# Chapter 7

# Option Basics

## Contents



*"Well, Beethoven, what is this?"*
—Attributed to Prince Anton Esterházy

Options give their holder the right to buy or sell some **underlying asset**. They form one of the most important classes of financial instruments and have wide-ranging applications in finance—In reality, most securities have option features. As far as explaining empirical data goes, the option pricing theory is the most successful theory in finance as well as economics [665]. The methodology developed by the theory also forms the cornerstone for the general theory of derivative pricing.

## 7.1 Introduction

There are two basic types of options: calls and puts. More complex option-like instruments can usually be decomposed into packages of calls and puts. A **call** option gives its holder the right to buy a specified number of the underlying asset by paying a specified **exercise** or **strike price** at or before **expiration**. A **put** option gives its holder the right to sell a specified number of the underlying asset by paying a specified strike price at or before expiration. The underlying asset may be stocks, stock indexes, foreign currencies, futures contracts, interest rates, fixed-income securities, prices of some fixed-income instruments, options, even winter temperatures, and countless others. When an option is embedded, it has to be traded along with the underlying asset. The case of embedded options is highly complex and will be treated separately. In this chapter, we focus mainly on exchange-traded stock options, noting that the basic insights are applicable to other kinds of underlying assets. As the value of an option depends on the price of its underlying asset, it is a **contingent claim** or **derivative security**.

The individual who issues an option is the **writer**. To acquire the option, the holder pays the writer a **premium**. When a call option is **exercised**, the holder pays the writer the strike price in exchange for the stock, and the option ceases to exist. When a put option is exercised, the holder receives from the writer the strike price in exchange for the stock, and the option ceases to exist. **Early exercise** refers to the act of exercising an option prior to expiration. At any trading date before expiration, the holder can also sell the option.

American and European options differ in when the holder can exercise them. **American** options can be exercised at any time up to the expiration date, while **European** options can only be exercised at expiration. An American option is worth at least as much as an otherwise identical European option because of this early exercise feature. Like the Holy Roman Empire, the terms "American" and "European" have nothing to do with geography.

Many strategies and analysis in the book depend on taking a short position. In stocks, short sales involve borrowing stock certificates and buying them back later; in a word, selling (what one does not own) precedes buying. The short seller is betting that the stock price will decline. Note that borrowed shares have to be paid back with shares, not dollars [699]. The short seller does *not* receive cash dividends; in fact, this individual must make matching dividend payments to the person to whom the shares were sold. Clearly, any dividend payout reduces a short seller's return.

It is easier to take a short position in derivatives. All one has to do is to find an investor who is willing to buy the derivative, that is, who is long. Since the underlying asset is not involved, it is not necessary that some asset holders lend their securities. For derivatives which do not deliver the underlying asset or those which are mostly settled by taking offset positions, outstanding derivative contracts may cover many times the underlying asset [50].

In this chapter, $C$ denotes the call option price, $P$ the put option price, $t$ the time to expiration, $X$ the strike price, $S$ the stock price, and $D$ the dividend. Subscripts are used to emphasize times to expiration, stock prices, or strike prices. The symbol $\mathrm{PV}(x)$ stands for the present value of $x$ at expiration.

## 7.2 Basic Payoff Patterns

An option does not oblige the holder to exercise the right; it can be allowed to expire worthless. Hence, an option will be exercised only when it is in the best interest of the holder to do so. Clearly, a call will be exercised only if the stock price is higher than the strike price. Similarly, a put will be exercised only if the stock price is less than the strike price. The value or **payoff** of a call at expiration is therefore $C = \max(0, S - X)$, and that of a put at expiration is $P = \max(0, X - S)$ (see Fig. 7.1). The payoff of a position, unlike profit, is its value regardless of the initial cost.

A call is said to be **in the money** if $S > X$, **at the money** if $S = X$, and **out of the money** if $S > X$. Similarly, a put is said to be **in the money** if $S < X$, **at the money** if

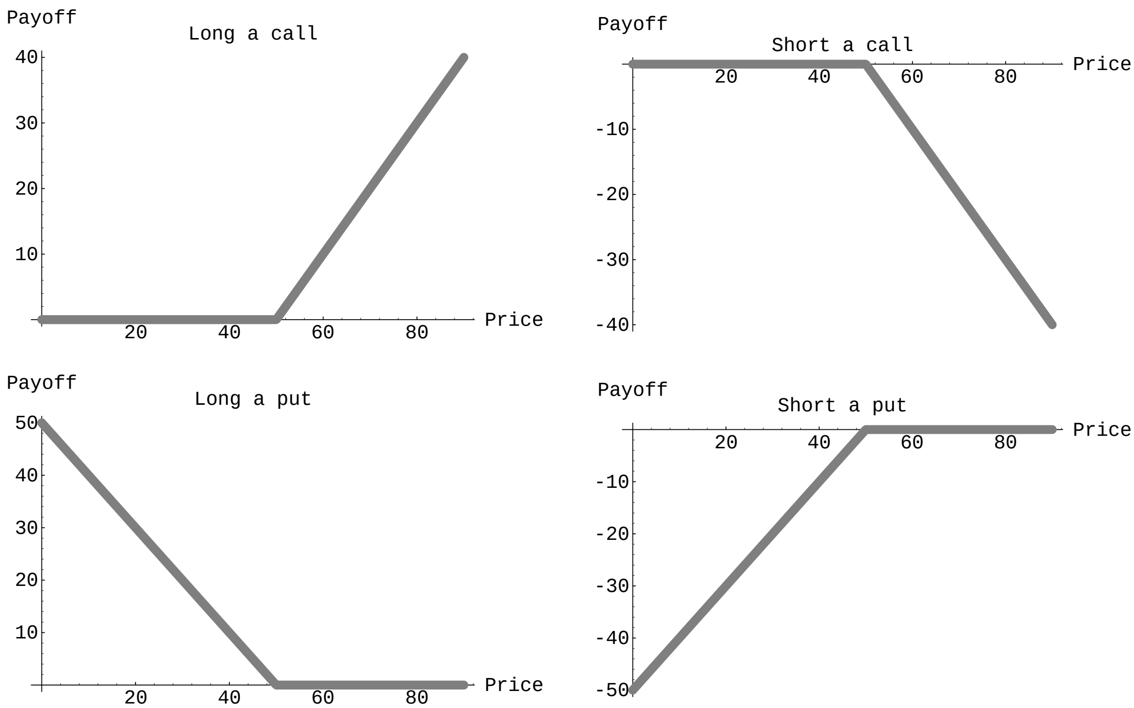


Figure 7.1: OPTION PAYOFFS. The payoffs of options at expiration with $X = 50$.

$S = X$, and **out of the money** if $S > X$. We call $\max(0, S - X)$ the intrinsic value of a call and $\max(0, X - S)$ the intrinsic value of a put. The **intrinsic value** is the value of an option if it is exercised immediately. The part of an option's value above its intrinsic value is called its **time value** and represents the possibility to become more valuable before the option expires. The option premium thus consists of the intrinsic value and the time value.

Although an option's terminal payoff is obvious, finding it value at any time before expiration is anything but. This problem will form the central theme in Chapter 9. Figure 7.2 plots the values of puts and calls prior to expiration. The payoff of a long position in stock is $S$, while the payoff of a short position in stock is $-S$. Figure 7.3 shows the payoffs of long and short positions in stock.

## 7.3 Exchange-Traded Options

The Chicago Board Options Exchange (CBOE) started the options trading on April 26, 1973. Now options are being traded in many exchanges such as the American Stock Exchange (AMEX) and the Philadelphia Stock Exchange (PHLX). Exchange-traded options standardize the terms of option contracts, create centralized trading and price dissemination facilities, and introduce the Options Clearing Corporation (OCC), all of which serve to reduce the costs and promote an active secondary market. The term "**listed option**" is also used to refer to an exchange-traded option [603].

Puts and calls first appeared in 1790. (Aristotle described a kind of call in *Politics* [23,

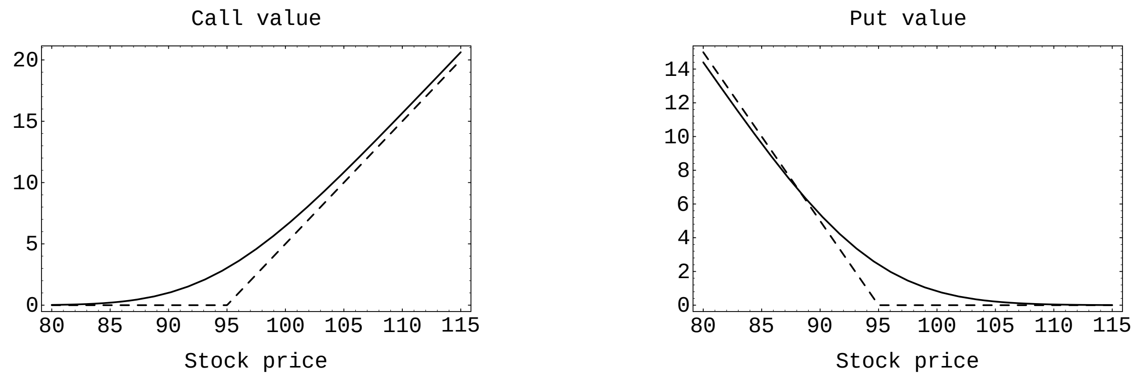


Figure 7.2: VALUES OF CALL AND PUT PRIOR TO EXPIRATION. Plotted are the general shapes of call and put values as a function of stock price before expiration. Dashed lines are the familiar option value diagrams at expiration.

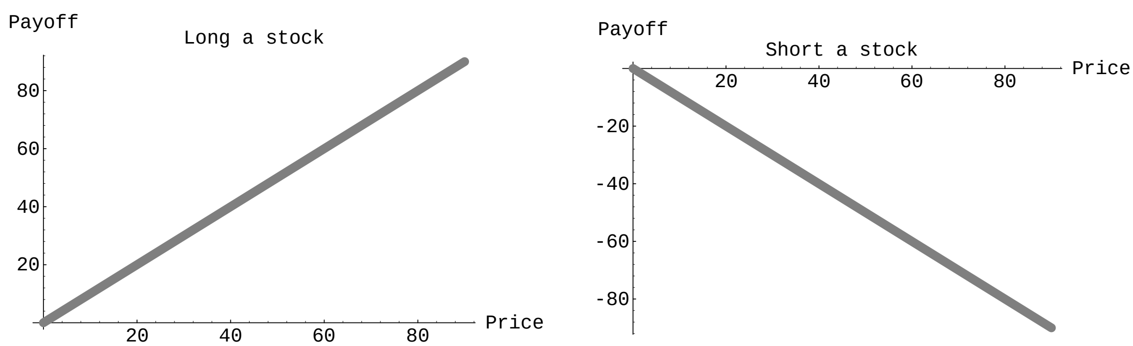


Figure 7.3: PAYOFF OF STOCK. The payoffs of long and short positions in stock.

Book 2, Chapter 11].) Prior to 1973, options were traded in **over-the-counter markets** where financial institutions and corporations trade directly with one another. The main distinction of an over-the-counter option is that it is customized. They are most popular in the area of foreign currencies and interest rates.

Terms on the exchange-traded stock options govern the expiration dates and the strike prices. The strike prices center around the current price of the underlying stock with fixed increments, depending on the price of the stock. Typical increments are $\$2\frac{1}{2}$ for a stock price less than \$25 per share, \$5 for a stock price between \$25 and \$200 per share, and \$10 for a stock price over \$200 per share. The expiration date is 10:59 P.M. Central Time on the Saturday after the third Friday of the expiration month. The last day on which options trade is the third Friday of the expiration month. Any stock typically has options outstanding expiring on three expiration dates. There are also rules regarding the days any option can be traded. The exchange limits the maximum number of options an individual can take on one side of the market. A contract normally covers 100 shares of stock. Option prices are quoted per unit of the underlying asset. For instance, although each stock option covers 100 shares, it is quoted as price per share of the underlying stock. For instance, the Merck July 35 call closed at $9\frac{1}{2}$ on March 20, 1995 (see Fig. 7.4). The total cost of the call is therefore \$950. Exchange-traded stock options are American.

Unlike over-the-counter options, exchange-traded stock options are not **cash dividend-protected** (or simply **protected**). This means the option contract is not adjusted for dividends. As the stock price typically falls by the amount roughly equal to the amount of the cash dividend per share in the U.S. as it goes ex-dividend, dividends are detrimental for calls. The converse is true for puts. But options are adjusted for stock splits. After an $n$-for-$m$ stock split, the strike price is only $m/n$ times its previous value, and the number of shares covered by one contract becomes $n/m$ times its previous value. Exchange-traded stock options are also adjusted for stock dividends. Unless otherwise qualified, options are assumed to be unprotected. Figure 7.4 shows a small sample of listed equity options.

**Example 7.3.1** For a call option to buy 100 shares of a company for \$50 per share, a 2-for-1 split would change the term to a strike price of \$25 per share for 200 shares. □

| | | | —Call— | | —Put— | |
|---|---|---|---|---|---|---|
| Option | Strike | Exp. | Vol. | Last | Vol. | Last |
| | | | ... | | | |
| **Exxon** | 60 | Apr | 1053 | 51/2 | 1000 | 3/16 |
| 65 | 65 | Apr | 951 | 15/16 | 830 | 11/16 |
| 65 | 65 | May | 53 | 17/16 | 10 | 11/16 |
| 65 | 65 | Oct | 32 | 23/4 | ... | ... |
| 65 | 70 | Jul | 2 | 1/4 | 40 | 51/4 |
| | | | ... | | | |
| **Merck** | 30 | Jul | 328 | 151/4 | ... | ... |
| 441/2 | 35 | Jul | 150 | 91/2 | 10 | 1/16 |
| 441/2 | 40 | Apr | 887 | 43/4 | 136 | 1/16 |
| 441/2 | 40 | Jul | 220 | 51/2 | 297 | 1/4 |
| 441/2 | 40 | Oct | 58 | 6 | 10 | 1/2 |
| 441/2 | 45 | Apr | 3050 | 7/8 | 100 | 11/8 |
| 441/2 | 45 | May | 462 | 13/8 | 50 | 13/8 |
| 441/2 | 45 | Jul | 883 | 115/16 | 147 | 13/4 |
| 441/2 | 45 | Oct | 367 | 23/4 | 188 | 21/16 |
| | | | ... | | | |
| **Microsoft** | 55 | Apr | 65 | 163/4 | 52 | 1/8 |
| 711/8 | 60 | Apr | 556 | 113/4 | 39 | 1/8 |
| 711/8 | 65 | Apr | 302 | 7 | 137 | 3/8 |
| 711/8 | 65 | Jul | 93 | 9 | 15 | 11/2 |
| 711/8 | 65 | Oct | 34 | 105/8 | 9 | 21/4 |
| 711/8 | 70 | Apr | 1543 | 31/8 | 162 | 11/2 |
| 711/8 | 70 | May | 42 | 41/4 | 2 | 21/8 |
| 711/8 | 70 | Jul | 190 | 53/4 | 61 | 3 |
| 711/8 | 70 | Oct | 94 | 71/2 | 1 | 4 |
| | | | ... | | | |

Figure 7.4: Option quotations from *The Wall Street Journal*, March 21, 1995.

For exchange-traded options, an option holder can **close out** or **liquidate** the position by issuing an **offsetting order** to sell the same option. An option writer can close out the position by issuing an offsetting order to buy the same option. This is called **settled by offset**, which is made possible by the Options Clearing Corporation. The **open interest** is simply the total number of contracts that have not been offset, exercised, or allowed to expire, in a word, the total number of long (short) positions.

One of the main reasons for the popularity and importance of options comes from the fact that they can provide payoff patterns that could not be obtained with stocks. This point will become clear in the following section.

## 7.4 Basic Option Strategies

A trading strategy aims at realizing a particular financial objective. Option strategies involve taking positions in options, the underlying assets, and borrowing or lending. They can be **bullish**, **bearish**, or **neutral** in terms of market outlook; they can be **aggressive**, **defensive**, or virtually **riskless** in terms of risk posture; they can be designed to profit in volatile or calm markets.

We have mentioned six **uncovered** (or **naked**) positions in the previous section: long stock, short stock, long call, short call, long put, and short put. For example, buying the stock is a bullish and aggressive strategy, bullish because it profits when the stock price goes up, and aggressive because the investor runs the risk of maximum loss, dollar for dollar, if the stock goes down. More aggressive strategies include buying stocks on margin. For instance, the Exxon April 60 call allows the holder to control a $65 stock for a mere $5.5 (see Fig. 7.4). Selling short is aggressive but bearish. In reality, asymmetry exists between buying the stock and shorting it because of the margin requirements for the latter. Consequently, buying a put might be better than shorting the stock.

### 7.4.1 Covered positions

There are three basic kinds of **covered** positions: **hedge**, **spread,** and **combination**. In covered positions, some securities protect the returns of other securities, and they are all related to the same underlying asset. Some strategies like **condor** and **seagull** do not fall into any of the above-mentioned categories [458].

