Chapter 11

Extensions and Applications of Options

Contents

As I never learnt mathematics,
so I have had to think.
—Joan Robinson (1903–1983) [707]

The underlying asset of an option spans stocks, stock indexes, currencies, futures contracts, interest rates, fixed-income securities, mortgage securities, to mention just a few [46, 302, 603]. Options can also be embedded in the securities, notable among them being callable bonds and mortgage-backed securities. This chapter samples various option instruments and presents some important applications of the option pricing theory.

11.1 Generalized Options

Consider a European-style derivative security whose payoff at expiration is a piecewise linear function $F(S)$ passing through the origin. If S_i are the breakpoints and $0 = S_0 < S_1 < \cdots < S_n$, then

$$F(S) = \begin{cases} 0 & \text{if } S < 0 \\ \alpha_i S + \beta_i & \text{if } S_i \leq S < S_{i+1} \text{ for } 0 \leq i < n \\ \alpha_n S + \beta_n & \text{if } S_n \leq S \end{cases}$$

where $\alpha_{i-1} S_i + \beta_{i-1} = \alpha_i S_i + \beta_i$ for continuity and $\beta_0 = 0$ for origin-crossing. Clearly, $F(0) = 0$ and

$$F(S_i) = \sum_{j=1}^{i} \alpha_{j-1} (S_j - S_{j-1})$$

for $i > 0$. A derivative security with the above payoff function $F(\cdot)$ is called a **generalized option**. See Fig. 11.1 for illustration.

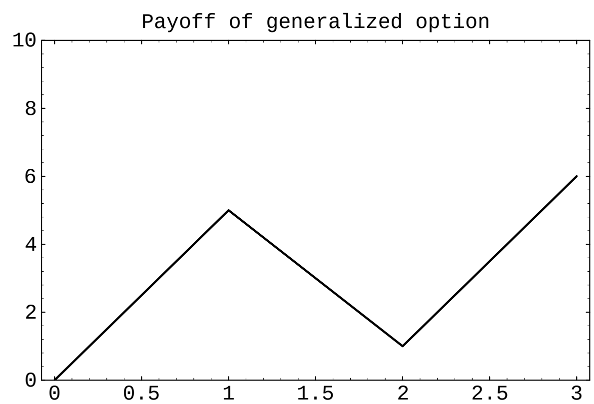

Figure 11.1: Payoff of generalized option.

A generalized option can be replicated by a package of European calls with the same expiration date. Consider a portfolio of options consisting of α_0 calls with strike price $S_0 = 0$, $\alpha_1 - \alpha_0$ calls with strike price S_1, $\alpha_2 - \alpha_1$ calls with strike price S_2, and so on. If the stock price S finishes between S_i and S_{i+1}, then the generalized option has a payoff of

$$F(S) = F(S_i) + \alpha_i (S - S_i) = \sum_{j=1}^{i} \alpha_{j-1} (S_j - S_{j-1}) + \alpha_i (S - S_i).$$

Among the options in the package, only those with the strike price not exceeding S_i finish in the money. The payoff is thus

$$\alpha_0 (S - S_0) + \sum_{j=1}^{i} (\alpha_j - \alpha_{j-1})(S - S_j) = \alpha_i (S - S_i) + \sum_{j=1}^{i} \alpha_{j-1} (S_j - S_{j-1}).$$

Now that the generalized option has the same payoff as the package of calls, they are equivalent.

Suppose the payoff function of the generalized option does not pass through the origin, say with an intercept of β. Simply add to the portfolio zero-coupon bonds with a total obligation in the amount of β at maturity and the same maturity date as the expiration date. In the most general case where the payoff function is any continuous function, simply employ a piecewise linear function with enough breakpoints to approximate its payoff to the desired accuracy.

11.2 Corporate Securities

With suitable reinterpretation of the variables, Black and Scholes observed that the option pricing methodology can be applied to corporate securities [80, 205, 570]. This profound insight led to a unified framework for pricing corporate securities. Here, the underlying asset is the **total value** of the firm instead of merely the value of the stock.

In the following analysis, it will be assumed that (1) a firm can finance payouts by the sale of assets, and (2) if a promised payment to an obligation other than stock is missed, the claim holders own the firm and the stockholders get nothing.

