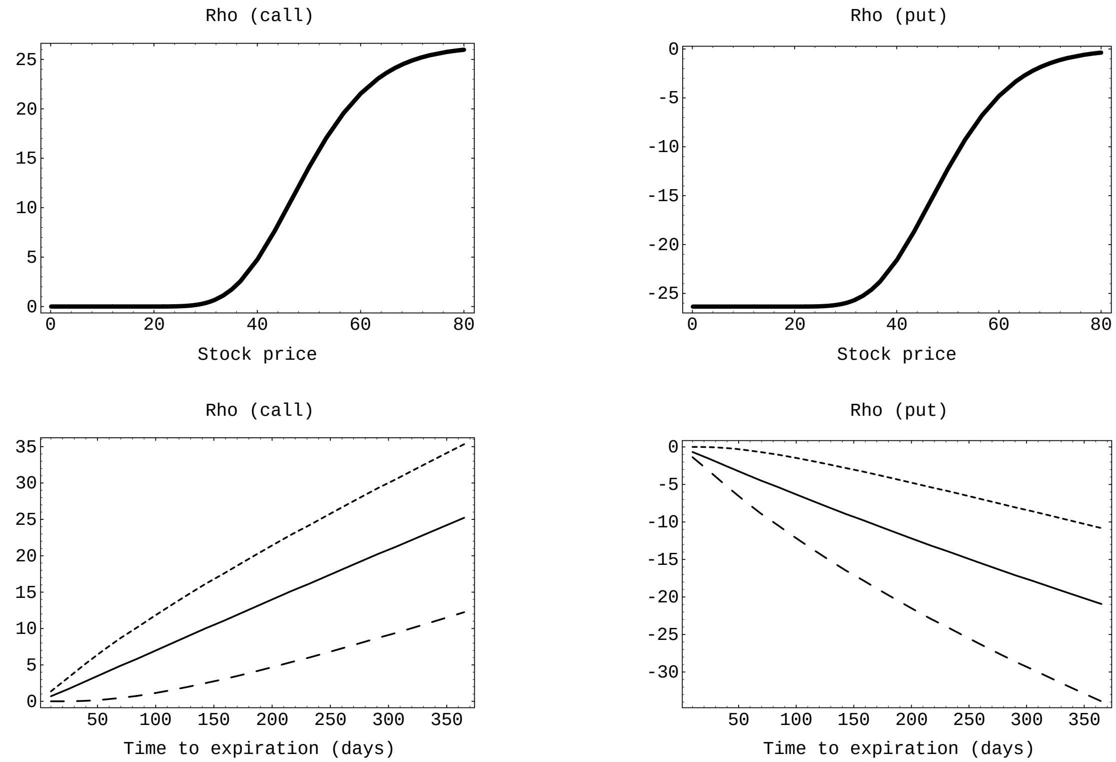

Figure 10.5: OPTION RHO. The default parameters are $S = 50$, $X = 50$, $\tau = 201$ (days), $\sigma = 0.3$, and $r = 8\%$. The dotted lines use $S = 60$ (in-the-money call or out-of-the-money put), the solid lines use $S = 50$ (at-the-money option), and the dashed lines use $S = 40$ (out-of-the-money call or in-the-money put).

tree algorithm. When the algorithm works backward in time and eventually reaches the end of the first period, C_u and C_d are computed. Recall that these option values correspond to stock prices Su and Sd, respectively, where S is today's stock price. Delta can therefore be approximated by

$$\frac{C_u - C_d}{Su - Sd},$$

the hedge ratio. The extra computational effort beyond the original binomial tree algorithm is essentially nil. Note that $Su \to Sd$ as the number of periods increases.

Other sensitivity measures can be similarly derived. Take gamma. At the stock price of $(Suu + Sud)/2$, delta is approximately $(C_{uu} - C_{ud})/(Suu - Sud)$, and at the stock price of $(Sud + Sdd)/2$, delta is approximately $(C_{ud} - C_{dd})/(Sud - Sdd)$. Now, gamma is computed as the rate of change in deltas between $(Suu + Sud)/2$ and $(Sud + Sdd)/2$, that is,

$$2\,\frac{\frac{C_{uu} - C_{ud}}{Suu - Sud} - \frac{C_{ud} - C_{dd}}{Sud - Sdd}}{Suu - Sdd}.$$

Contrast it with the numerical differentiation,

$$\frac{f(S + \Delta S) - 2f(S) + f(S - \Delta S)}{(\Delta S)^2}.$$

Strictly speaking, the delta and gamma thus computed are the delta at the end of the first period and the gamma at the end of the second period. In other words, they are not the sensitivity measures at the present time but at times τ/n and $2(\tau/n)$ from now, respectively, where n denotes the number of periods into which the time to expiration τ is partitioned. However, as n increases, such values should approach delta and gamma well. Theta, similarly, can be computed as

$$\Theta \approx \frac{C_{ud} - C}{2(\tau/n)}.$$

As for vega and rho, there is no alternative but to run the binomial tree algorithm twice. In a coming chapter, theta will be shown to be computable from delta and gamma.