### 7.4.2 Hedge

A hedge refers to combining an option with its underlying stock in such a way that one protects the other against loss. A hedge that combines a long position in stock with a long position in puts is called a **protective put**. A hedge that combines a long position in stock with a short call is called a **covered call**. Covered call may be the most common option strategy used by institutional investors to generate extra income in a flat market. See Fig. 7.5 for the profits of protective puts and covered calls. Since both strategies break even only if the stock price rises, their market outlook is bullish. The strategies are also defensive: The investor owns the stock already anyway in covered calls, whereas the protective put guarantees a minimum value for the portfolio. A **reverse hedge** is a hedge in the opposite direction: a short position in stock combined with a short put or a long call.

A **ratio hedge** combines two short calls against each share of stock. It profits as long as the stock price does not move far in either direction. See Fig. 7.6 for illustration. The

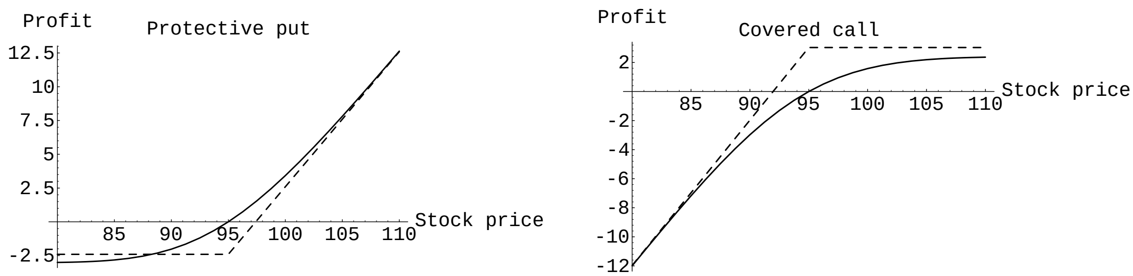


Figure 7.5: PROFITS OF PROTECTIVE PUT AND COVERED CALL. Plotted are the profit diagrams of a protective put and a covered call with a strike price of \$95, assuming a current stock price of \$95. Dashed lines are the respective portfolio profits at expiration. **Profit diagrams** do not take into account the time value of the money used in setting up the position.

profit pattern of a ratio hedge is hard to replicate without options.

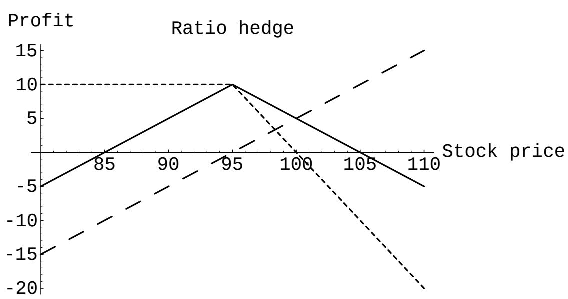


Figure 7.6: PROFIT OF RATIO HEDGE. Plotted is the profit diagram of a ratio hedge at expiration with a strike price of \$95, assuming a current stock price of \$95. The dashed line is the stock profit, the dotted line is the option profit, and the solid line is the portfolio profit.

Writing a **cash secured put** means writing naked puts while putting aside enough money to cover the strike price if the puts are exercised. The payoff pattern is similar to the covered call. The maximum profit is $X - (\mathrm{PV}(X) - P)$, while the maximum loss is $P - \mathrm{PV}(X)$, which occurs when the stock becomes worthless.

### 7.4.3 Spread

A spread consists of options of the *same* type on the same underlying asset but with different strike prices or expiration dates. They are of great interest to options market makers and sophisticated investors. We use $X_L$, $X_M$, and $X_H$ to denote three strike prices with $X_L < X_M < X_H$.

A **bull call spread** consists of a long $X_L$ call and a short $X_H$ call with the same expiration date. The initial investment is $C_L - C_H$. Note that $C_H < C_L$. The maximum profit is $(X_H - X_L) - (C_L - C_H)$, while the maximum loss is $C_H - C_L$. The risk posture is obviously defensive. See Fig. 7.7 for illustration. Such a spread is called **price spread**, **money spread**, or **vertical spread** (vertical, because it involves options on different rows of the same vertical column, as is obvious from Fig. 7.4) [88]. Similar results can be achieved by writing a high strike price put and buying a lower strike price put at the same expiration date, creating a **bull put spread**. A **bear spread** amounts to selling a bull spread. It profits from declining stock prices.

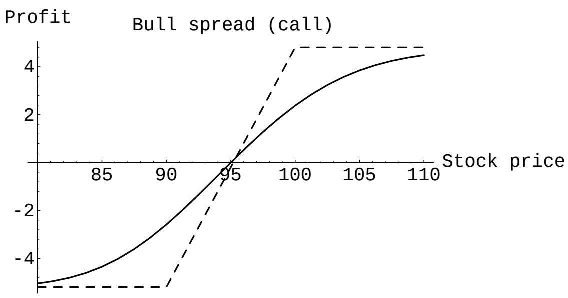


Figure 7.7: PROFIT OF BULL SPREAD. Plotted is the profit diagrams of a bull spread at expiration (dashed line) and at one month before expiration (solid line). Here, the strike price is \$95, and the current stock price is \$95.

**Example 7.4.1** An investor bought a call. Afterwards, the market moved in her favor, and she was able to write a call for the same premium but at a higher strike price. She ended up with a bull spread and a terminal payoff that could never be negative. □

Three calls or three puts with different strike prices and the same expiration date create the so-called **butterfly spread**. Specifically, we long one $X_L$ call, long one $X_H$ call, and short two $X_M$ calls. The first two calls form the **wings**. See Fig. 7.8 for illustration. Notice that a butterfly spread pays a positive amount at expiration only if the asset price falls between $X_L$ and $X_H$. Therefore, a butterfly spread with a small $X_H - X_L$ approximates a **state contingent claim**, which pays off \$1 only when a particular state takes place [302]. State contingent claim is also called **Arrow security** in recognition of Arrow's seminal contribution in 1953 [740].

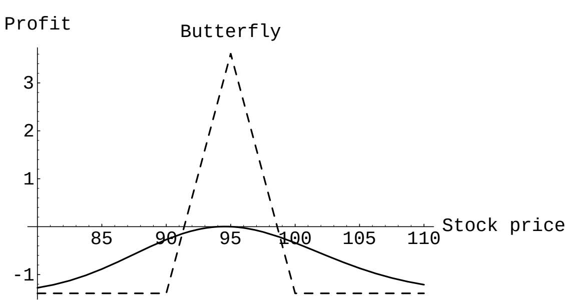


Figure 7.8: PROFIT OF BUTTERFLY. Plotted is the profit diagram of a butterfly at expiration (dashed line) and at one month before expiration when it is initially set up (solid line). Here, the strike prices are \$90, \$95, and \$100, and the current stock price is \$95.

A **horizontal spread**, also called **time spread** or **calendar spread**, involves two options with the same strike price but different expiration dates [88]. A typical time spread consists of a long call with a far expiration date and a short call with a nearer expiration date. Its profit pattern arises from the difference in the rate of time decay between options expiring on different dates. See Fig. 7.9 for illustration. A **diagonal spread** involves two options with different strike prices and different expiration dates.

A riskless portfolio can be created by simultaneously buying the stock, writing a call, and buying a put. The net result is known profit at any time up to expiration regardless of the stock price fluctuation. See Fig. 7.10 for illustration.

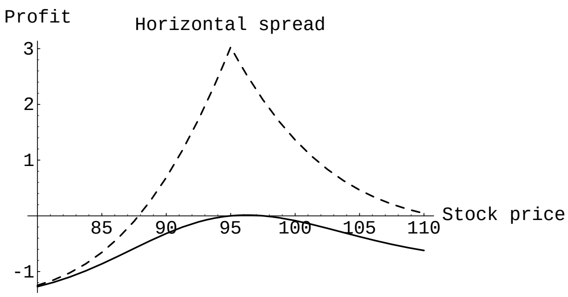


Figure 7.9: PROFIT OF HORIZONTAL SPREAD. Plotted is the profit diagram of a horizontal spread at expiration of the nearer call (dashed line) and at the time when it is initially set up (solid line). Here, the strike price is \$95, the current stock price is \$95, one month remains until the first expiration date, and two months remain until the second expiration date.

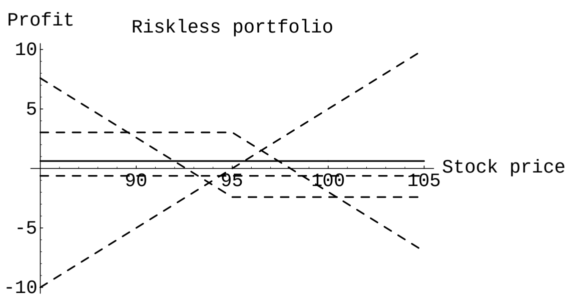


Figure 7.10: PROFIT OF RISKLESS PORTFOLIO. Plotted is the profit diagram of the riskless portfolio mentioned in the text. Here, the strike price is \$95, and the current stock price is \$95.

### 7.4.4 Combination

A combination consists of options of *different* types on the same underlying asset, and they are either both bought or both written. A **straddle** is created by a long call and a long put with the same strike price and expiration date. A straddle is neutral on price with limited risk, and it profits from high volatility. A person who buys a straddle is said to be **long volatility** [557]. Selling a straddle benefits from low volatility with a maximum profit of $C + P$. See Fig. 7.11 for illustration.

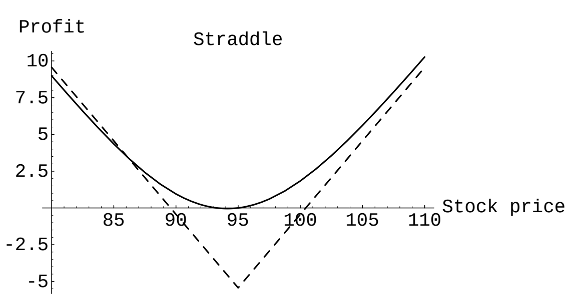


Figure 7.11: PROFIT OF STRADDLE. Plotted is the profit diagram of a straddle at expiration (dashed line) and at one month before expiration when it is initially set up (solid line). Here, the strike price is \$95, and the current stock price is \$95.

A **strip** consists of a long call and two long puts with the same strike price and expiration date. A **strap** consists of a long put and two long calls with the same strike price and expiration date. The profit patterns of strip and strap are very much like that of straddle except that they are not symmetrical around the strike price. Hence, although they also bet on volatile price movements, one direction is deemed more likely than the other.

A **strangle** is identical to a straddle except that the call's strike price is higher than the put's strike price. Comparing selling a straddle and selling a strangle, the latter seems less risky because the region for profitability is wider. Figure 7.12 illustrates the profit pattern of a strangle.

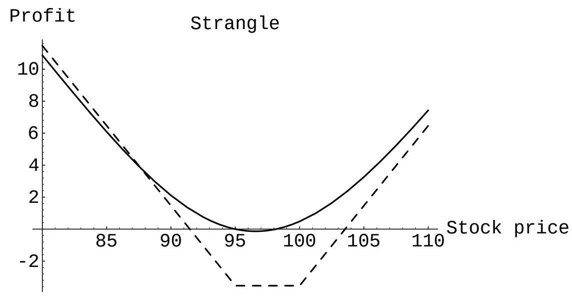


Figure 7.12: PROFIT OF STRANGLE. Plotted is the profit diagram of a strangle at expiration (dashed line) and at any time before expiration when it is set up (solid line). Here, the strike prices are \$95 (for the put) and \$100 (for the call), the current stock price is \$95, and there is one month to expiration.

# Chapter 8

# Arbitrage in Option Pricing

## Contents



*All general laws are attended with inconveniences,*
*when applied to particular cases.*
—David Hume,
"Of the Rise and Progress of the Arts and Sciences" [437, p. 76]

The principle of arbitrage says roughly that there should not be free lunch. Simple as it is, this principle supplies the essential argument for option pricing. After presenting the argument, several important option pricing relationships are established in this rather technical chapter.

## 8.1 The Arbitrage Argument

The **principle of arbitrage** is the basic tool in the valuation of options. Riskless arbitrage opportunity is a situation that enables one to earn positive returns without any initial investment. In an efficient market, such opportunities should not exist. This principle is behind many modern theories of option pricing if not a concept that unifies all of finance [80, 263, 757]. The related **portfolio dominance** principle says portfolio A should be priced higher than portfolio B if its payoff is at least as good under all circumstances and better under some circumstances [572]. This principle can be traced to Pascal (1623–1662), philosopher, theologian, founder of probability and decision theories [363]. In the 1950s, Miller and Modigliani made it a pillar of financial theory [55].

A portfolio yielding zero returns in every possible situation must have zero present value. This is true because any value other than zero implies net gains are obtainable without risk, by shorting it if its present value is positive and buying it if negative. This is a simple application of the arbitrage principle. Let's follow up with a few more examples.

**Example 8.1.1** An American option cannot be worth less than its intrinsic value. For, otherwise, it can be bought, exercised, and the stock sold with a net profit. □

**Example 8.1.2** A put or a call must have a non-negative value for, otherwise, one can buy it for a positive cash inflow now and end up with a non-negative amount at expiration. □

**Example 8.1.3** In a world where only instantaneous parallel shifts in the spot rate curve are permissible, one can earn arbitrage profits by buying two zero-coupon bonds and shorting a third with the same present value and with a maturity equal to the Macaulay duration of the long position by immunization considerations (see §5.10.2) [451]. Consequently, investors would own only bonds of the shortest and longest maturities, a conclusion inconsistent with the reality. □

**Example 8.1.4** The present value of a riskless ordinary annuity of $100 per year for five years is

$$\sum_{i=1}^{5} 100 \times d(i), \tag{8.1}$$

where $d(i)$ is, we recall, the price of a riskless zero-coupon bond with $1 par value and maturing exactly $i$ years from now. Arbitrage considerations guarantee that, if the price were anything else, riskless gains would be generated by trading in the marketplace. The price is thus unique. In contrast, the method used in Example 3.2.1 led to a multitude of prices with different interest rate assumptions. The arbitrage approach also does *not* depend on the interest rates being non-stochastic [156]. The same arbitrage argument supports (5.2): $P = \sum_{i=1}^{n} C_i d(i)$. □

## 8.2 Relative Option Prices

We shall derive in this and the following sections arbitrage relationships that option values must satisfy in equilibrium under non-negative interest rates. All are independent of assumptions regarding the probabilistic behavior of stock prices. We do assume, among others, that there are no transactions costs or margin requirements, and borrowing and lending are available at the riskless interest rate. Furthermore, people are prepared to take advantage of any arbitrage opportunities so quickly that there are essentially no arbitrage opportunities.

Recall that $\text{PV}(x)$ stands for the present value of $x$ dollars at expiration. Hence, $\text{PV}(x) = x d(\tau)$ if $\tau$ represents the time from now when $x$ dollars are available. To simplify the presentation, assume the current time is time zero.

The following lemma says option value rises with time to expiration. The intuition is that there are more opportunities for profitable exercise if the time to expiration is longer.

**Lemma 8.2.1** *An American call (put) option with a longer time to expiration cannot be worth less than an otherwise identical call (put) with a shorter time to expiration.*

PROOF: We prove the lemma for the call only. Suppose $C_{t_1} > C_{t_2}$ instead, where $t_1 < t_2$. We can buy $C_{t_2}$ and sell $C_{t_1}$ to generate a net cash inflow of $C_{t_1} - C_{t_2}$ at time zero. Up to the moment when the time to $t_2$ is $\tau$ and the short call option either expires or is exercised, the position is worth $C_\tau - \max(S_\tau - X, 0)$. If this value is positive, close out the position with a profit by selling the remaining call. Otherwise, $\max(S_\tau - X, 0) > C_\tau > 0$, and the short call is exercised. In this case, simply exercise the remaining call and have a net cash flow of zero. In both cases, the total payoff is positive without initial investment, signifying a riskless arbitrage opportunity. □

The quotations in Fig. 7.4 can be easily checked against the above proposition. We remark that the above proposition may not hold for European options.

**Lemma 8.2.2** *A call (put) option with a higher (lower) strike price cannot be worth more than an otherwise identical call (put) with a lower (higher) strike price.*

PROOF: We prove the lemma for the call only. This proposition certainly holds at expiration; hence, it is valid for European calls. Let the two strike prices be $X_1 < X_2$. If $C_{X_1} < C_{X_2}$, then we buy the low-priced $C_{X_1}$ and write the high-priced $C_{X_2}$, generating a positive return. If the holder of $C_{X_2}$ exercises it before expiration, just exercise the long call to generate a positive cash inflow of $X_2 - X_1$. □

**Lemma 8.2.3** *The difference in the values of two otherwise identical options cannot be greater than the difference in their strike prices.*

PROOF: We shall consider the call only. Let the two strike prices be $X_1 < X_2$. Assume, instead, that $C_{X_1} - C_{X_2} > X_2 - X_1$. The strategy is to buy the lower-priced $C_{X_2}$ and write the higher-priced $C_{X_1}$, generating a positive return. Deposit $X_2 - X_1$ in a riskless bank account.

Suppose the holder of $C_{X_1}$ exercises the option before expiration. There are two scenarios. If $C_{X_2} > S - X_1$, then sell $C_{X_2}$ to realize a cash flow of $C_{X_2} - (S - X_1) > 0$. Otherwise, exercise $C_{X_2}$ and realize a cash flow of $X_1 - X_2 < 0$. In both scenarios, close out the position with the money in the bank and have non-negative total payoff.