11.2.1 Risky zero-coupon bonds and stock

Consider a firm called XYZ with a very simple capital structure. This firm has only two kinds of securities outstanding: n shares of its own common stock and zero-coupon bonds with a par value of X. This firm does not pay dividends. The central question is, what are the values of the bonds, B, and stock?

On the maturity date, if the total value of the firm V^* is less than the bondholders' claims X, the firm declares bankruptcy and the stock becomes worthless. On the other hand, if $V^* > X$, then the bondholders get X and the stockholders $V^* - X$. The following table shows their respective payoff patterns.

	$V^* \leq X$	$V^* > X$
Bonds	V^*	X
Stock	0	$V^* - X$

It is clear that the stock has the same payoff as a call option on the total value of the firm with a strike price of X and an expiration date equal to the maturity date of the bonds. It is this call option that provides limited liability for the shareholders. Similarly, the bonds are like covered calls on the total value of the firm with the same parameters.

If we let C stand for such a call and V the current value of the firm, then $nS = C$ and $B = V - C$. Since $V = nS + B$, calculating C amounts to knowing how the value of the firm is distributed between stockholders and bondholders. Whatever the value of C, the total value of the stock and bonds at maturity remains V^*. Hence, the relative size of debt and equity is irrelevant to the firm's current value, which is the expected present value of V^*. This is a special case of the Modigliani-Miller irrelevance theorems.

Similar equivalence relationships exist between options and other financial instruments such as seniority, call provisions and sinking fund arrangements on debts, convertible bonds, commodities, interest rate floors, interest rate caps, and warrants. The option pricing theory thus provides a unified theory for the pricing of corporate securities [570].

From Theorem 9.3.3 and the put-call parity, we have

$$nS = VN(x) - Xe^{-r\tau}N\left(x - \sigma\sqrt{\tau}\right) \tag{11.1}$$

$$B = VN(-x) + Xe^{-r\tau}N\left(x - \sigma\sqrt{\tau}\right) \tag{11.2}$$

where

$$x \equiv \frac{\ln(V/X) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

The continuously compounded yield to maturity of the firm's corporate bond is $(1/\tau)\ln(X/B)$. Another interesting measure is the **default premium**, defined as $(1/\tau)\ln(X/B) - r$, which is equal to

$$-\frac{1}{\tau}\ln\left(N(-z) + \frac{1}{\omega}N\left(z - \sigma\sqrt{\tau}\right)\right), \tag{11.3}$$

where

$$\omega = \frac{Xe^{-r\tau}}{V} \quad \text{and} \quad z = \frac{\ln\omega}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau} = -x + \sigma\sqrt{\tau}\,.$$

Note that ω above is the **debt-to-total-value ratio**. The volatility of the value of the firm, σ, can be looked upon as a measure of **operating risk**. Thus the default premium depends only on the firm's capital structure, operating risk, and debt maturity.

Numerical illustrations

Suppose XYZ's asset consists of 1,000 shares of Merck as of March 20, 1995. For convenience, we extract the part of Fig. 7.4 relevant to our discussion in Fig. 11.2. Merck's market value per share is \$44.5. XYZ's securities consist of 1,000 shares of common stock and 30 zero-coupon bonds maturing on July 21, 1995. Each bond promises to pay \$1,000 at maturity. Therefore, $n = 1000$, $V = 1000 \times 44.5 = 44500$, and $X = 1000 \times 30 = 30000$. The option for Merck relevant to our question is the July call with a strike price of $X/n = 30$ per share. Fortunately, such an option exists and is selling for \$15.25. We conclude that XYZ's stock should be worth $15.25 \times 1000 = 15250$. The entire bond issue is therefore worth $B = 44500 - 15250 = 29250$, or \$975 per bond.

The bond should sell for less than an otherwise identical default-free bond. In fact, the bond is equivalent to a default-free zero-coupon bond paying X at maturity plus $n = 1000$ written European puts on Merck at the strike price of \$30 by the put-call parity. Hence, the difference between \$975 and the price of the default-free bond is the value of n European puts on Merck at the strike price of \$30. Figure 11.3 shows the total current market value of the stock and bonds under various debt amounts X. For example, if the promised payment to bondholders is \$45,000, the relevant option is the July call with a strike price of $45000/1000 = 45$. Since that option is selling for \$1 15/16, the current market value of the stock is therefore $(1 + 15/16) \times 1000 = 1937.5$. We can readily see that the current market value of stock decreases as the debt-equity ratio increases.