10.2.1 Reasons numerical differentiation fails

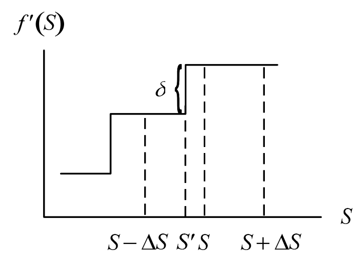

Figure 10.6: NUMERICAL DIFFERENTIATION FOR DELTA AND GAMMA.

A careful inspection of (9.10) reveals why numerical differentiation fails. First, the option value is a continuous piecewise linear function of the current stock price S. Kinks develop at $S = Xu^{-j}d^{-(n-j)}$ $(j = 0, \ldots, n)$. As a result, if ΔS is suitably small, delta computed by numerical differentiation will be a ladder-like function of S, hence not differentiable, at the kinks. This bodes ill for numerical gamma. In fact, if ΔS is suitably small, gamma computed through numerical differentiation will be zero most of the time since $f'(S - \Delta S) = f'(S) = f'(S + \Delta S)$ except for S near a kink. When S is near a kink, another problem arises, however. Assume S is to the right of the kink at S' and $S - \Delta S < S' < S$. Hence, $f'(S) = f'(S + \Delta S)$ and $f'(S) - f'(S - \Delta S) = \delta$ for some constant $\delta > 0$. Numerical gamma now equals

$$\frac{f(S + \Delta S) - 2f(S) + f(S - \Delta S)}{(\Delta S)^2} = \frac{\delta(S' - S + \Delta S)}{(\Delta S)^2}.$$

This number can be huge as ΔS decreases toward $S - S'$. Therefore, the standard practice of reducing the step size does not help at all. See Fig. 10.6 for illustration.

10.2.2 Extended binomial tree algorithms

In the recommended numerical scheme based on the binomial tree for computing delta and gamma, they are computed not at the current time but one and two periods from now, respectively. An improved method starts the binomial tree two periods *before* now as in Fig. 10.7. Delta is then computed as

$$\frac{C_{u/d} - C_{d/u}}{(Su/d) - (Sd/u)}.$$

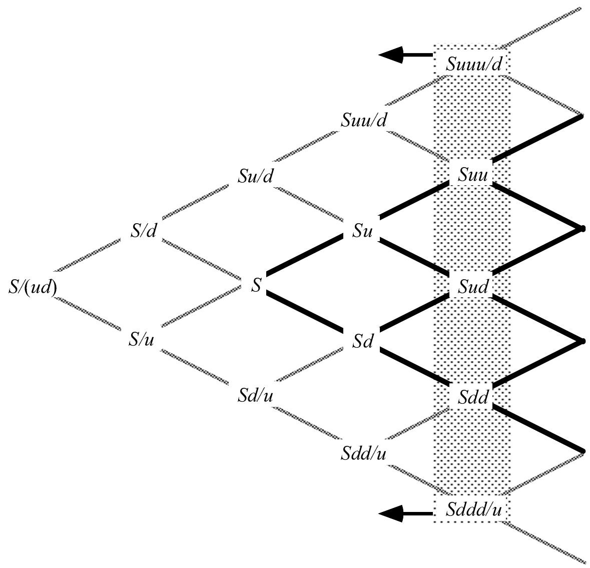

Figure 10.7: EXTENDED BINOMIAL TREE. The extended binomial tree is constructed from the original binomial tree (in bold lines) but with time extended beyond the present by two periods.

Similarly, gamma is computed as

$$2\,\frac{\frac{C_{u/d}-C}{(Su/d)-S}-\frac{C-C_{d/u}}{S-(Sd/u)}}{(Su/d)-(Sd/u)}.$$

See [624] for more information.

Programming assignment 10.2.1 Implement the extended binomial tree algorithm for numerical delta and gamma. Compare the results against numerical differentiation and closed-form solution. ◊