Now, consider the case in which the holder of $C_{X_1}$ does not exercise the option early. At the expiration date, our payoff is $0$, $X_1 - S < 0$, and $X_1 - X_2 < 0$, respectively, if $S \leq X_1$, $X_1 < S < X_2$, and $X_2 \leq S$. The total payoff remains non-negative after adding the money in the bank account, which is at least $X_2 - X_1$. □

## 8.3 Put-Call Parity and Consequences

Assume either the stock pays no dividends or the options are protected so that the option value is insensitive to cash dividends. Note that any analysis for options on non-dividend paying stocks holds for protected options on dividend-paying stocks by definition. Hence, theorems for protected options will not be listed separately.

Consider the portfolio of one short European call, one long European put, one share of stock, and a loan of $\mathrm{PV}(X)$. All options are assumed to carry the same strike price and time to expiration, $\tau$. The initial cash flow is therefore $C - P - S + \mathrm{PV}(X)$. At expiration, if the stock price $S_\tau$ is at most $X$, the put will be worth $X - S_\tau$ and the call will expire worthless. Exercise the put to receive $X$ and repay the debt. On the other hand, if $S_\tau > X$, then the call will be worth $S_\tau - X$, and the put will expire worthless. So $X$ will be received when the call is exercised, and there will be enough proceeds to repay the debt. The net future cash flow is hence zero in either case. The arbitrage principle implies that the initial investment to set up the portfolio must have zero value as well. We therefore have demonstrated the following **put-call parity** for the European options, which seems to be due to Castelli in 1877 and rediscovered many times hence [137].

**Theorem 8.3.1** *For European options on stocks that pay no dividends,*

$$C = P + S - \mathrm{PV}(X) \tag{8.2}$$

*holds.* □

The put-call parity shows that there is essentially only one kind of European option because the other can be replicated from it in combination with the underlying stock, and riskless lending or borrowing.

Rearranging (8.2) as

$$S = C - P + \mathrm{PV}(X),$$

we see that a long position in stock is equivalent to a portfolio containing a long call, a short put, and lending $\mathrm{PV}(X)$. Combinations such as these create **synthetic securities**. Other ways of rearranging the put-call relationship are also possible. Consider

$$C - P = S - \mathrm{PV}(X),$$

that is, a long call and a short put amounts to a long position in the underlying stock and borrowing the present value of the strike price—in a word, buying the stock on margin. This might be the preferred way to take a levered long position in the stock, as buying the stock on margin in the stock market is subject to strict margin requirements.

Suppose the present value of the dividends whose ex-dividend dates occur prior to the expiration date is $D$. The put-call-parity relationship can then be generalized to

$$C = P + S - D - \mathrm{PV}(X). \tag{8.3}$$

The put-call parity relationship implies

$$C = (S - X) + (X - \mathrm{PV}(X)) + P \geq S - X.$$

Since $C \geq 0$, it must hold that $C \geq \max(S - X, 0)$. Furthermore, an American option cannot be worth less than its intrinsic value because it can be exercised immediately. Hence, the following lemma follows.

**Lemma 8.3.2** *An American call or a European call on a non-dividend-paying stock is never worth less than its intrinsic value* $\max(S - X, 0)$. □

The put-call parity relationship implies

$$P = (X - S) + (\mathrm{PV}(X) - X + C).$$

It is therefore *not* true that a European put must sell for more than its intrinsic value. In Fig. 7.2, for example, the put value becomes less than its intrinsic value when the option is deep in the money. This can be explained as follows. As the put goes deeper in the money, the call option value drops to zero. Hence, $P$ becomes approximately $(X - S) + \mathrm{PV}(X) - X < X - S$, its intrinsic value, if the interest rate is positive. A less technical explanation is that a deep-in-the-money European put could have been earning interest if it were exercised immediately. Being priced at less than its intrinsic value therefore represents an opportunity loss.

## 8.4 Early Exercise Feature of American Calls

Assume interest rates are positive for this subsection. It turns out that it never pays to exercise an American call before expiration if the underlying stock does not pay dividends; in other words, selling an American call is preferred to exercising it. The argument goes like this. From Exercise 8.3.3, we know $C \geq \max(S - \mathrm{PV}(X), 0)$. If the call is exercised, the value is only $S - X$. The disparity comes from two sources: (1) the loss of the insurance against subsequent stock price declines once the call is exercised, and (2) the time value of money because $X$ is paid upon exercise. This somewhat surprising result is due to Merton.

**Theorem 8.4.1** *An American call on non-dividend-paying stock should not be exercised before expiration.* □

As a consequence, every pricing relationship for European calls holds for American calls as well when the underlying stock pays no dividends. Note that the above theorem should not be interpreted as saying that holders of such American calls should keep them until expiration. For instance, a call option that is deep in the money today might turn out to be out of the money at expiration, hence worthless. In this case, keeping the option might be less profitable than exercising it earlier (with the immediate sale of the stock). What the theorem really means is that, at any time when early exercise is being considered, there is always a *better*, not just equally good, alternative: Sell it.

The put-call parity only holds for European options. For American ones on a non-dividend-paying stock, we have

$$P \geq C + \mathrm{PV}(X) - S. \tag{8.4}$$

This is because (1) the American call has the same value as the European call by Theorem 8.4.1, and (2) the American put is at least as valuable as its European counterpart.

Early exercise may become optimal for American calls on dividend-paying stocks. The reason has to do with the fact that the holder of an uncovered call not only does not receive any cash dividends, but the stock price also declines once the stock goes ex-dividend. Other things being equal, the longer the time to expiration, the more the stock price will be reduced by cash dividends. Beyond a certain point, this effect will dominate, and it pays to exercise the option early. Take the extreme example of a firm that plans to pay out all its assets as cash dividends. The stock, and the option too, certainly has no value after the ex-dividend date, and in-the-money calls should therefore be exercised.

If the underlying stock pays dividends, an American call should be exercised only at expiration or *just before an ex-dividend date*. To prove this, we first argue that $C \geq S - X$ must hold at any time other than the expiration date or just before an ex-dividend date. Assume otherwise: $C < S - X$. Now, buy the call, short the stock, and lend $X d(\tau)$, where $\tau$ is the next dividend date. The initial cash inflow is positive because $X > X d(\tau)$. We subsequently close out the position just before the next ex-dividend date by calling the loan, worth $X$, and selling the call, worth at least $\max(S_\tau - X, 0)$ by Lemma 8.3.2. The proceeds are sufficient to buy the stock at $S_\tau$; thus the initial cash flow represents an arbitrage profit. Now that the value of a call exceeds its intrinsic value between ex-dividend dates, selling is better than exercising.

**Theorem 8.4.2** *An American call can only be exercised profitably at expiration or just before an ex-dividend date. Hence, the only time to consider early exercise for American calls is just before an ex-dividend date.* □

Consider a scenario in which, at all times before expiration, the present value of the dividends to be paid until expiration is exceeded by that of the interest that can be earned on the strike price, i.e., $X - \text{PV}(X) > D$. Take a date just before an ex-dividend date. If a call holder exercises the option, the holdings just after an ex-dividend date will be worth $S - X + D'$ with $D'$ denoting the dividend due for holding the stock through the ex-dividend date. Note that $D' \leq D$. If the holder chooses not to exercise the call, on the other hand, the holdings will then be worth, by definition, $C$ after the dividend date. From (8.3), we conclude that

$$C \geq S - \text{PV}(X) - (D - D') > S - X + D'.$$

Hence, it is better to sell the call than to exercise it just before an ex-dividend date. Combining this conclusion with Theorem 8.4.2, we have the following proposition.

**Lemma 8.4.3** *If, at any time before expiration, the present value of the interest from the strike price exceeds that of all the future dividends, the call should not be exercised before expiration.* □

## 8.5 Early Exercise Feature of American Puts

Unlike American calls on non-dividend-paying stocks, it might be optimal to exercise an American put even if the underlying stock does not pay dividends. Part of the reason for such a diametrically opposing conclusion lies in the fact that the time value of money now works *for* early exercise: Exercising a put generates an immediate cash income equal to the strike price. One consequence is that early exercise becomes more profitable as the interest rate increases, other things being equal.

The existence of dividends tends to offset the benefits of early exercise in the case of American puts. Consider a stock that is currently worthless, $S = 0$. If the holder of a put exercises the option, $X$ is tendered. If the holder sells the option, he receives $P \leq X$ (see Lemma 8.6.1) and keeps the stock. Doing nothing generates no income. If the stock will remain worthless till expiration, exercising now is one of the optimal strategies. We conclude that it is no longer true that we only consider a few points for the early exercise of the put. Contrast this with Theorem 8.4.2. Consequently, concrete results regarding early exercise of American puts are also scarcer and weaker.

## 8.6 Miscellaneous Bounds

This section contains more price bounds. Unless otherwise stated, all the bounds hold even if the underlying stock pays dividends.

**Lemma 8.6.1** *A call option is never worth more than the stock price. An American put is never worth more than the strike price, and a European put is never worth more than the present value of the strike price.*

PROOF: If the call value exceeded the stock price, a covered call position could earn arbitrage profits. If the put value exceeded the strike price, writing a cash secured put would earn arbitrage profits. The tighter bound holds for European puts because the cash can earn riskless interest until expiration (this conclusion also follows from the put-call parity). □

The put-call parity can be used to prove the following inequality for put options.

**Lemma 8.6.2** *The value of a European put satisfies* $P \geq \max(\mathrm{PV}(X) - S, 0)$. □

A tighter bound, $P \geq \max(X - S, 0)$, holds for the American put since its value must be at least the intrinsic value.

**Lemma 8.6.3** *American options on non-dividend-paying stocks satisfy* $C - P \geq S - X$.

PROOF: Assume otherwise, $C - P < S - X$. Write the put, buy the call, sell the stock short (hence the need for the no-dividend assumption), and place $X$ in a bank account. This generates a positive cash inflow. If the short put is exercised prior to expiration, withdraw the money from the bank account to pay for the stock, which is then used to close out the short sale. □

The above lemma and (8.4) can be combined to imply that American options on non-dividend-paying stocks satisfy

$$C - S + X \geq P \geq C - S + \mathrm{PV}(X).$$

## 8.7 Convexity of Option Prices

The convexity of option prices is stated and proved in the following lemma.

**Lemma 8.7.1** *Take three otherwise identical call options with strike prices $X_1 < X_2 < X_3$. Then*

$$\begin{aligned} C_{X_2} &\leq wC_{X_1} + (1-w)\,C_{X_3} \\ P_{X_2} &\leq wP_{X_1} + (1-w)\,P_{X_3} \end{aligned}$$

*Here, $w \equiv (X_3 - X_2)/(X_3 - X_1)$. (Equivalently, $X_2 = wX_1 + (1-w)\,X_3$.)*

PROOF: We prove the lemma for the call only. Suppose the lemma were wrong. Then, write $C_{X_2}$ and buy $wC_{X_1}$ and $(1-w)\,C_{X_3}$, generating a positive cash inflow. If the short call is not exercised before expiration, then simply hold the calls until expiration and have the payoff as shown in the following table.

| | $S \leq X_1$ | $X_1 < S \leq X_2$ | $X_2 < S < X_3$ | $X_3 \leq S$ |
|---|---|---|---|---|
| Call written at $X_2$ | 0 | 0 | $X_2 - S$ | $X_2 - S$ |
| $w$ calls bought at $X_1$ | 0 | $w\,(S - X_1)$ | $w\,(S - X_1)$ | $w\,(S - X_1)$ |
| $1 - w$ calls bought at $X_3$ | 0 | 0 | 0 | $(1-w)\,(S - X_3)$ |
| Total payoff | 0 | $w\,(S - X_1)$ | $w\,(S - X_1) + (X_2 - S)$ | 0 |

Each total payoff in the above table is at least zero with the help of $X_2 = wX_1 + (1-w)\,X_3$.

On the other hand, suppose the short call is exercised early when the stock price is $S$. If $w\,C_{X_1} + (1-w)\,C_{X_3} > S - X_2$, sell the long calls to generate a cash flow of $wC_{X_1} + (1-w)\,C_{X_3} - (S - X_2) > 0$. Otherwise, exercise the long calls and deliver the stock. The net cash flow is $-wX_1 - (1-w)\,X_3 + X_2 = 0$. □

The above lemma says the butterfly spread requires a positive initial investment. Market makers can use butterfly spreads to quote bids and offers on options at strike prices for which there is no trading in either calls or puts. In contrast, the put-call parity is applicable only when there *is* trading in either calls or puts. Lemma 8.2.3 says $(C_{X_1} - C_{X_2})/(X_2 - X_1) \leq 1$, which means the slope of the call or put value, when plotted against the strike price, is at most one. The above lemma further says that the general shape is convex.

**Example 8.7.2** The prices of the Merck July 30 call, July 35 call, and July 40 call are \$15.25, \$9.5, and \$5.5 from Fig. 7.4. These prices satisfy the convexity property since $9.5 \times 2 < 15.25 + 5.5$. Look up the prices of the Microsoft April 60 put, April 65 put, and April 70 put. The prices are \$0.125, \$0.375, and \$1.5, respectively, again satisfying the convexity property. □

## 8.8 The Option Portfolio Property

The popular stock index options are fundamentally options on a portfolio. Option on the S&P 100 Index is currently the most actively traded option contract in the United States and is settled in cash not stocks [132, 650, 768]. This option is American. Option on the S&P 500 Index is also available. It is European. The Standard & Poor's 500 Composite Stock Price Index historically represents about 70% of the value of all U.S. common stocks. Options on the most followed Dow Jones Industrial Average, DJIA (ticker symbol DJX), were introduced in 1997. (DJX is actually DJIA divided by 100.) Other popular stock market indexes include the Russell 2000 Index for small company stocks and the broadest based Wilshire 5000 Index. Figure 8.1 tabulates some indexes as of January 7, 1999.

| | High | Low | Close | Net Chg. | From Dec. 31 | %Chg. |
|---|---|---|---|---|---|---|
| DJ Indus (DJX) .... | 95.42 | 94.26 | 95.38 | −0.07 | +3.57 | +3.9 |
| S&P 100 (OEX) .... | 633.03 | 624.91 | 632.76 | +1.76 | +28.73 | +4.8 |
| S&P 500 -A.M.(SPX) | 1272.34 | 1257.64 | 1269.73 | −2.61 | +40.50 | +3.3 |
| Russell 2000 (RUT) . | 428.15 | 423.88 | 427.83 | +0.04 | +5.87 | +1.4 |
| Major Mkt (XMI) ... | 1022.89 | 1008.53 | 1017.83 | −5.40 | +27.68 | +2.8 |
| NYSE (NYA) ....... | 611.01 | 604.47 | 609.19 | −1.82 | +13.38 | +2.3 |
| Value Line (VLE) ... | 951.64 | 942.92 | 948.11 | −3.53 | +20.27 | +2.2 |

Figure 8.1: SAMPLE STOCK INDEX QUOTATIONS. Source: *The Wall Street Journal*, January 8, 1999.

The result to follow demonstrates that an option on a portfolio of stocks is cheaper than a portfolio of options. Hence, it is cheaper to hedge against market movements as a whole with index options than with options on individual stocks. This theorem is due to Merton.

**Theorem 8.8.1** *Consider a portfolio of assets with prices $S_i$ and weights $w_i$. Let $X_i$ denote the strike price of a call, worth $C_i$, on asset $i$. The index call on the portfolio with a strike price $X \equiv \sum_i w_i X_i$ has a value at most $\sum_i w_i C_i$. The same result holds for puts as well. All options refer to either European options or American calls on non-dividend-paying stocks and expire on the same date.*

PROOF: First, assume $S_i \leq X_i$ for all $i$ at expiration. The portfolio of calls will be worth zero because all the calls end up being out of the money. On the other hand, $\sum_i w_i S_i \leq \sum_i w_i X_i = X$, and the index call is also out of the money, hence worthless.

Next, consider $S_i > X_i$ for all $i$ at expiration. The portfolio of calls will be worth $\sum_i w_i C_i = \sum_i w_i (S_i - X_i)$ because all the calls end up being in the money. On the other hand, $\sum_i w_i S_i > \sum_i w_i X_i = X$, and the index call is also in the money with the same value $\sum_i w_i (S_i - X_i)$.

Finally, without loss of generality, suppose $S_i > X_i$ for $i \leq k$ and $S_i \leq X_i$ for $i > k$. The portfolio of calls will be worth $\sum_{i=1}^k w_i (S_i - X_i)$. On the other hand, the index call is worth

$$\max\left(\sum_{i=1}^n w_i (S_i - X_i), 0\right) \leq \sum_{i=1}^k w_i (S_i - X_i).$$

Since American calls on non-dividend-paying stocks can be treated as European calls, the above theorem holds for such American calls. □

It is clear from the the proof of the above theorem that a portfolio of options and an option on a portfolio have the same payoff if all the underlying securities finish in the money or out of the money. Their payoffs diverge only when the underlying securities are not perfectly correlated with each other [496]; the degree of the divergence tends to increase the more the underlying securities are uncorrelated. Note that the validity of the theorem rests not upon the diversification of risk so familiar to modern portfolio theory, but rather on the principle of hedging.

A call on a portfolio will be exercised only if the portfolio rises in value. Similarly, a put on a portfolio will be exercised only if the portfolio falls in value. Hence, an option on a portfolio can be used much like ordinary options for protective purposes. In contrast, a portfolio of calls might lose value even if the underlying portfolio does not change in value. Similarly, a portfolio of puts might gain value even if the underlying portfolio does not change in value. Hence, a portfolio of, say, puts does not provide the same service as a protective put in the single stock case.