			—Call—		—Put—	
Option	Strike	Exp.	Vol.	Last	Vol.	Last
Merck	30	Jul	328	151/4	...	...
441/2	35	Jul	150	91/2	10	1/16
441/2	40	Apr	887	43/4	136	1/16
441/2	40	Jul	220	51/2	297	1/4
441/2	40	Oct	58	6	10	1/2
441/2	45	Apr	3050	7/8	100	11/8
441/2	45	May	462	13/8	50	13/8
441/2	45	Jul	883	115/16	147	13/4
441/2	45	Oct	367	23/4	188	21/16

Figure 11.2: MERCK OPTION QUOTATIONS FROM *The Wall Street Journal*, MARCH 21, 1995.

Promised payment to bondholders X	Current market value of bonds B	Current market value of stock nS	Current total value of firm V
30,000	29,250.0	15,250.0	44,500
35,000	35,000.0	9,500.0	44,500
40,000	39,000.0	5,500.0	44,500
45,000	42,562.5	1,937.5	44,500

Figure 11.3: DISTRIBUTION OF CORPORATE VALUES UNDER ALTERNATIVE CAPITAL STRUCTURES. Numbers are based on Fig. 11.2.

The above example is unrealistically simple because the firm in question is merely a holding company. For more complex firms, the total value has to be indirectly estimated as the total market value of all the securities.

Conflicts between stockholders and bondholders

Options and corporate securities have one important difference: A firm can change the capital structure, while an option's terms cannot be changed after it is issued. This means parameters such volatility, dividend, and strike price are under partial control of the firm's stockholders.

Suppose XYZ issues 15 more bonds with the same terms in order to buy back stock. The total debt is now $X = 45,000$. Figure 11.3 says the total market value of the bonds should be \$42,562.5. The *new* bondholders therefore pay $42562.5 \times (15/45) = 14187.5$, which is used by XYZ to buy back some shares. The remaining stock is worth \$1,937.5. The stockholders therefore gain

$$14187.5 + 1937.5 - 15250 = 875.$$

The *original* bondholders lose an equal amount of

$$29250 - \frac{30}{45} \times 42562.5 = 875. \tag{11.4}$$

This simple calculation demonstrates the inherent conflicts of interest between stockholders and bondholders.

As another example, suppose the stockholders distribute \$14,833.3 cash dividends by selling $(1/3) \times 1000$ Merck shares. The stockholders now have \$14,833.3 in cash plus a call on $(2/3) \times 1000$ Merck shares with a total strike price of $X = 30000$. This is equivalent to owning two thirds of a call on 1,000 Merck shares with a total strike price of \$45,000. Since 1,000 such calls are worth \$1,937.5 from Fig. 11.2, the total market value of XYZ's stock is $(2/3) \times 1937.5 = 1291.67$. The market value of the bonds is hence $(2/3) \times 1000 \times 44.5 - 1291.67 = 28375$. As a result, the stockholders have a total gain of

$$14833.3 + 1291.67 - 15250 \approx 875,$$

and the bondholders watch their value drop from \$29,250 to \$28,375, a net loss of \$875.

Bondholders usually loathe the stockholders taking unduly risky investments. The option theory can explain it by noticing that higher volatility increases the likelihood that the call option will be exercised, to the financial detriment of the bondholders.

			—Call—		—Put—	
Option	Strike	Exp.	Vol.	Last	Vol.	Last
Microsoft	55	Apr	65	16 3/4	52	1/8
71 1/8	60	Apr	556	11 3/4	39	1/8
71 1/8	65	Apr	302	7	137	3/8
71 1/8	70	Apr	1543	3 1/8	162	1 1/2

Figure 11.4: MICROSOFT OPTION QUOTATIONS FROM *The Wall Street Journal*, MARCH 21, 1995. This table is extracted from Fig. 7.4.