Institutions such as pension funds and mutual funds hold huge portfolios with returns highly correlated with the overall market. Some may like to generate additional income by writing calls against their portfolios, or they may like to protect themselves against a decline in value by purchasing puts against their portfolios. Writing calls or buying puts against many of the securities in their portfolios may be too expensive and may not closely match their goal. The index option alternative may provide more adequate protection. Good justification for the existence of index options and proposals can be found in [205, §8.3]. Consult [277, 302] for more information.

## 8.9 Concluding Remarks

Bounds derived above were model-free and should be satisfied by any model claiming to be valid. Observe that they were all *relative* price bounds. In the next chapter, we will consider models for stock prices in order to obtain option values.

# Chapter 9

# Option Pricing Models

## Contents



*Life can only be understood backwards;*
*but it must be lived forwards.*
—Søren Kierkegaard (1813–1855)

Although it is rather easy to price an option at expiration, pricing it at a prior moment is anything but. The principle of arbitrage, albeit valuable in deriving various bounds, is insufficient to pin down the exact option value. Further assumptions about the probabilistic behavior of stock prices are needed for that purpose.

It is the major task of this chapter to develop option pricing formulae under reasonable models for stock prices. The discrete-time binomial option pricing model in particular will be the focus of this chapter. The coverage of the continuous-time Black-Scholes option pricing model, to which the binomial model converges in the limit, will be postponed until Chapter 15.

## 9.1 Introduction

Option pricing theory can be traced to Louis Bachelier's Ph.D. thesis submitted in 1900, *Mathematical Theory of Speculation*. This French mathematician (1870–1946) developed much of the mathematics underlying modern economic theories on efficient markets, random walk price models, Brownian motion (ahead of Einstein by five years), and martingales [238, 250, 298, 570, 677]. He remained obscure until about 1960 when his major work was translated into English. His career problem seemed to stem from, on the one hand, some technical errors and, on the other hand, the topic of his dissertation[1] [549]. This is not the first time for ideas in economics to influence other sciences [382, 572], the most celebrated of them being Malthus's simultaneous influence on Darwin and Wallace in 1838 [215, 560].

The major obstacle toward an explicit option pricing model is that it seems to depend on the probability distribution for the price of the underlying asset and the risk-adjusted interest rate to discount the option's expected payoff. Neither factor can be observed directly. After many attempts, some of which came very close, the breakthrough arrived in 1973 when Fischer Black (1938–1995) and Myron Scholes with help from Merton published[2] their celebrated option pricing model now universally known as the **Black-Scholes option pricing model** [80]. One of the crown jewels of finance theory, this research has far-reaching implications in practice. It also contributed greatly to the success of the Chicago Board Options Exchange [572]. The 1997 Nobel Prize in Economic Sciences was awarded to Merton and Scholes for their work on "the valuation of stock options."

The mathematics of the Black-Scholes model is formidable because the model allows the price to move to any one of an infinite number of prices in any finite period of time. The alternative **binomial option pricing model** (**BOPM**) limits the price movement to *two* choices in a period, simplifying the mathematics tremendously at the expense of realism. All is not lost, however, since the binomial model converges to the Black-Scholes model as the period length goes to zero. The binomial model also suggests efficient numerical algorithms to price options.

Although other researchers also came up with similar ideas, the binomial option pricing

[1] "The topic is somewhat remote from those our candidates are in the habit of treating," wrote his advisor, Poincaré (1854–1912) [238].

[2] Their paper, "The Pricing of Options and Corporate Liabilities," was sent in 1970 to *Journal of Political Economy* and was rejected immediately by the editors [55, 57].

model is generally attributed to Sharpe in 1975 [669] and appeared in his popular textbook, *Investments* (1978) [699]. We will follow the ideas independently put forth by Cox, Ross, and Rubinstein [204] and Rendleman and Bartter [643].

Throughout this chapter, $C$ denotes the call option price, $P$ the put option price, $X$ the strike price, $S$ the stock price, and $D$ the dividend amount. Subscripts are used to emphasize times to expiration, stock prices, or strike prices. The symbol $\mathrm{PV}(x)$ stands for the present value of $x$ at expiration unless stated otherwise. Positive interest rates are assumed.

## 9.2 The Binomial Option Pricing Model (BOPM)

In this model, time is discrete, measured in periods. The central idea of the Black-Scholes analysis says five pieces of information are sufficient to determine the value of an option lasting for a single period based on arbitrage considerations. The five pieces of information are the current stock price, the two possible prices in the next period, the option's strike price, and the riskless interest rate. The way to prove it is truly ingenious: Replicate the option by a portfolio of stocks and riskless bonds. What may seem surprising is that we need to know neither the probability that the stock price will rise or fall in the next period nor the expected growth rate of the stock price.

Let $r > 0$ denote the constant, continuously compounded riskless interest rate per period and $R$ the gross return (so $R \equiv e^r$). Note that $R \equiv 1 + r$ if compounding is periodic. Denote the **binomial distribution** with parameters $n$ and $p$ by

$$b(j; n, p) \equiv \binom{n}{j} p^j (1-p)^{n-j} = \frac{n!}{j!\,(n-j)!}\, p^j (1-p)^{n-j}.$$

Recall that $n! = n\,(n-1)\cdots 2 \cdot 1$. The convention is $0! = 1$. Hence, $b(j; n, p)$ is the probability of getting $j$ heads when tossing a coin $n$ times with $p$ being the probability of getting a head. The **complementary binomial distribution function** with parameters $n$ and $p$ is defined as

$$\Phi(k; n, p) \equiv \sum_{j=k}^{n} b(j; n, p).$$

$\Phi(k; n, p)$ denotes the probability of getting at least $k$ heads when tossing a coin $n$ times. It is not difficult to see that

$$1 - \Phi(k; n, p) = \Phi(n-k+1; n, 1-p). \tag{9.1}$$

Under the binomial option pricing model, if the current stock price is $S$, it can go to $Su$ with probability $q$ and $Sd$ with probability $1-q$, where $1 > q > 0$ and $u > d$. See the illustration in Fig. 9.1. In fact, $u > R > d$ must hold to rule out arbitrage profits.

### 9.2.1 Calls and puts on non-dividend-paying stocks: single period

As the first step, assume the expiration date is one period from now. Let $C$ be the current call price, $C_u$ be the price one period from now if the stock price moves to $Su$, and $C_d$ be

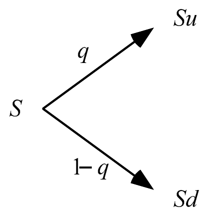


Figure 9.1: BINOMIAL MODEL (PROCESS) FOR STOCK PRICES.

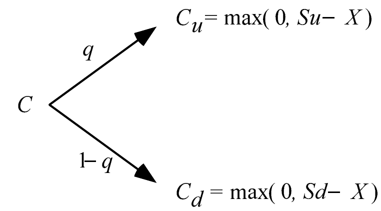


Figure 9.2: VALUE OF ONE-PERIOD CALL IN BINOMIAL OPTION PRICING MODEL.

the price one period from now if the stock price moves to $Sd$. Clearly,

$$C_u = \max(0, Su - X) \quad \text{and} \quad C_d = \max(0, Sd - X).$$

See Fig. 9.2 for illustration.

Now, set up a portfolio of $h$ shares of stock and $\$B$ in riskless bonds. This costs $hS+B$. The value of this portfolio in a period is depicted in Fig. 9.3. Now, take the key step in choosing $h$ and $B$ such that the portfolio has the same payoff as the call,

$$hSu + RB = C_u \quad \text{and} \quad hSd + RB = C_d.$$

Solve the above equations to get

$$h = \frac{C_u - C_d}{(u-d)\,S} \geq 0 \tag{9.2}$$

$$B = \frac{uC_d - dC_u}{(u-d)\,R} \tag{9.3}$$

Hence, an **equivalent portfolio** that replicates the call's payoff has been created. This portfolio can be said to be a **synthetic call option**. Other terms for equivalent portfolio include **replicating portfolio** and **hedging portfolio**.

Note that $q$ is not involved at all, and it is not necessary to specify the underlying asset's expected (gross) return, $qSu + (1-q)\,Sd$. Instead, we employ the equivalent portfolio to price the option relative to the price of the underlying asset. The arbitrage argument only assumes that more deterministic wealth is preferred to less. The expected return therefore has only indirect influence on the option value, by way of $S$, $u$, and $d$.

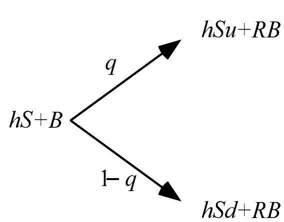


Figure 9.3: VALUE OF EQUIVALENT PORTFOLIO IN ONE PERIOD.

By the arbitrage principle, the equivalent portfolio should cost the same as the call if the call is not exercised immediately. Since

$$uC_d - dC_u = \max(0, Sud - Xu) - \max(0, Sud - Xd) < 0,$$

the portfolio is a levered long position in stocks. We call $h$ the **hedge ratio** or **delta**.

After substitution and rearrangement, we find that

$$hS + B = \frac{\left(\frac{R-d}{u-d}\right) C_u + \left(\frac{u-R}{u-d}\right) C_d}{R}. \tag{9.4}$$

So $hS + B \geq 0$. The above equation can be rewritten as

$$hS + B = \frac{pC_u + (1-p)\, C_d}{R}, \tag{9.5}$$

where

$$p \equiv \frac{R-d}{u-d}, \quad \text{and} \quad 1 - p = \frac{u-R}{u-d}. \tag{9.6}$$

We have thus replicated the call option as a levered long position in stocks—with one exception. A call, if it is American, can be exercised immediately. In contrast, the equivalent portfolio mirrors the call's payoff *if* the option is not exercised now. If $hS + B \geq S - X$, then the call will not be exercised immediately; thus $C = hS + B$. On the other hand, if $hS + B < S - X$, then the option should be exercised immediately, for we can take the proceeds $S - X$ to buy the equivalent portfolio plus some more bonds. Hence, the call option is worth $S - X$. We conclude that

$$C = \max(hS + B, S - X). \tag{9.7}$$

In the case of European options, early exercise is not possible; hence $C = hS + B$. In the case of American calls on stocks that do not pay dividends, Theorem 8.4.1 already proves that early exercise is not optimal; hence $C = hS + B$ holds as well (see also Exercise 9.2.3). As a result, (9.7) is simplified to

$$C = hS + B \tag{9.8}$$

for both European and American calls on stocks that pay no dividends.

### 9.2.2 Risk-neutral valuation

The call value $C$ is independent of $q$, the probability of an upward movement in price, hence the expected return of the stock as well. The option therefore does not directly depend on investors' **risk preferences**, and it will be priced the same regardless of how risk-averse an investor is. The option value is determined uniquely by $S$, $u$, $d$, $X$, $r$, and, as we shall see later, the number of periods to expiration, among which only $S$ is stochastic. In particular, the option value depends on the size of price changes, $u$ and $d$, and the investors must agree upon their magnitudes.

As $0 < p < 1$, it may be interpreted as a probability. Under the binomial model, the expected stock price in the next period is $qSu + (1-q)\, Sd$. The expected return for the stock is equal to the riskless interest rate, or $qSu + (1-q)\, Sd = RS$, when $q = p$. In other words, if the probability of an upward price movement is $p$, the equivalent portfolio earns the riskless interest rate. This insight turns out to be critical.

Call an investor **risk-neutral** if that person is indifferent between a certain rate of return and an uncertain rate of return with the same expected value. Risk-neutral investors care only about expected returns. The expected rate of return of all securities must be the riskless interest rate $r$ when investors are risk-neutral. For this reason, $p$ is sometimes called the **risk-neutral probability** or **risk-neutralized probability**. Since risk preferences and $q$ are not directly involved in pricing options, *any* risk attitude, including risk neutrality, should give the same result. In a **risk-neutral economy**, all securities earn the riskless rate of interest. We can therefore interpret the value of a call as the expectation of its discounted future payoff in a risk-neutral economy. So it turns out that the rate used for discounting the expected future value is the riskless interest rate without any risk adjustment in a risk-neutral economy. Risk-neutral valuation is perhaps the most important tool for the analysis of derivative securities.

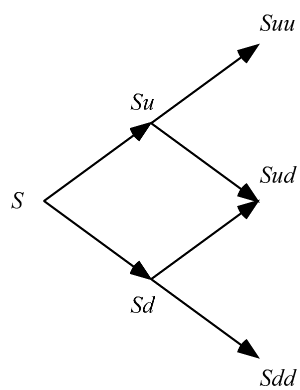


Figure 9.4: STOCK PRICES IN TWO PERIODS.

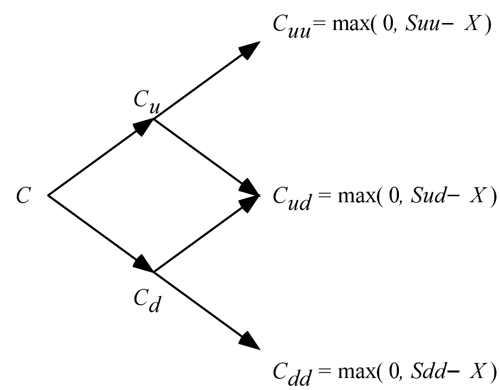


Figure 9.5: VALUE OF A TWO-PERIOD CALL PRIOR TO EXPIRATION. This graph is called a **binomial tree** although **binomial lattice** or **closed lattice** is a better term since real tree branches do not merge.

### 9.2.3 Calls and puts on non-dividend-paying stocks: multi-period

We now proceed to consider a call with two periods remaining before expiration. We shall move *backward* in time in order to derive the call value. For this reason, this procedure is called **backward induction** [21, 58]. Under the binomial model, the stock can take on three possible prices after two periods, $Suu$, $Sud$, and $Sdd$ (see Fig. 9.4). Note that at any moment in time, the next two stock prices only depend on the current price, not the prices of earlier times. This memoryless property is typically taken for granted and is the key feature of an efficient market,[3] an original idea due to Bachelier. In the terminology of probability, we may say the stock price is taking a **random walk** [298].

[3]Specifically, the **weak form of efficient markets hypothesis**, which says current prices fully embody all information contained in historical prices [277]. This form of market efficiency implies that technical analysts cannot make above-average returns by reading charts of historical stock prices, and its validity has been amply documented [547].

Let $C_{uu}$ be the call's value two periods from now if the stock price moves to $Suu$,

$$C_{uu} = \max(0, Suu - X).$$

$C_{ud}$ and $C_{dd}$ can be defined analogously, as

$$C_{ud} = \max(0, Sud - X) \text{ and } C_{dd} = \max(0, Sdd - X).$$

See Fig. 9.5 for illustration. Applying the same logic as leads to (9.8), we obtain the call values at the end of the current period as

$$C_u = \frac{pC_{uu} + (1-p)\, C_{ud}}{R} \text{ and } C_d = \frac{pC_{ud} + (1-p)\, C_{dd}}{R}.$$

Denote the deltas as $h_u$ and $h_d$ if the current stock price is $S_u$ and $S_d$, respectively. Deltas can be derived from (9.2)–(9.3).

We now reach the current period. An equivalent portfolio of $h$ shares of stock and $\$B$ in riskless bonds can be set up for the call that costs $C_u$ ($C_d$) if the stock price goes to $Su$ ($Sd$). The values of $h$ and $B$ can again be derived from (9.2)–(9.3). Since this delta $h$ may not be the same as the deltas in the following period ($h_u$ and $h_d$), the maintenance of an equivalent portfolio is a dynamic process. By construction, the value of the portfolio at the end of the current period ($C_u$ or $C_d$) is exactly the amount needed to set up the next portfolio; it is the proportion in risky stocks that changes. This trading strategy is **self-financing** as there is neither injection nor withdrawal of funds over the time horizon. In other words, changes in portfolio value are due entirely to capital gains [373]. This dynamic maintaining of an equivalent portfolio does *not* depend on our correctly predicting the stock price movements. We are covered whether the stock price goes up or down, independent of its probability of doing so.

Since the option will not be exercised one period from now by Exercise 9.2.3, we have $C_u > Su - X$ and $C_d > Sd - X$; therefore,

$$hS + B = \frac{pC_u + (1-p)\, C_d}{R} > \frac{(pu + (1-p)\, d)\, S - X}{R} = S - \frac{X}{R} > S - X.$$

Hence, the call will again not be exercised in the current period even if it is American, and

$$C = hS + B = \frac{pC_u + (1-p)\, C_d}{R}. \tag{9.9}$$

It is straightforward to extend the above argument to the cases of more than two periods to expiration. Given a price $S$ and its two successors, $Su$ and $Sd$, all that is needed is the guarantee that $C_u > Su - X$ and $C_d > Sd - X$ to ensure $C > S - X$. This has been established for $C$ at one and two periods to expiration. It is not hard to show inductively that $C > S - X$ for earlier periods as well. Hence, early exercise is never optimal for American calls on non-dividend-paying stocks.

From (9.9) above and the formulae for $C_u$ and $C_d$, we conclude that

$$\begin{aligned} C &= \frac{p^2 C_{uu} + 2p\,(1-p)\, C_{ud} + (1-p)^2 C_{dd}}{R^2} \\ &= \frac{p^2 \times \max\left(0, Su^2 - X\right) + 2p\,(1-p) \times \max\left(0, Sud - X\right) + (1-p)^2 \times \max\left(0, Sd^2 - X\right)}{R^2}. \end{aligned}$$

The above formula can be extended to the general case with $n$ periods to expiration,

$$C = \frac{\displaystyle\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, Su^j d^{n-j} - X\right)}{R^n}, \tag{9.10}$$

which says the value of a call on a non-dividend-paying stock is the expected discounted value of the payoff at expiration in a risk-neutral economy. This is the only option value consistent with no arbitrage opportunities. Option values thus derived are often called **arbitrage values**. A similar argument can be employed to show that the value of a European put is

$$P = \frac{\displaystyle\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max\left(0, X - Su^j d^{n-j}\right)}{R^n}.$$

We summarize the findings below.

**Lemma 9.2.1** *For non-dividend-paying stocks, the value of a call, be it European or American, and the value of a European put equal the respective expected discounted future payoff at expiration in a risk-neutral economy.* □

The above lemma no longer holds once optimal early exercise becomes possible, as in the case of American puts and unprotected American calls on dividend-paying stocks. Still, the thesis that a European-style option or any other derivative can be evaluated as if the economy were risk-neutral remains valid. Mathematically, this means the value of a European-style derivative with the terminal payoff function $\mathcal{D}$ equals

$$e^{-rn} E^{\pi}[\mathcal{D}], \tag{9.11}$$

where $E^{\pi}$ means the expectation is taken over the risk-neutral probability.

Let $a$ be the minimum number of upward price moves in the next $n$ periods for the call to finish in the money at expiration. Since $a$ is the smallest non-negative integer such that $Su^a d^{n-a} \geq X$, we have

$$a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil. \tag{9.12}$$

The call value of (9.10) now becomes

$$\begin{aligned} C &= \frac{\displaystyle\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \left(Su^j d^{n-j} - X\right)}{R^n} && (9.13) \\ &= S \sum_{j=a}^{n} \binom{n}{j} \frac{(pu)^j ((1-p)\, d)^{n-j}}{R^n} - \frac{X}{R^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \\ &= S \sum_{j=a}^{n} b\left(j; n, pue^{-r}\right) - Xe^{-rn} \sum_{j=a}^{n} b(j; n, p) && (9.14) \end{aligned}$$

Note that the option values depend on $S$, $X$, $r$, $u$, $d$, and $n$.

**Comment 9.2.2** If $a > n$, the call will always finish out of the money, and (9.13) gives a value of zero, consistent with our intuition. Now consider the $u \to d$ scenario, signifying zero volatility in stock prices. As $u > R > d$, we must have $R \to d$ as well. Equations (9.12) and (9.14) imply $C = 0$ if $X > Sd^n \to SR^n$ (hence $a \to \infty$). In other words, the call option is worthless if $\text{PV}(X) \equiv XR^{-n} > S$. On the other hand, if $\text{PV}(X) \leq S$ (hence $a \to 0$), then $C = S - \text{PV}(X)$. □

The put-call parity can be used for European puts. These findings are summarized below.

**Theorem 9.2.3** *The value of a call and the value of a European put, both on a non-dividend-paying stock, are*

$$\begin{aligned} C &= S\Phi\left(a; n, pue^{-r}\right) - Xe^{-rn}\Phi(a; n, p) \\ P &= Xe^{-rn}\Phi(n-a+1; n, 1-p) - S\Phi\left(n-a+1; n, 1-pue^{-r}\right), \end{aligned}$$

*respectively, where* $p \equiv (e^r - d)/(u-d)$. □

The option value for the put above was simplified with the help of (9.1). It can also be derived from the same logic as underlies the steps for the call but with $\max(0, S-X)$ replaced by $\max(0, X-S)$ at expiration. It is noteworthy that, with the random variable $\boldsymbol{S}$ denoting the stock price at expiration, the above theorem's formulae can be written as

$$C = S \times \text{Prob}_1[\,\boldsymbol{S} \geq X\,] - XR^{-n} \times \text{Prob}_2[\,\boldsymbol{S} \geq X\,] \tag{9.15}$$

$$P = XR^{-n} \times \text{Prob}_2[\,\boldsymbol{S} \leq X\,] - S \times \text{Prob}_1[\,\boldsymbol{S} \leq X\,] \tag{9.15$'$}$$

where $\text{Prob}_1$ uses $pu/R$ and $\text{Prob}_2$ uses $p$ for the probability that the stock price moves from $S$ to $Su$, respectively.

There are $n+1$ possible states of the world at expiration, corresponding to stock prices $Su^i d^{n-i}$ for $0 \leq i \leq n$. This means $n+1$ state contingent claims are needed to make the market complete if only a single round of trade is allowed. A market is **complete** if every contingent claim is attainable [373]. For example, to replicate a European call requires $n+1$ types of state contingent claims the $i$th of which pays one dollar at expiration if the stock price (state) is $Su^i d^{n-i}$ and zero otherwise. A portfolio of state contingent claims with $Su^i d^{n-i}$ units of claim $i$ for $a \leq i \leq n$ replicates the call. Such a buy-and-hold strategy is clearly self-financing.

However, if trading is allowed for each period (the **continuous trading** case), as is in the replication of options using stocks and bonds above, *two* securities suffice to replicate every possible claim (see Exercise 9.2.9) [250, 389]. Risk-neutral valuation is possible because there exists a self-financing trading strategy that replicates the option.

**Comment 9.2.4** The possibility of risk-neutral valuation is usually taken to define the absence of arbitrage in the model in the sense that no self-financing trading strategies exist that earn arbitrage profits. In fact, this postulate is mathematically correct for discrete-time models. The converse proposition that the absence of arbitrage profits implies the existence of risk-neutral probability can be rigorously proved. This probability measure is unique for complete markets. The equivalence between arbitrage freedom in a model and the existence of a risk-neutral probability is sometimes called the **fundamental theorem of asset pricing** [332, 373, 589]. □

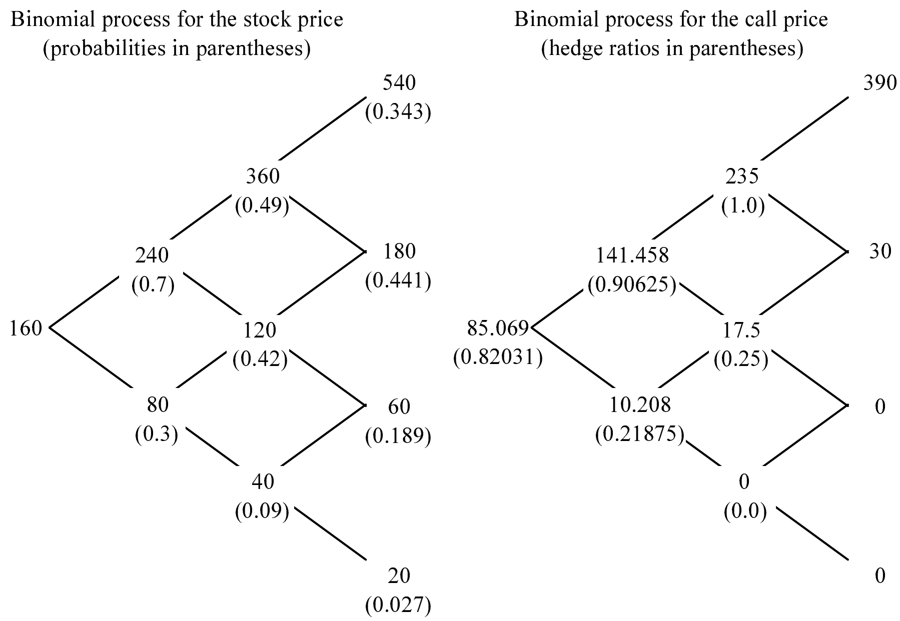


Figure 9.6: STOCK PRICE MOVEMENTS AND THE CORRESPONDING EUROPEAN CALL PRICES. The parameters are: $S = 160$, $X = 150$, $n = 3$, $u = 1.5$, $d = 0.5$, $R = e^{0.18232} = 1.2$, $p = (R-d)/(u-d) = 0.7$, $h = (C_u - C_d)/S(u-d) = (C_u - C_d)/S$, and $C = (pC_u + (1-p)\,C_d)/R = (0.7 \times C_u + 0.3 \times C_d)/1.2$.

### 9.2.4 A numerical example

Take a stock whose current price is \$160 per share and which does not pay dividends. Assume the stock price follows the following random walk: Its price can go from $S$ to either $S \times 1.5$ or $S \times 0.5$. Assume also a riskless bond with a continuously compounded interest rate of 18.232% per period exists. We would like to derive the price of a European call on this stock with a strike price of \$150 and three periods to expiration. The price movements for the stock price and the call value are depicted in Fig. 9.6. The call value is \$85.069. As expected, the delta changes as the stock price fluctuates. Alternatively, the call value is the present value of the expected payoff at expiration,

$$\frac{390 \times 0.343 + 30 \times 0.441}{(1.2)^3} = 85.069,$$

by Lemma 9.2.1.

Any mispricing in the options markets will lead to arbitrage profits. This can be illustrated by an example. Suppose the option is overpriced, selling for \$90 instead. Let us trace the stock prices and see how to respond at each point in time.

**Time 0.** Sell the overpriced call for \$90 and hedge by investing \$85.069, the fair price, in a portfolio with 0.82031 shares of stock as required by the delta. To accomplish this, we borrow $0.82031 \times 160 - 85.069 = 46.1806$. Put the unutilized proceeds, $90 - 85.069 = 4.931$, in a bank. This will be our arbitrage profit.

**Time 1.** Suppose the stock price moves to \$240. The new delta is 0.90625. Buy $0.90625 -$

$0.82031 = 0.08594$ more shares at the cost of $0.08594 \times 240 = 20.6256$ financed by borrowing. Our debt now totals $20.6256 + 46.1806 \times 1.2 = 76.04232$.

**Time 2.** Suppose the stock price plunges to \$120. The new delta is 0.25. Sell $0.90625 - 0.25 = 0.65625$ shares for an income of $0.65625 \times 120 = 78.75$. Use this income to reduce the debt to $76.04232 \times 1.2 - 78.75 = 12.5$.

**Time 3 (the case of rising price).** The stock price moves to \$180. The call we wrote finishes in the money. For a loss of $180 - 150 = 30$, we can close out the position by either buying back the call or buying a share of stock for delivery. Financing this loss with borrowing brings the total debt to $12.5 \times 1.2 + 30 = 45$, which is covered exactly by selling the 0.25 shares of stock for $0.25 \times 180 = 45$. Withdraw the riskless arbitrage profit $4.931 \times (1.2)^3 = 8.521$ from the bank.

**Time 3 (the case of declining price).** The stock price moves to \$60. The call we wrote is worthless. Sell the 0.25 shares of stock for a total of $0.25 \times 60 = 15$ to repay the debt of $12.5 \times 12 = 15$. Withdraw the riskless arbitrage profit $4.931 \times (1.2)^3 = 8.521$ from the bank.

So, we keep the same arbitrage profit however the stock price moves during the process. Note that in dynamically hedging the short call position, the number of calls is not changed; only the number of shares and bonds varies. See [205] for the possible perils of adjusting the number of short calls instead.

### 9.2.5 Numerical algorithms

#### Binomial tree algorithms

The binomial option pricing model leads naturally to a class of algorithms for pricing options called **binomial tree algorithms**. The algorithm presented in Fig. 9.7 prices calls on non-dividend-paying stocks. Since such calls should not be exercised early, the algorithm is fairly straightforward. The underlying idea is illustrated in Fig. 9.8. To adapt the algorithm in Fig. 9.7 to price European puts on non-dividend-paying stocks, simply replace $\max\left(0, Su^{n-i}d^i - X\right)$ in Step 1 with $\max\left(0, X - Su^{n-i}d^i\right)$. The algorithm appears in Fig. 9.9.

The algorithm in Fig. 9.7 (similarly, Fig. 9.9) is easy to analyze. The first loop can be made to take $O(n)$ steps: Start with $u^n$, then iteratively compute $u^i d^{n-i}$ from $u^{i+1}d^{n-i-1}$ by multiplying the latter by $d/u$. The ensuing double loop takes $O(n^2)$ steps. The total running time is therefore quadratic. The memory requirement is also quadratic.

The binomial tree algorithm works in stages. It starts from the last period and gradually works toward the current period. This suggests one can reduce the memory requirement by reusing the space. Specifically, replace $C[\,n+1\,][\,n+1\,]$ in Fig. 9.7 with a one-dimensional array of size $n+1$, $C[\,n+1\,]$. Then replace Step 1 with

$$C[\,i\,] := \max\left(0, Su^{n-i}d^i - X\right);$$

Binomial tree algorithm for pricing calls on non-dividend-paying stocks:

```
input:   S, u, d, X, n, r  (u > e^r > d, r > 0);
real     R, p, C[n+1][n+1];
integer  i, j;
R := e^r;
p := (R - d)/(u - d);
for i = 0 to n
         1. C[n][i] := max(0, S u^(n-i) d^i - X);
for j = n - 1 down to 0
         for i = 0 to j
                 2.1. C[j][i] := (p × C[j+1][i]
                                + (1 - p) × C[j+1][i+1])/R;
return C[0][0];
```

Figure 9.7: BINOMIAL TREE ALGORITHM FOR CALLS ON NON-DIVIDEND-PAYING STOCKS. The $C[j][i]$ entry represents the call value at time $j$ if the stock price makes $i$ downward movements out of a total of $j$ movements.

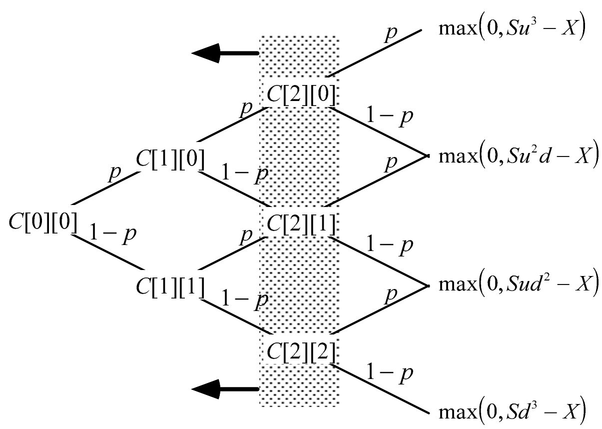


Figure 9.8: IDEAS BEHIND BINOMIAL TREE ALGORITHMS. Binomial tree algorithms start with terminal values. Such terminal values are computed in Step 1 of the algorithm in Fig. 9.7. They then sweep a line backward in time to compute values at intermediate times until the starting time is reached.

Step 2.1 should now be modified as follows,

$$C[i] := (p \times C[i] + (1-p) \times C[i+1])/R;$$

Finally, $C[0]$ is returned instead of $C[0][0]$. The memory size is linear now. The one-dimensional array $C$ essentially captures the strip in Fig. 9.8.

Further improvements can be made by observing that, if $C[j+1][i]$ and $C[j+1][i+1]$ are both zeros, then $C[j][i]$ is zero, too. So we only need to let the $i$-loop within the double loop run from zero to $\min(n-a, j)$ instead of $j$, where $a$ is defined in (9.12). This makes the algorithm run in $O(n\,(n-a))$ steps, which may be much smaller than $O(n^2)$ when $a$ is large. The space requirement can be similarly reduced to $O(n-a)$ with a smaller one-dimensional array, $C[n-a+1]$. See Fig. 9.10 for the idea (the one-dimensional array implements the strip in that figure).

**Programming assignment 9.2.1** Implement the algorithms in Figs. 9.7 and 9.9. ◇

**Programming assignment 9.2.2** Implement the algorithm mentioned in the text that runs in time $O(n\,(n-a))$ and space $O(n-a)$. Compare its speed with the standard $O\left(n^2\right)$-time binomial tree algorithm. ◇

Binomial tree algorithm for pricing European puts on non-dividend-paying stocks:

```
input:   S, u, d, X, n, r  (u > e^r > d and r > 0);
real      R, p, P[n+1][n+1];
integer  i, j;
R := e^r;
p := (R - d)/(u - d);
for i = 0 to n
         1. P[n][i] := max(0, X - S u^{n-i} d^i);
for j = n - 1 down to 0
         for i = 0 to j
               2.1. P[j][i] := (p × P[j+1][i]
                         +(1 - p) × P[j+1][i+1])/R;
return P[0][0];
```

Figure 9.9: Binomial tree algorithm for European puts on non-dividend-paying stocks. The $P[j][i]$ entry represents the put value at time $j$ if the stock price makes $i$ downward movements out of a total of $j$.

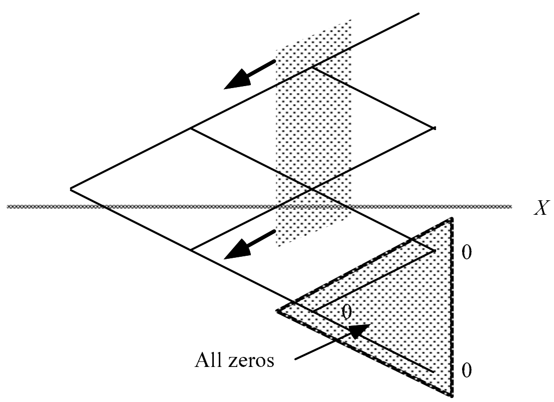


Figure 9.10: Skipping zero nodes to improve efficiency. Zeros at the terminal nodes propagate through the tree depicted here for a call. Such nodes can be skipped by binomial tree algorithms. The savings in time and space can be substantial. The stock finishes below the strike price if it finishes below the horizontal line. In fact, *all* the nodes below the horizontal line have stock prices less than $X$ if $ud = 1$ holds. This is because all nodes on the same horizontal line have identical stock price.