Subordinated debts

Suppose XYZ adds **subordinated** (or **junior**) **debt** with face value X_j to its capital structure. The original debt, with a face value of X_s, then becomes the **senior debt** and takes priority over the subordinated debt in case of default. Let both debts have the same maturity. The following table shows the revised payoff.

	$V^* \leq X_s$	$X_s < V^* \leq X_s + X_j$	$X_s + X_j < V^*$
Senior debt	V^*	X_s	X_s
Junior debt	0	$V^* - X_s$	X_j
Stock	0	0	$V^* - X_s - X_j$

The above table demonstrates that subordinated debt has the same payoff as a portfolio consisting of a long X_s call and a short $X_s + X_j$ call, a bull spread, in a word.

11.2.2 Warrants

Warrants give their holders the right to buy shares of the underlying stock from the corporation. A corporation issues warrants against its own stock. Most warrants have terms between five and ten years although perpetual warrants exist. Unlike a call, *new* shares are issued when a warrant is exercised. Warrants are typically protected against stock splits and cash dividends.

Consider a simple corporation with n shares of stock and m European warrants. Each warrant can be converted into one share of newly issued stock upon payment of the strike price X. The total value of the corporation is therefore $V = nS + mW$ with W denoting the current value of each warrant. At expiration, if it becomes profitable to exercise the warrants, the value of each warrant should be equal to the value of the newly issued stocks minus the cost of the conversion, that is,

$$W^* = \frac{1}{n+m}(V^* + mX) - X = \frac{1}{n+m}(V^* - nX),$$

where V^* denotes the total value of the corporation just prior to the conversion.

It will be optimal to exercise the warrants if and only if $V^* > nX$. One can readily see that a European warrant is like a European call on one $(n+m)$th of the total value of the corporation with a strike price of $Xn/(n+m)$. Hence, a European warrant is equivalent to $n/(n+m)$ European calls on one nth of the total value of the corporation (or $S+(m/n)W$) with a strike price of X. So,

$$W = \frac{n}{n+m} C(W),$$

where $C(W)$ is the Black-Scholes formula for European calls but with the stock price S replaced by $S + (m/n)W$ and the volatility σ denoting that of the total value, not just the stock price. The value W can be numerically solved. In the above analysis, it was assumed tacitly that all the warrant holders exercise simultaneously.

11.2.3 Callable bonds

A corporation issues callable bonds because, if future interest rates fall or the corporation's financial situation improves, it can refinance the debts under better terms. Consider a corporation with two classes of obligations: n shares of common stock and a single issue of callable bonds. The bonds have an aggregate face value of X. The stockholders have the right to call the bonds at any time for a total price of X_c. The corporation pays no dividends. Now, whenever the bonds are called prior to maturity, the payoff to the stockholders is $V - X_c$. Under these conditions, the stock is equivalent to an American call on the total value of the firm with a strike price of X_c prior to expiration and X at expiration. In a sense, the bondholders own the bonds and sell the bond issuer the call options.

11.2.4 Convertible bonds

A **convertible bond** is like an ordinary bond except that they can be converted into new shares at the discretion of its owner. Consider a non-dividend-paying corporation with two classes of obligations: n shares of common stock and m zero-coupon convertible bonds. Each convertible bond can be converted into k newly issued shares at maturity (k is called the **conversion ratio**). If the bonds are not converted, their holders will receive X at maturity. Assume the corporation value follows a continuous path without jumps for the analysis.

The bondholders will own a fraction $\lambda \equiv (mk)/(n+mk)$ of the firm if conversion is chosen (λ is called the **dilution factor**). Hence, it makes sense to convert only if the part of the total market value of the corporation due the bondholders after conversion, λV^*, exceeds X, i.e., $V^* > X/\lambda$. The payoff of the convertible bond at maturity can be described as follows: (1) if $V^* \leq X$, the payoff is V^*; (2) if $X < V^* \leq X/\lambda$, the payoff is X since it will not be converted; (3) if $X/\lambda < V^*$, the payoff is λV^*.

Convertible bonds with call provisions

Many convertible bonds contain call provisions. When the bonds are called, their holders can either convert the bond or redeem it at the call price. The call strategy intends to minimize the value of the bonds.