### An optimal algorithm

We can further reduce the running time to $O(n-a)$ and the memory requirement to $O(1)$. Consider the following program segment to compute $b(j;n,p) \equiv \binom{n}{j} p^j (1-p)^{n-j}$.

```
b[a] := (n choose a) p^a (1 - p)^{n-a};
for j = a + 1 to n
      b[j] := b[j - 1] × p × (n - j + 1)/((1 - p) × j);
```

It clearly runs in $O(n-a)$ steps. What gets computed in $b[j]$ is $b(j;n,p)$ for $a \leq j \leq n$ because

$$\begin{aligned} b(j;n,p) &= \frac{p\,(n-j+1)}{(1-p)\,j} \frac{n!}{(j-1)!\,(n-j+1)!} p^{j-1} (1-p)^{n-j+1} \\ &= \frac{p\,(n-j+1)}{(1-p)\,j} b(j-1;n,p). \end{aligned}$$

With the $b(j;n,p)$s available, the risk-neutral valuation formula (9.13) is trivial to compute. The case for puts is similar. So, pricing European options on non-dividend-paying stocks can be computed in linear time. As for the memory requirement, we only need a single variable instead of a whole array to store the $b(j;n,p)$'s when they are being sequentially computed. The algorithm appears in Fig. 9.11.

Linear-time, constant-space algorithm for pricing calls on non-dividend-paying stocks:

```
input:   S, u, d, X, n, r  (u > e^r > d and r > 0);
real     R, p, a, b, D, C;
integer  j;
a := ⌈ln(X/Sd^n)/ln(u/d)⌉;
p := (e^r − d)/(u − d);
R := e^{nr};
b := p^a (1 − p)^{n−a};
D := S × u^a d^{n−a}; // b(a; n, p).
C := b × (D − X)/R;
for j = a + 1 to n {
        1. b := b × p × (n − j + 1)/((1 − p) × j);
        2. D := D × u/d;
        3. C := C + b × (D − X)/R;
}
return C;
```

Figure 9.11: OPTIMAL ALGORITHM FOR EUROPEAN CALLS ON STOCKS THAT DO NOT PAY DIVIDENDS. Variable $b$ stores $b(j;n,p)$ for $j = a, a+1, \ldots, n$, in that order, and variable $C$ accumulates the summands in (9.13) by adding up $b(j;n,p) \times \left(Su^j d^{n-j} - X\right)/R^n$.

The above linear-time algorithm basically computes the discounted expected value of $\max(S - X, 0)$. In fact, it can be adapted to price any European-style derivative with a call-like payoff function, say $\max\left(\sqrt{S - X}, 0\right)$: Simply replace $D - X$ with $\sqrt{D - X}$. It is straightforward to modify it to price European puts.

The algorithm described above cannot be extended to incorporate early exercise and dividend policies. Another serious problem with the above approach regards the limited precision of digital computers. For moderately large $n$,

$$\frac{n!}{a!\,(n-a)!}\, p^a\, (1-p)^{n-a} \approx 0;$$

hence all subsequent $b$ values are nearly zeros, too. One approach is to compute $\ln b(j;n,p)$ instead of $b(j;n,p)$. The needed changes to the algorithm are

```
        ...
b := ln n! − ln a! − ln(n − a)! + a × ln p + (n − a) × ln(1 − p);
        ...
    1. b := b + ln p + ln(n − j + 1) − ln(1 − p) − ln j;
        ...
    3. C := C + e^b × (D − X)/R;
        ...
```

### Monte Carlo method

Now is a good time to introduce the Monte Carlo method. Look at (9.10) again. It can be interpreted as the expected value of the random variable $Z$ defined by

$$Z = \frac{\max\left(0, Su^j d^{n-j} - X\right)}{R^n} \quad \text{with probability } b(j;n,p) \text{ for } 0 \le j \le n.$$

Consider the following approximation scheme. Throw $n$ coins with $p$ being the probability of getting a head. Assign the value

$$\frac{\max\left(0, Su^j d^{n-j} - X\right)}{R^n}$$

if the experiment generates $j$ heads. Finally, repeat the procedure $m$ times and take the average. This average clearly has the right expected value, $E[\,Z\,] = C$. Furthermore, its variance, $\text{Var}[\,Z\,]/m$ from elementary statistics, converges to zero as $m$ increases.

Pricing European options may be too trivial a problem to apply the Monte Carlo method. In §18.2, we will see that the Monte Carlo method is an invaluable tool in pricing European-style derivative securities and mortgage-backed securities. It is also one of the most important elements of studying econometrics [482].

**Programming assignment 9.2.3** Implement the Monte Carlo method. Observe its convergence rate as the sampling size $m$ increases. ◇

## 9.3 The Black-Scholes Formula

The binomial option pricing model on the surface suffers from two unrealistic assumptions: The stock price only takes on two possible values in a period, and trading occurs at discrete points in time. We shall show that such objections are more apparent than real by shortening the elapsed time of a period. As the number of periods from now to expiration increases, the stock price ranges over ever larger numbers of possible values, and trading takes place nearly continuously. What remains to be done is to achieve it with proper calibration of the various parameters in the binomial option pricing model so that the result makes sense as a period takes up ever shorter time.

### 9.3.1 Distribution of the rate of return

Let $\tau$ denote the time to expiration of an option. With $n$ periods during the life of the option, each period represents a time interval of $\tau/n$. Our task is to adjust the period-based variables, $u$, $d$, and $r$, to obtain empirically realistic results as $n$ goes to infinity.

We will focus on the European call on non-dividend-paying stocks in deriving the formula. As before, the interest rate $r$ is positive with continuous compounding and with the same time unit as $\tau$. Almost without exceptions, $\tau$ will be measured by the year, and $r$ is an annual rate. Let $\hat{r}$ denote the interest rate per period and $\widehat{R} \equiv e^{\hat{r}}$ the period gross return. Since the period gross return can be expressed either as $e^{r\tau/n}$ or as $\widehat{R}$,

$$\hat{r} = \frac{r\tau}{n}$$

must hold.

We proceed to derive $u$ and $d$. With continuous compounding, $\ln u$ and $\ln d$ denote the stock's continuously compounded rate of return per period if its price moves from $S$ to

$Su$ and $Sd$, respectively. The rate of return per period is the following Bernoulli random variable,

$$B = \begin{cases} \ln u & \text{with probability } q \\ \ln d & \text{with probability } 1-q \end{cases}$$

Let $S_\tau$ denote the stock price at expiration. It is a random variable. In fact, $\ln(S_\tau/S)$ is the sum of $n$ such Bernoulli random variables. The continuously compounded rate of return over a period of $\tau$ is therefore

$$\ln \frac{S_\tau}{S} = \ln\left(\frac{Su^j d^{n-j}}{S}\right) = j\ln(u/d) + n\ln d \tag{9.16}$$

if the stock price makes $j$ upward movements during the $n$ periods. Note that

$$E\left[\ln \frac{S_\tau}{S}\right] = E[\,j\,] \times \ln(u/d) + n\ln d \quad \text{and} \quad \mathrm{Var}\left[\ln \frac{S_\tau}{S}\right] = \mathrm{Var}[\,j\,] \times \ln^2(u/d).$$

Since each upward price movement occurs with probability $q$, the expected number of upward price movements in $n$ periods is $E[\,j\,] = nq$, and its variance is

$$\mathrm{Var}[\,j\,] = n\left(q\,(1-q)^2 + (1-q)(0-q)^2\right) = nq\,(1-q).$$

In summary, define

$$\widehat{\mu} \equiv E\left[\ln \frac{S_\tau}{S}\right]/n \quad \text{and} \quad \widehat{\sigma}^2 \equiv \mathrm{Var}\left[\ln \frac{S_\tau}{S}\right]/n.$$

That is, $\widehat{\mu}$ is the *period* continuously compounded rate of return's expected value, and $\widehat{\sigma}^2$ is its variance. Then,

$$\begin{aligned} n\widehat{\mu} &= n\,(q\ln(u/d) + \ln d) \\ n\widehat{\sigma}^2 &= nq\,(1-q)\ln^2(u/d) \end{aligned}$$

Neither $n\widehat{\mu}$ nor $n\widehat{\sigma}^2$ should be infinitely large or zero as $n$ goes to infinity. Still, this leaves some room in setting up the binomial model. Although $q$ does not affect the option value directly, it needs to be specified in maintaining some probabilistic similarity between the stock price under the binomial model and the empiric stock price. Now, let the binomial model converge to the expectation, $\mu\tau$, and variance, $\sigma^2\tau$, of the stock's actual continuously compounded rate of return. The term $\sigma$ is called the (annualized) **volatility** of the stock. This implies the following two conditions,

$$n\widehat{\mu} = n\,(q\ln(u/d) + \ln d) \to \mu\tau \tag{9.17}$$

$$n\widehat{\sigma}^2 = nq\,(1-q)\ln^2(u/d) \to \sigma^2\tau \tag{9.18}$$

as $n$ goes to infinity. If we impose the third condition $ud = 1$, then the following assignments satisfy (9.17)–(9.18),

$$u = e^{\sigma\sqrt{\tau/n}}, \quad d = e^{-\sigma\sqrt{\tau/n}}, \quad \text{and} \quad q = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{\tau}{n}}. \tag{9.19}$$

In fact, these assignments lead to

$$\begin{aligned} n\widehat{\mu} &= \mu\tau \\ n\widehat{\sigma}^2 &= \left(1 - \left(\frac{\mu}{\sigma}\right)^2 \frac{\tau}{n}\right) \sigma^2\tau \to \sigma^2\tau. \end{aligned}$$

The inequalities $u > \widehat{R} > d$ may not always hold under these assignments. In fact, the risk-neutral probability may even lie outside $[\,0, 1\,]$. One solution can be found in Exercise 9.3.1 and another in §12.4.6. In any case, the inequalities hold for suitably large $n$. A consequence of the $ud = 1$ condition is that nodes on the same horizontal level of the tree have identical price (review Fig. 9.10).

What emerges as the limiting probabilistic distribution for the continuously compounded rate of return? The Central Limit Theorem says that, under certain weak conditions, sum of independent random variables such as $\ln(S_\tau/S)$ converges to the normal distribution,

$$\text{Prob}\left[\frac{\ln(S_\tau/S) - n\widehat{\mu}}{\sqrt{n}\,\widehat{\sigma}} \leq z\right] \to N(z). \tag{9.20}$$

A simple condition for (9.20) to hold is the **Ljapunov condition** [90],

$$\frac{q\,|\ln u - \widehat{\mu}\,|^3 + (1-q)\,|\ln d - \widehat{\mu}\,|^3}{n\widehat{\sigma}^3} \to 0.$$

After substitutions, the Ljapunov condition becomes

$$\frac{(1-q)^2 + q^2}{n\sqrt{q\,(1-q)}},$$

which indeed goes to zero as $n$ goes to infinity.

The continuously compounded rate of return, $\ln(S_\tau/S)$, thus approaches the normal distribution with mean $\mu\tau$ and variance $\sigma^2\tau$. As a result, $\ln S_\tau$ approaches the normal distribution with mean $\mu\tau + \ln S$ and variance $\sigma^2\tau$. This means $S_\tau$ has a lognormal distribution in the limit. The significance of using the continuously compounded rate is now clear: It is normally distributed.

The lognormality of stock price has a number of implications. It implies that the stock price is positive if it starts positive. Furthermore, although there is no upper bound on the stock price, large increases or decreases are unlikely. Finally, equal movements in the rate of return about the mean is equally likely due to the symmetry of the normal distribution: Two prices $S_1$ and $S_2$ are equally likely to be the final price at expiration if $S_1/S = S/S_2$.

**Exercise 9.3.1** Although we want the price volatility of the resulting binomial tree to match that of the actual stock in the limit, the underlying stock's expected return, hence $q$ as well, does not play a direct role in the binomial option pricing model. Therefore, there is more than one way to assign $u$ and $d$. Suppose we require $q = 0.5$. (1) Show that

$$u = \exp\left[\frac{\mu\tau}{n} + \sigma\sqrt{\frac{\tau}{n}}\right] \quad \text{and} \quad d = \exp\left[\frac{\mu\tau}{n} - \sigma\sqrt{\frac{\tau}{n}}\right]$$

satisfy (9.17)–(9.18) as *equalities*. (2) Is it valid to use the probability 0.5 during backward induction under these new numbers? $\diamondsuit$

**Comment 9.3.1** Recall that the Monte Carlo method in §9.2.5 used a *biased* coin. The scheme in Exercise 9.3.1, in contrast, employed a *fair* coin, which may be easier to program. The original choice, however, has the advantage that $ud = 1$, which can lead to faster algorithms. Alternative choices of $u$ and $d$ are expected to have only slight, if at all, impacts on the convergence of binomial tree algorithms [99]. Unless otherwise stated, binomial tree algorithms shall adopt the original choice of $u$ and $d$. □

### 9.3.2 Toward the Black-Scholes formula

We are now ready to take the final steps toward the Black-Scholes formula as $\tau$ is divided into ever more periods with the given choices of $u$, $d$, $\hat{r}$, and $q = p$, the risk-neutral probability. Remember that these choices make the stock price converge to the lognormal distribution. We first state the Black-Scholes formulae for European calls and puts on stocks that do not pay dividends.

**Theorem 9.3.2**

$$\begin{aligned} C &= SN(x) - Xe^{-r\tau}N\left(x - \sigma\sqrt{\tau}\right) \\ P &= Xe^{-r\tau}N\left(-x + \sigma\sqrt{\tau}\right) - SN(-x) \end{aligned}$$

*where*

$$x \equiv \frac{\ln(S/X) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

□

Before proving the theorem, we plot the call and put values as functions of the current stock price, time to expiration, volatility, and interest rate in Fig. 9.12. Note that the option value for at-the-money options is essentially a linear function of volatility.

Since the put-call parity can be employed to prove the formula for a European put from that for a call, we shall prove the Black-Scholes formula for the call only. The binomial option pricing formula in Theorem 9.2.3,

$$C = S\Phi\left(a; n, pue^{-\hat{r}}\right) - Xe^{-\hat{r}n}\Phi(a; n, p) = S\Phi\left(a; n, pue^{-\hat{r}}\right) - Xe^{-r\tau}\Phi(a; n, p),$$

has apparent similarity to the Black-Scholes formula for the call. Clearly, we are done if

$$\Phi\left(a; n, pue^{-\hat{r}}\right) \to N(x) \text{ and } \Phi(a; n, p) \to N\left(x - \sigma\sqrt{\tau}\right). \tag{9.21}$$

We will only prove $\Phi(a; n, p) \to N\left(x - \sigma\sqrt{\tau}\right)$; the other part can be derived analogously.