Consider the same corporation again. In particular, the total face value of the bonds is X, the total call price is P, and $P \geq X$. The bondholders will own a fraction λ of the firm after conversion. Let V^* denote the firm value at maturity. We first argue that it is not optimal to call the bonds when $\lambda V < P$ as follows. Not calling the bonds leaves the bondholders at maturity with a value of V^*, X, or λV^*, as the table below shows.

	$V^* < X$	$X \leq V^* < X/\lambda$	$X/\lambda \leq V^*$
Immediate call (present value)	P	P	P
No call throughout (future value at maturity)	V^*	X	λV^*
No call throughout (present value)	V	$\text{PV}(X)$	λV

The present values in all three cases are either less than or equal to P. Calling the bonds immediately is hence not optimal.

We now argue that it is not optimal to call the bonds *after* $\lambda V = P$ happens. Calling the bonds leaves the bondholders with λV^* at maturity. If the bonds are not called now when $\lambda V = P$, the bondholders' terminal wealth is tabulated below.

	$V^* < X$	$X \leq V^* < X/\lambda$	$X/\lambda \leq V^*$
No call throughout	V^*	X	λV^*
Call sometime *in the future*	λV^*	λV^*	λV^*

Not calling the bonds now hence may result in a higher terminal value for the bondholders than immediate call. Hence, waiting is not optimal.

The optimal call strategy is therefore to call the bonds the first time $\lambda V = P$ happens. More general settings with coupon interests and dividends will be covered in §15.4.8.

11.3 Barrier Options

Options whose payoff depends on whether the underlying asset's price reaches a certain level are called **barrier options**. For example, a **knock-out option** is like an ordinary European option except that it ceases to exist if a certain barrier H is reached by the price of its underlying asset. A call knock-out option is sometimes called a **down-and-out option** if $H < X$. A put knock-out option is sometimes called an **up-and-out option** when $H > X$. A **knock-in option**, in contrast, comes into existence if a certain barrier is reached. A **down-and-in option** is a call knock-in option that comes into existence only when the barrier is reached and $H < X$. An **up-and-in option** is a put knock-in option that comes into existence only when the barrier is reached and $H > X$. Barrier options have been traded in the U.S. since 1967 and are probably the most popular among the over-the-counter options [326, 794].

The value of a European down-and-in call is

$$S e^{-q\tau} \left(\frac{H}{S} \right)^{2\lambda} N(x) - X e^{-r\tau} \left(\frac{H}{S} \right)^{2\lambda-2} N\left(x - \sigma\sqrt{\tau} \right) \tag{11.5}$$

where

$$\begin{aligned} x &= \frac{\ln\left(H^2/(SX) \right) + \left(r - q + \sigma^2/2 \right) \tau}{\sigma\sqrt{\tau}} \\ \lambda &= \frac{r - q + \sigma^2/2}{\sigma^2} \end{aligned}$$

given $S \geq H$ and the stock's dividend yield q (see Fig. 11.5). A European down-and-out call can be priced via the **in-out parity** (see Comment 11.3.1). The value of a European up-and-in put is

$$X e^{-r\tau} \left(\frac{H}{S} \right)^{2\lambda-2} N\left(-x + \sigma\sqrt{\tau} \right) - S e^{-q\tau} \left(\frac{H}{S} \right)^{2\lambda} N(-x)$$

given $S \leq H$. A European up-and-out call can be priced via the in-out parity (see Exercise 11.3.2). The above formulae are due to Merton [572]. See [139] and Exercise 17.3.5 for proofs.

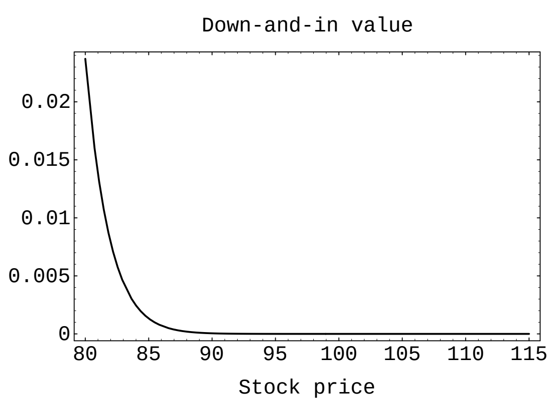

Figure 11.5: VALUE OF DOWN-AND-IN OPTION. Plotted is the down-and-in option value as a function of the stock price with barrier $H = 80$. The other parameters are identical to those used for the call in Fig. 7.2: $X = 95$, $\sigma = 0.25$, $\tau = 1/12$, and $r = 0.08$. Note the dramatic difference between the two plots.

The binomial model can be applied to price barrier options. As the binomial tree algorithm works backward in time, it checks if the barrier price is reached by the underlying asset and, if so, replaces the option value with an appropriate value (see Programming assignment 11.3.6). In many cases in practice, the barrier is the price at the end of the trading day, and the algorithm should reflect that.

11.3.1 Bonds with safety covenants

Bonds with safety covenants can be evaluated with the help of knock-out options. Suppose a firm is required to pass its ownership to the bondholders if its value falls below a specified barrier H which may be a function of time. The bondholders therefore receive V the first time the firm's value falls below H. At maturity, the bondholders receive X if $V > X$ and V if $V < X$, where X is the par value. The value of the bonds therefore equals that of the firm minus a down-and-out option with a strike price X and barrier H.

11.3.2 Exponential barrier

Consider the generalized barrier option with the barrier $H(t) = He^{-\rho\tau}$, where $\rho \geq 0$ and $H \leq X$. The standard barrier option corresponds to $\rho = 0$. The value of a European down-and-in call is

$$S\left(\frac{H(t)}{S}\right)^{2\lambda} N(x) - Xe^{-r\tau}\left(\frac{H(t)}{S}\right)^{2\lambda-2} N\left(x - \sigma\sqrt{\tau}\right),$$

where

$$\begin{aligned} x &= \frac{\ln\left(H(t)^2/(SX)\right) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}} \\ \lambda &= \frac{r - \rho + \sigma^2/2}{\sigma^2} \end{aligned}$$

given $S \geq H(t)$. This result is due to Merton [572, §8.9].

11.3.3 Other types of barrier options

Barrier options can have other variations [138]. If the barrier is active only during an initial period, the option is called a **partial barrier option**, while if the barrier is active only during the latter part of the option's life, it is called a **forward starting barrier option**. **Double barrier options** have two barriers. In **rolling options**, a sequence of barriers is specified, and, for calls (puts), the strike price is lowered (raised) each time a barrier is hit, and the option is knocked out at the last barrier.

11.4 Interest Rate Caps and Floors

Most exchange-traded options are short-term, while the risk exposure to be covered may span several years. A long-term floating-rate debt presents such a problem. Here, the borrower is concerned about a rate rise, and the lender may be worried about a rate decline. They can seek protection in **interest rate caps** and **floors**, respectively.

Interest rate caps and interest rate floors are over-the-counter options. The writer of a cap pays the purchaser each time the contract's **reference rate** is above the contract's **ceiling rate** or **cap rate** on a settlement date. A cap therefore provides a *multi-period* hedge against increases in interest rates. The writer of a floor, in contrast, pays the holder each time the contract's reference rate is below the contract's **floor rate** on a settlement date. Unlike stock options, caps and floors are settled in cash. The predetermined interest rate level such as the cap rate and the floor rate are called the **strike rate** [283]. The premium is expressed as a percentage of the notional principal on which the cap or floor is written. For example, for a notional principal of $10 million and 20 basis points as the premium, the cost is $10 \times 20/10000 = 0.02$ million. The full premium is usually paid up front.

This is how the cap works; the floor works similarly. At each settlement date, the cap holder receives

$$\max(\textsf{reference rate} - \textsf{cap rate}, 0) \times \textsf{notional principal} \times t, \tag{11.6}$$

where t is the length of the payment period. For example, if the reference rate is the six-month LIBOR, then t is typically either 181/360 or 184/360 since LIBOR uses the "actual/360" day count convention. The "actual/360" convention is also called **Banker's Rule** [479].