Recall that $\Phi(a; n, p)$ is the probability of at least $a$ successes in $n$ independent trials with success probability $p$ for each trial. Let $j$ denote the number of successes (upward price movements) in $n$ such trials. This random variable, a sum of $n$ Bernoulli variables, has mean $np$ and variance $np(1-p)$ and satisfies

$$1 - \Phi(a; n, p) = \text{Prob}[j \leq a - 1] = \text{Prob}\left[\frac{j - np}{\sqrt{np(1-p)}} \leq \frac{a - 1 - np}{\sqrt{np(1-p)}}\right]. \tag{9.22}$$

It was shown before in (9.16) that $\ln(S_\tau/S) = j\ln(u/d) + n\ln d$. Define

$$\widehat{\mu}_p \equiv p\ln(u/d) + \ln d \text{ and } \widehat{\sigma}_p^2 \equiv p(1-p)\ln^2(u/d) \tag{9.23}$$

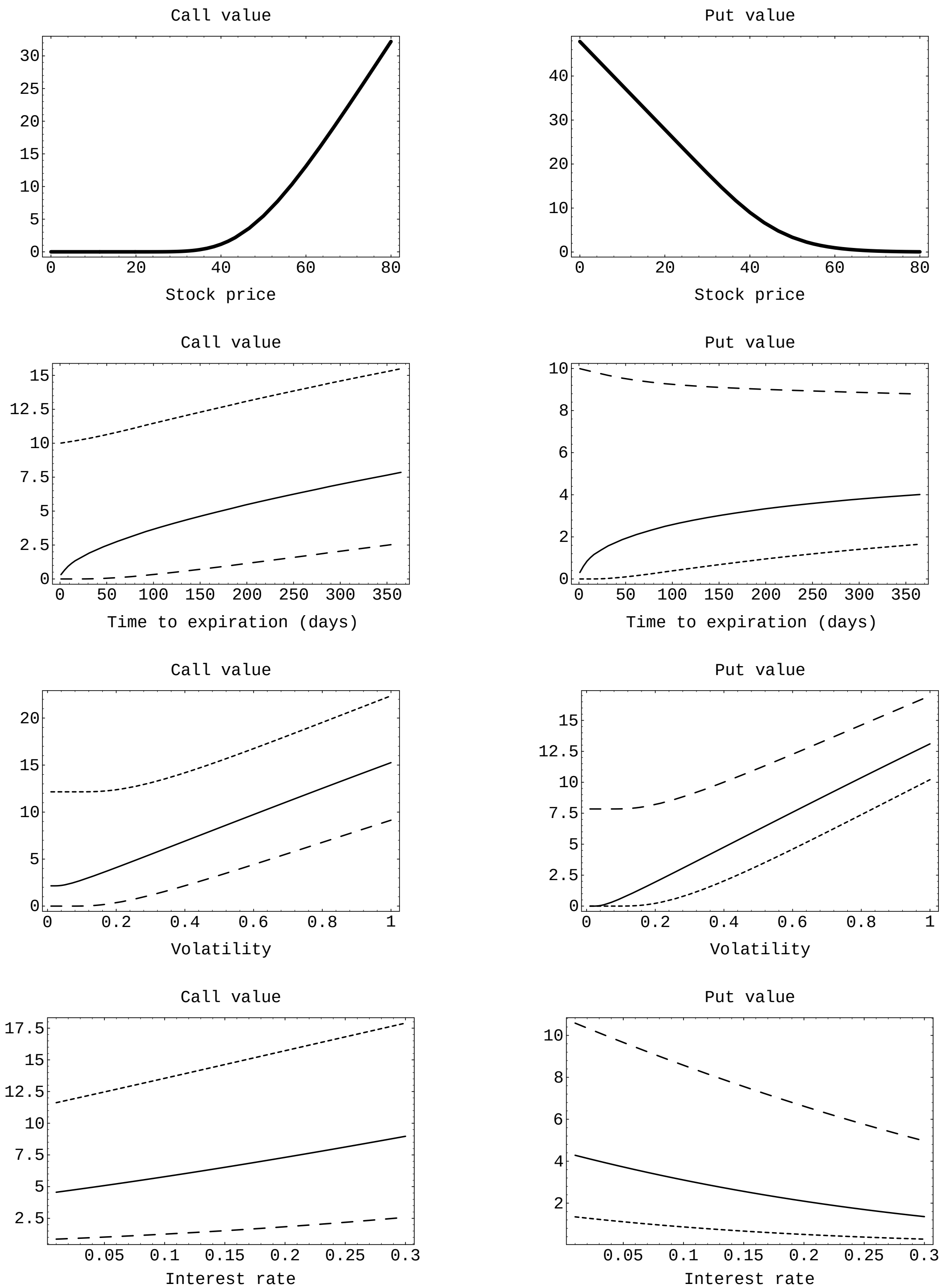


Figure 9.12: VALUES OF EUROPEAN OPTION AS A FUNCTION OF PARAMETERS. The parameters used here are $S = 50$, $X = 50$, $\sigma = 0.3$, $\tau = 201$ (days), and $r = 8\%$. When three curves are graphed on the same plot, the dashed line uses $S = 40$ (out-of-the-money call or in-the-money put), the solid line uses $S = 50$ (at the money), and the dotted line uses $S = 60$ (in-the-money call or out-of-the-money put).

in analogy with the similar formulae for $\widehat{\mu}$ and $\widehat{\sigma}$ earlier. It can be verified that

$$\frac{j-np}{\sqrt{np\,(1-p)}}=\frac{\ln(S_\tau/S)-n\widehat{\mu}_p}{\sqrt{n}\,\widehat{\sigma}_p}.$$

Now,

$$a-1=\frac{\ln(X/Sd^n)}{\ln(u/d)}-\epsilon$$

for some $0<\epsilon\leq 1$. Combine this equality with the definitions for $\widehat{\mu}_p$ and $\widehat{\sigma}_p$ to obtain

$$\frac{a-1-np}{\sqrt{np\,(1-p)}}=\frac{\ln(X/S)-n\widehat{\mu}_p-\epsilon\ln(u/d)}{\sqrt{n}\,\widehat{\sigma}_p}.$$

So (9.23) becomes

$$1-\Phi(a;n,p)=\text{Prob}\left[\frac{\ln(S_\tau/S)-n\widehat{\mu}_p}{\sqrt{n}\,\widehat{\sigma}_p}\leq\frac{\ln(X/S)-n\widehat{\mu}_p-\epsilon\ln(u/d)}{\sqrt{n}\,\widehat{\sigma}_p}\right].$$

Since

$$\frac{p\,|\ln u-\widehat{\mu}_p\,|^3+(1-p)\,|\ln d-\widehat{\mu}_p\,|^3}{n\widehat{\sigma}_p^3}=\frac{(1-p)^2+p^2}{n\sqrt{p\,(1-p)}}\to 0$$

with the help of (9.25) below, the Ljapunov condition is satisfied, and the Central Limit Theorem is applicable . It only remains to evaluate $n\widehat{\mu}_p$ and $\sqrt{n}\,\widehat{\sigma}_p$ as $n$ goes to infinity.

Applying $e^y=1+y+\left(y^2/2!\right)+\cdots$ to $p\equiv\left(e^{r\tau/n}-d\right)/(u-d)$, we obtain

$$p\to\frac{1}{2}+\frac{1}{2}\,\frac{r-\sigma^2/2}{\sigma}\sqrt{\frac{\tau}{n}}.\tag{9.24}$$

From (9.24) and (9.25),

$$\begin{aligned}n\widehat{\mu}_p &= n\sigma\sqrt{\frac{\tau}{n}}\,(2p-1)\to\tau\left(r-\frac{\sigma^2}{2}\right) \qquad (9.25)\\ \sqrt{n}\,\widehat{\sigma}_p &= \sigma\sqrt{\tau}\sqrt{1-\left(\frac{r-\sigma^2/2}{\sigma}\right)^2\frac{\tau}{n}}\to\sigma\sqrt{\tau}\end{aligned}$$

Since $\ln(u/d)=2\sigma\sqrt{\tau/n}\to 0$, we have

$$\frac{\ln(X/S)-n\widehat{\mu}_p-\epsilon\ln(u/d)}{\sqrt{n}\,\widehat{\sigma}_p}\to z\equiv\frac{\ln(X/S)-\tau\left(r-\sigma^2/2\right)}{\sigma\sqrt{\tau}}$$

and, hence,

$$1-\Phi(a;n,p)\to N(z)=N\left(\frac{\ln\,(X/S)-r\tau}{\sigma\sqrt{\tau}}+\frac{1}{2}\,\sigma\sqrt{\tau}\right).$$

The desired identity,

$$\Phi(a;n,p)\to N(-z)=N\left(\frac{\ln(S/X)+r\tau}{\sigma\sqrt{\tau}}-\frac{1}{2}\,\sigma\sqrt{\tau}\right)=N\left(x-\sigma\sqrt{\tau}\right),$$

finally emerges, where

$$x\equiv\frac{\ln(S/X)+\left(r+\sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

Hidden in the proof of (9.26) is the following useful result.

**Corollary 9.3.3** *In a risk-neutral economy, the continuously compounded rate of return,* $\ln(S_\tau/S)$, *approaches the normal distribution with mean* $\left(r - \sigma^2/2\right)\tau$ *and variance* $\sigma^2\tau$. □

The above corollary about the distribution of stock prices together with (6.16) implies the expected stock price at time $t$ in a risk-neutral economy is $Se^{rt}$. The stock's expected annual rate of return is thus the riskless rate.

### 9.3.3 Use of binomial tree algorithms

The Black-Scholes formulae as described in Theorem 9.3.3 have five parameters: $S$, $X$, $\sigma$, $\tau$, and $r$. Binomial tree algorithms such as the one in Fig. 9.7 take six inputs: $S$, $X$, $u$, $d$, $\hat{r}$, and $n$. The connection between the two follows: $u = e^{\sigma\sqrt{\tau/n}}$, $d = e^{-\sigma\sqrt{\tau/n}}$, and $\hat{r} = r\tau/n$. The binomial tree algorithm converges reasonably fast (see Fig. 9.14).

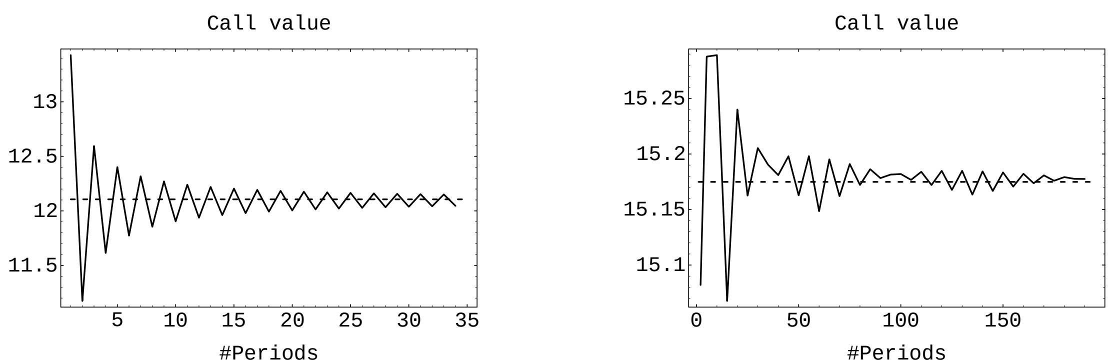


Figure 9.13: CONVERGENCE RATES OF BINOMIAL TREE ALGORITHMS. Plotted are the call values as computed by the binomial tree algorithm against the number of time partitions $n$. The options in question are European calls. The parameters used are $S = 100$, $X = 100$ (left) and 95 (right), $r = 8\%$, $\sigma = 0.2$, and $\tau = 1$ (year). The analytical values, 12.1058 and 15.1749, are displayed for reference. Oscillations are inherent of binomial models [608].

**Example 9.3.4** Consider a three-month option when the interest rate is 8% per annum and the volatility is 30% per annum. This means $\tau = 0.25$, $r = 0.08$, and $\sigma = 0.3$. Assume the binomial tree algorithm uses $n = 5$. Then, it should be executed with $u = e^{0.3\sqrt{0.25/5}} = 1.0694$ and $d = e^{-0.3\sqrt{0.25/5}} = 0.9351$. □

## 9.4 Using the Black-Scholes Formula

Five parameters are needed for the Black-Scholes formula.

### 9.4.1 Interest rate

The riskless interest rate $r$ should be the spot rate with maturity near the option's expiration date. It should also be a continuously compounded rate. (In practice, the specific rate depends on the investor; it can be the Treasury bill rate, the Treasury bill repo rate, or the broker call money rate [195].) This choice of interest rate can be justified as follows.

Suppose interest rates may not be a constant but they can only change predictably. Let $r_i$ denote the continuously compounded, one-period interest rate, measured in periods for period $i$. The bond maturing on the option's expiration date is then priced at $\exp[-\sum_{i=1}^{n} r_i]$ per dollar of face value. This implies $e^{-r\tau} = \exp[-\sum_{i=1}^{n} r_i]$. Hence, a single discount bond price with maturity at period $n$ encompasses all the information needed for interest rates, that is, the $n$-period spot rate. In the limit, $-\sum_{i=0}^{n-1} r_i \to -\int_0^{\tau} r(t)\,dt$, where $r(t)$ is the short rate at time $t$. The relevant annualized interest rate is thus

$$r \equiv \frac{\int_0^{\tau} r(t)\,dt}{\tau}.$$

Interest rate uncertainty may not be very critical to valuing options with lives under one year. Plots in Fig. 9.12 also suggest that small changes in interest rates do not move the option value significantly.

### 9.4.2 Estimating the volatility from historical data

The volatility parameter $\sigma$ is the sole parameter not directly observable and has to be estimated. The Black-Scholes formula assumes stock prices are lognormally distributed. In other words, the $n$ continuously compounded rates of return per period,

$$u_i \equiv \ln\left(\frac{S_i}{S_{i-1}}\right), \quad 1 \leq i \leq n,$$

are independent samples from a normal distribution with mean $\mu\tau/n$ and variance $\sigma^2\tau/n$. Here, $S_i$ denotes the stock price at time $i$. A good estimate of the standard deviation of the rate of return per period is

$$s \equiv \sqrt{\frac{\sum_{i=1}^{n}(u_i - \overline{u})}{n-1}}, \tag{9.26}$$

where $\overline{u} \equiv (1/n)\sum u_i = (1/n)\ln(S_n/S_0)$. The $\overline{u}$ and $s^2$ are in fact maximum likelihood estimators of $\mu$ and $\sigma^2$, respectively (see §20.2.2). The estimator in (9.27) may be biased in practice, however. Notable among the reasons are the bid-ask spreads and the discreteness of stock prices [39, 172].

Sometimes, the percent return (or **simple rate of return**), $(S_i - S_{i-1})/S_{i-1}$, is used in place of $u_i$ to avoid logarithm. The result thus obtained is only approximately correct because $\ln x \approx x - 1$ only when $x$ is small, and a small error here can mean huge differences in the option value [129].

If a period contains an ex-dividend date, its sample rate of return should be slightly modified to be

$$u_i = \ln\left(\frac{S_i + D}{S_{i-1}}\right),$$

where $D$ is the amount of the dividend. If an $n$-for-$m$ split occurs in the period, the sample rate of return should be modified to be

$$u_i = \ln\left(\frac{nS_i}{mS_{i-1}}\right).$$

Since the standard deviation of the rate of return equals $\sigma\sqrt{\tau/n}$, the estimate for $\sigma$ is $s/\sqrt{\tau/n}$. This value is called **historical volatility**. Empirical evidence suggests that days when stocks were not traded should be excluded from the calculation. Some even count only trading days in $\tau$, the time to expiration [458]. Since the volatility is the standard deviation of the stock's annualized rate of return, a volatility figure of, say, 0.19 is commonly called "a volatility of 19 percent."

As in the case of interest rate, volatility can be allowed to change in time as long as it is predictable. In the context of the binomial model, this means $u$ and $d$ now depend on time. The variance of $\ln(S_\tau/S)$ is now $\int_0^\tau \sigma^2(t)\,dt$ rather than $\sigma^2\tau$, and the volatility becomes $\sqrt{\int_0^\tau \sigma^2(t)\,dt/\tau}$. We do caution that there is evidence suggesting that the volatility is stochastic (see §15.6). Estimators that utilize high and low prices can be superior theoretically in terms of lower variance [328].

### 9.4.3 Implied volatility

The Black-Scholes formula can be used to compute the market's opinion of the volatility. This is achieved by solving for $\sigma$ given $C$, $S$, $X$, $\tau$, and $r$ with the numerical methods in §3.5.2. The volatility thus obtained is called **implied volatility**, the volatility implied by the market price of the option. Volatility numbers are often stored in a table indexed by maturity and strike price [422, 433].

Implied volatility is often preferred to historical volatility in practice, but it is not perfect. For instance, options written on the same underlying asset usually do not produce the same implied volatility. A typical pattern is a "**smile**" in relation to strike price, whereby the implied volatility is lowest for at-the-money options and becomes higher the further the option is in- or out-of-the-money [132]. This pattern is especially strong for short-term options [37]. Such biases cannot be accounted for by the early exercise feature of American options [87]. To address this issue, volatilities are often combined to produce a composite implied volatility. This practice is not theoretically sound. In fact, the existence of non-identical implied volatilities for options on the same underlying asset shows the Black-Scholes option pricing model cannot be literally true. See [129, §9.3.5] and [458, §8.7.2] for more discussions on this issue. Section 15.6 surveys approaches that try to explain the "smile."

### 9.4.4 Tabulating the values

Rewrite the Black-Scholes formula for the European call as

$$\begin{aligned} C &= Xe^{-r\tau}\left(\frac{S}{Xe^{-r\tau}}N(x) - N\left(x-\sigma\sqrt{\tau}\right)\right) \\ x &= \frac{\ln\left(S/\left(Xe^{-r\tau}\right)\right)}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2} \end{aligned}$$

If we construct a table containing entries

$$\frac{S}{Xe^{-r\tau}}\,N(x) - N\left(x - \sigma\sqrt{\tau}\right)$$

indexed by $S/\left(Xe^{-r\tau}\right)$ and $\sigma\sqrt{\tau}$, then a person can look up the value in the table based on $S$, $X$, $r$, $\tau$, and $\sigma$. The call value is then a simple multiplication of the looked-up value by $Xe^{-r\tau}$. A precomputed table of judiciously selected option values can actually be used to price options via interpolation [466].

## 9.5 American Puts on Non-Dividend-Paying Stocks

Pricing American puts has one extra factor to consider, its early exercise feature. Since the person who exercises a put option receives the strike price and earns the time value of money, there is incentive for early exercise. On the other hand, early exercise may render the put holder worse off if the stock subsequently increases in value. These two factors hence conflict with each other. In general, the first factor tends to dominate and favor early exercise if the interest rate is high and the volatility low.

Even with this consideration, the binomial option pricing model still offers a completely correct solution, although its justification is delicate [12, 209, 505]. As before, we construct an equivalent portfolio that is dynamically adjusted to replicate the American put. The only difference is that early exercise may be beneficial. Specifically, start with the payoffs $\max\left(0, X - Su^j d^{n-j}\right)$ at expiration and work with backward induction. At each intermediate node, create an equivalent portfolio that replicates the American put. Now, (9.7) should be replaced by $P = \max(hS + B, X - S)$. The complete algorithm appears in Fig. 9.15, whose difference from the algorithm for European puts in Fig. 9.9 is minimal. Figure 9.16 plots the value of an American put against its European counterpart.

Binomial tree algorithm for pricing American puts on non-dividend-paying stocks:

```
input:   S, u, d, X, n, r  (u > e^r > d and r > 0);
real     R, p, P[n+1][n+1], v;
integer  i, j;
R := e^r;
p := (R - d)/(u - d);
for i = 0 to n
        P[n][i] := max(0, X - S u^(n-i) d^i);
for j = n - 1 down to 0
        for i = 0 to j
                // Possible early exercise.
                P[j][i] := max((p × P[j+1][i] + (1 - p) × P[j+1][i+1])/R, X - S u^(j-i) d^i);
return P[0][0];
```

Figure 9.14: BINOMIAL TREE ALGORITHM FOR AMERICAN PUTS ON NON-DIVIDEND-PAYING STOCKS. The $P[j][i]$ entry represents the put value at time $j$ if the stock price makes $i$ downward movements out of a total of $j$ movements.

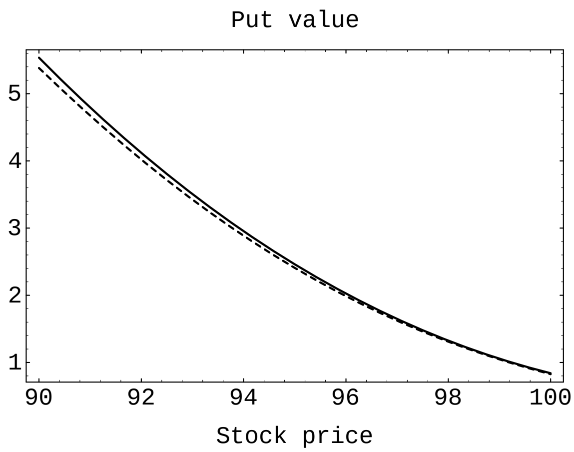


Figure 9.15: AMERICAN PUT VS. EUROPEAN PUT. Plotted is the American put price at one month before expiration. The strike price is \$95, and the riskless rate is 8%. The volatility of the stock is assumed to be 0.25. The corresponding European put is also plotted (dotted line) for comparison.

### 9.5.1 A numerical example

Assume the parameters are $S = 160$, $X = 130$, $n = 3$, $u = 1.5$, $d = 0.5$, and $R = e^{0.18232} = 1.2$. One can then verify that $p = (R - d)/(u - d) = 0.7$, $h = (P_u - P_d)/S\,(u - d) = (P_u - P_d)/S$, and $P = (pP_u + (1-p)\,P_d)/R = (0.7 \times P_u + 0.3 \times P_d)/1.2$. Consider node A in Fig. 9.17. The equivalent portfolio's value is

$$\frac{0.7 \times 0 + 0.3 \times 70}{1.2} = 17.5,$$

greater than the intrinsic value $130 - 120 = 10$. Hence, the option should not be exercised even if it is in the money, and the put value is 17.5. Now consider node B. The equivalent portfolio's value is

$$\frac{0.7 \times 70 + 0.3 \times 110}{1.2} = 68.33,$$

less than the intrinsic value $130 - 40 = 90$. Hence, the option should be exercised, and the put value is 90.

**Programming assignment 9.5.1** Implement the algorithm in Fig. 9.15 for American puts. ◇

## 9.6 Options on Stocks That Pay Dividends

The binomial option pricing model remains correct if dividends are predictable. Only unprotected options will be considered here. Algorithms presented here typically run in $O\left(n^2\right)$ time and $O\left(n^2\right)$ space. Using a linear array to sweep backward in time as in Fig. 9.8 can cut the memory requirement to $O(n)$.

### 9.6.1 European options on stocks that pay known dividend yields

A known **dividend yield** means that the dividend income forms a constant percentage of the security price. If $\delta$ represents the dividend yield, then the stock pays out $S\delta$ on each ex-dividend date. Cast in the binomial model, this implies that the stock price will go from $S$ to $Su\,(1-\delta)$ or $Sd\,(1-\delta)$ in a period which includes an ex-dividend date. If a period

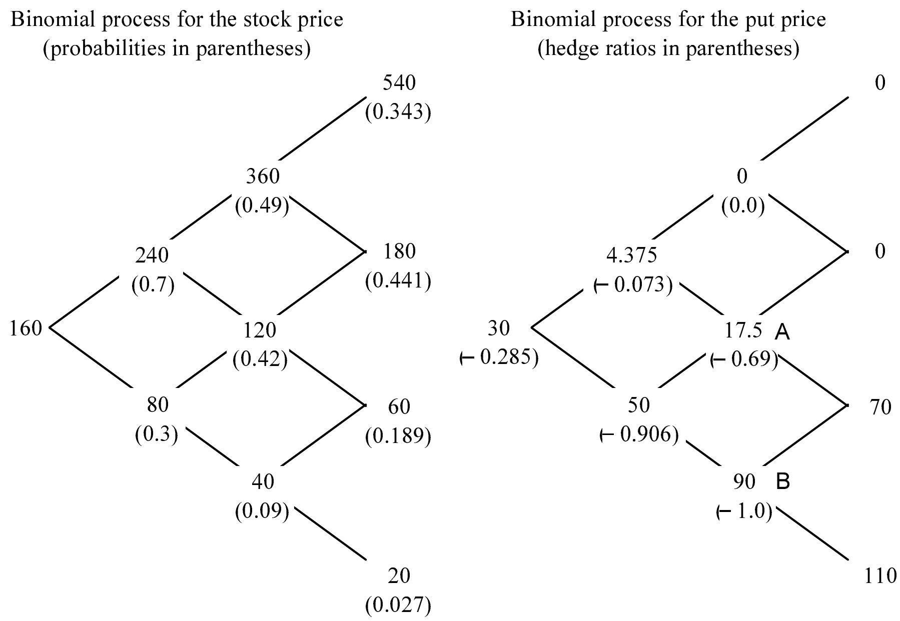


Figure 9.16: Stock price movements and American put prices.

does not include any ex-dividend date, on the other hand, the binomial model remains the same as before. See Fig. 9.18.

$$\begin{array}{ccccccc}
 & & & & & & Suuu\,(1-\delta) \\
 & & & & & \nearrow & \\
 & & & & Suu\,(1-\delta) & & \\
 & & & \nearrow & & \searrow & \\
 & & Su & & & & Suud\,(1-\delta) \\
 & \nearrow & & \searrow & & \nearrow & \\
S & & & & Sud\,(1-\delta) & & \\
 & \searrow & & \nearrow & & \searrow & \\
 & & Sd & & & & Sudd\,(1-\delta) \\
 & & & \searrow & & \nearrow & \\
 & & & & Sdd\,(1-\delta) & & \\
 & & & & & \searrow & \\
 & & & & & & Sddd\,(1-\delta)
\end{array}$$

Figure 9.17: Binomial model for stocks that pay known dividend yields. The dividend yield is $\delta$, and the ex-dividend date occurs during the second period.

For European options, only the number of ex-dividend dates matters, not their specific dates. This can be seen as follows. Let $m$ denote the number of ex-dividend dates before expiration. Then, the stock price at expiration is of the form $(1-\delta)^m S u^j d^{n-j}$ independent of the timing of the dividends. As a consequence, we can simply use the relevant binomial tree algorithms for options on non-dividend-paying stocks with the current stock price $S$ replaced by $(1-\delta)^m S$. Pricing can thus be done in linear time and constant space.

### 9.6.2 American options on stocks that pay known dividend yields

American calls should only be exercised just before ex-dividend dates or at expiration (see Theorem 8.4.2). The binomial model for calls works as follows. With one period to go before expiration, the stock price can go from $S$ to either $Su\,(1-\delta)^v$ or $Sd\,(1-\delta)^v$, where $\delta$ is the dividend yield and $v \in \{0, 1\}$, depending on whether the period contains an ex-dividend date. The option values under these two possibilities are $C_u = \max(Su\,(1-\delta)^v - X, 0)$ and $C_d = \max(Sd\,(1-\delta)^v - X, 0)$. We can create an equivalent portfolio with $\frac{C_u - C_d}{(u-d)\,S}$ stocks and $B = \frac{uC_d - dC_u}{(u-d)\,R}$ dollars in bonds. The call value is then

$$C = \max\left(S - X, \frac{pC_u + (1-p)\,C_d}{R}\right).$$

The algorithm is shown in Fig. 9.19. It can be easily modified to value American puts.

Early exercise might be optimal when $v = 1$. Suppose $Sd\,(1-\delta) > X$. Since $u > d$, we must have $C_u = Su\,(1-\delta) - X$ and $C_d = Sd\,(1-\delta) - X$. One can verify that

$$\frac{pC_u + (1-p)\,C_d}{R} = (1-\delta)\,S - \frac{X}{R},$$

which is exceeded by $S - X$ for sufficiently large $S$. This proves early exercise before expiration might be optimal.

---

Binomial tree algorithm for pricing American calls on stocks that pay known dividend yields:

**input**: $S, u, d, X, n, \delta\ (1 > \delta > 0), m, r\ (u > e^r > d$ and $r > 0)$;
**real** $R, p, C[n+1][n+1]$;
**integer** $i, j$;
$R := e^r$;
$p := (R-d)/(u-d)$;
**for** $i = 0$ **to** $n$
    $C[n][i] := \max\left(0, Su^{n-i}d^i\,(1-\delta)^m - X\right)$;
**for** $j = n-1$ **down to** $0$
    **for** $i = 0$ **to** $j$ {
        **if** [ period $j$ contains an ex-dividend date ] $m := m - 1$;
        $C[j][i] := \max\left((p \times C[j+1][i] + (1-p) \times C[j+1][i+1])/R, Su^{j-i}d^i\,(1-\delta)^m - X\right)$;
    }
**return** $C[0][0]$;

---

Figure 9.18: BINOMIAL TREE ALGORITHM FOR AMERICAN CALLS ON STOCKS PAYING DIVIDEND YIELDS. The $C[j][i]$ entry represents the put value at time $j$ if the stock price makes $i$ downward movements out of a total of $j$ movements. For brevity, the ex-dividend dates are not shown as input parameters. Recall that $m$ initially stores the total number of ex-dividend dates before expiration.

### 9.6.3 Options on stocks that pay known dividends

Although companies may try to maintain a constant dividend yield in the long run, a constant dividend is satisfactory in the short run [205]. Most exchange-traded options are short-term.

Compared with stocks that pay constant dividend yields, the case of constant dividend introduces complications. Use $D$ to denote the amount of the dividend and consider Fig. 9.20. Suppose an ex-dividend date falls in the first period. At the end of that period, the possible stock prices are $Su-D$ and $Sd-D$. Follow the stock price for one more period. The number of possible stock prices is no longer three but four: $(Su-D)\,u$, $(Su-D)\,d$, $(Sd-D)\,u$, and $(Sd-D)\,d$. In other words, the binomial tree no longer combines. Such a tree is called an **open lattice**. The fundamental reason is that timing of the dividends now becomes important; $(Su-D)\,u$ is different from $Suu-D$. It is not hard to see that $m$ ex-dividend dates will give rise to at least $2^m$ leaves. Therefore, the known-dividends case consumes tremendous computation time and memory.[4]

$$
\begin{array}{ccccc}
 & & & & (Su-D)\,u \\
 & & & \nearrow & \\
 & & Su-D & & \\
 & \nearrow & & \searrow & \\
 & & & & (Su-D)\,d \\
S & & & & \\
 & & & & (Sd-D)\,u \\
 & \searrow & & \nearrow & \\
 & & Sd-D & & \\
 & & & \searrow & \\
 & & & & (Sd-D)\,d
\end{array}
$$

Figure 9.19: BINOMIAL MODEL FOR STOCKS THAT PAY KNOWN DIVIDENDS. The amount of the dividend is $D$, and the ex-dividend date occurs during the first period.

### A simplifying assumption

One way to adjust for dividends is to use the Black-Scholes formula with the stock price reduced by the present value of the anticipated dividends. This procedure is valid if the stock price can be decomposed into a sum of two components, a riskless one paying the known dividends during the life of the option and a risky one. The riskless component at any time is the present value of all the dividends during the life of the option discounted from the ex-dividend dates to the present at the riskless rate. The Black-Scholes formula is then applicable with $S$ set equal to the risky component of the stock price and $\sigma$ the volatility of the process followed by the risky component. So, the stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution. This means that the Black-Scholes formula can be used provided that the stock price is reduced by the present value of all the dividends during the life of the option. We note that uncertainty about dividends is rarely an important issue for options lasting less than one year.

That the above assumption leads to efficient valuation can be better understood with the binomial model. The assumption means we can start with the current stock price minus the present value of all the dividends and develop the binomial tree for the stock price as if the stock paid no dividends. This done, we add to each stock price on the tree the

[4] We can experience such exponential explosion with the *Mathematica* program in [576]. The program runs into glacial response time with thirteen time steps and up. It is not without reason that computer scientists call problems requiring exponential time intractable.

present value of all its future dividends. Now compute the European option price as before on this tree of stock prices. It should be evident that the efficiency comes basically from the fact that these assumptions make the binomial tree combine. Such models are said to be **computationally simple** [601]. As for American options, the same procedure applies except for the need to test for early exercises at each node.

### 9.6.4 Options on stocks that pay continuous dividend yields

In the **continuous payout model**, dividends are paid continuously. Such a model approximates broad-based stock market index portfolio in which some company will pay a dividend nearly every day. Foreign currencies also pay daily dividends in the form of interest, hence well approximated by the continuous payout model.

The payment of a **continuous dividend yield** at rate $q$ reduces the growth rate of the stock price by $q$. In other words, a stock that grows from $S$ to $S_\tau$ with a continuous dividend yield of $q$ would grow from $Se^{-q\tau}$ to $S_\tau$ without dividends. Hence, a European option on a stock with price $S$ paying a continuous dividend yield of $q$ has the same value as the corresponding European option on a stock with price $Se^{-q\tau}$ that pays no dividends. Black-Scholes formulae can be employed with $S$ replaced by $Se^{-q\tau}$. Hence, the following formulae hold,

$$C = Se^{-q\tau} N(x) - Xe^{-r\tau} N\left(x - \sigma\sqrt{\tau}\right) \tag{9.27}$$

$$P = Xe^{-r\tau} N\left(-x + \sigma\sqrt{\tau}\right) - Se^{-q\tau} N(-x) \tag{9.28$'$}$$

where

$$x = \frac{\ln(S/X) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

The above formulae, due to Merton [572], remain valid even if the dividend rate is not a constant as long as it is predictable and $q$ is replaced by the average annualized dividend yield during the life of the option [422, 650].

#### Binomial tree algorithms

To run the binomial tree algorithm, simply pick the risk-neutral probability as

$$\frac{e^{(r-q)\,\Delta t} - d}{u - d}, \tag{9.28}$$

where $\Delta t \equiv \tau/n$. The quick reason is that the stock price grows at the expected rate of $r-q$ in the risk-neutral economy. Other than this change, the binomial tree algorithm remains the same as if there were no dividends. Note that $u$ and $d$ refer to stock *price* movements, hence net of the dividends. This arrangement is identical to retaining $p \equiv \frac{e^{r\Delta t}-d}{u-d}$ but with $u$ and $d$ multiplied by $e^{q\Delta t}$.

## 9.7 Concluding Remarks and Additional Reading

The basic Black-Scholes formulae are derived under several assumptions. For instance, it is assumed that the underlying stock pays no cash dividends during the life of the option and the option is European. Margin requirements, taxes, and transactions costs are ignored. The interest rate and the volatility of the stock are generally postulated to be constant, although two extensions were mentioned in §9.4.1 and §9.4.2. Finally, only very small changes in the stock price are allowed in a very short period of time. Consult [205, 422] for various extensions to the basic model and [135, 591] for analytical results concerning American options. The paper [529] considers the case when returns are predictable. Related formulae are collected in [300, 794]. Many methods for valuing American options are benchmarked in [114]. Consult [302, §13.2] for more information regarding estimating volatility from historical data.

The Black-Scholes formula can be derived in at least four other ways [250]. Some of them will be mentioned after we cover the subject of stochastic differential equations. The speed of binomial tree algorithms may be improved by various methods. These alternatives will be covered in Chapters 17 and 18. Many excellent textbooks cover options [205, 277, 302, 422].

Option pricing theory forms one of the pillars for finance [570]. As the necessarily short list of applications in Chapter 11 shows, option-like features pervade almost every part of the field. The option pricing methodology has been applied to the valuation of non-corporate financial arrangements including government loan guarantees, pension fund insurance and deposit insurance, employee compensation packages such as stock options, guaranteed wage floors, business strategies, and even tenure for university faculty [341, 570]. Interested reader can also read [449, 714] for the intellectual developments prior to Black and Scholes's breakthrough. To learn more about Black as a scientist, financial practitioner, and person, consult [301, 573].

Option pricing and information are available on the World Wide Web. Comprehensive option pricing programs for the book are at

`www.csie.ntu.edu.tw/~lyuu/Capitals/capitals.htm`

Another site for option pricing is

`www.numa.com/derivs/ref/calculat/calculat.htm`

As of 1998, this site prices only European options, but it also does calculations for warrants and convertible bonds. *QuoteCom* provides option quotations at `www.quote.com`. Site locations—or **URL**s (**Uniform Resource Locators**), in technical jargon—may change.