Comment 11.4.1 LIBOR (London Interbank Offered Rate) refers to the lending rates on short-term loans the *offered* interest rate on U.S. dollar deposits (**Eurodollars**) between large banks in London. Many short-term debts and floating-rate loans are priced off LIBOR in that the interest rates are quoted at a fixed margin above LIBOR. Differences between the LIBOR rate and the domestic rate are due to risk, government regulations, and taxes [666]. Although non-U.S. dollar LIBORs such as German mark LIBOR are also quoted [557], we shall limit ourselves to Eurodollars. See Fig. 11.6 for sample quotations on LIBOR rates from *The Financial Times*. □

	One month	Three months	Six months	One year
$ LIBOR FT London Interbank Fixing	61/16	61/8	61/8	63/16

Figure 11.6: LIBOR RATE QUOTATIONS FROM *The Financial Times*, MAY 19, 1995. They can be found under *World Interest Rates*.

The payoff in (11.6) denotes a European call on the interest rate with a strike price equal to the cap rate. Hence, a cap can be seen as a package of European calls (**caplets**) on the underlying interest rate. Similarly, a floor can be seen as a package of European puts (**floorlets**) on the underlying interest rate. Like ordinary options, the factors that determine the fair value of such a package of options are the current interest rate, the cap or floor rate (strike price), the volatility of the reference rate, and the time to each cash settlement [557].

Interest rate caps impose an upper limit on the cost of floating-rate debts. Suppose a firm issues a floating-rate note, paying six-month LIBOR plus 90 basis points. The firm's financial situation cannot allow paying an annual rate beyond 11%. Hence, it can purchase an interest rate cap with a cap rate of 10.1%. Thereafter, any time the rate moves above 10.1% and the firm pays more than 11%, the excess will be compensated exactly by the dealer who sells the cap. A cap effectively limits the exposure to interest rate risk. Similarly, interest rate floors place a floor on the interest income from a floating-rate asset. A firm can also buy an interest rate cap and simultaneously sell an interest rate floor to create an **interest rate collar**. When the reference rate rises above the cap rate, the firm is compensated by the cap seller the difference. When the reference rate dips below the floor rate, the firm pays the floor purchaser the difference. The interest rate cost is therefore bounded between the floor rate and the cap rate.

11.5 Stock Index Options

A stock index is a mathematical expression of the value of a portfolio of stocks. The New York Stock Exchange Composite Index (ticker symbol NYA), for example, is a weighted average of the prices of all the stocks on the New York Stock Exchange (NYSE), the weights being proportional to the total market values of their outstanding shares. Buying the index is equivalent to buying a portfolio of all the common stocks traded on the NYSE. The Dow Jones Industrial Average, Standard & Poor's 100 Index (S&P 100 Index, ticker symbol OEX), Standard & Poor's 500 Index (S&P 500 Index, ticker symbol SPX), and Major Market Index (ticker symbol XMI) are four more examples of popular stock indexes. The SPX is plotted in Fig. 11.7. The SPX, the OEX, and the NYA are **capitalization-weighted** averages, whereas the Dow Jones Industrial Average and the Major Market Index are **price-weighted** averages [88]. Capitalization-weighted indexes are also called (**market**) **value-weighted** indexes. A third, less popular, weighting method is **geometric weighting**. In this method, every stock has the same influence on the index. The Value Line Index (ticker symbol VLE) is a geometrically weighted index [557]. Stock indexes are usually not adjusted for cash dividends [422].

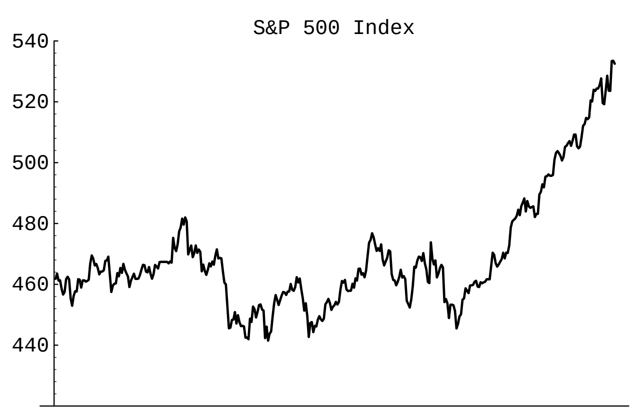

Figure 11.7: DAILY S&P 500 INDEX FROM AUGUST 30, 1993 TO JUNE 2, 1995. Non-trading dates have been deleted.

Mathematically, a price-weighted index is calculated as $\sum_i P_i/\alpha$, where P_i is the price of stock i in the index and α is an adjustment factor that takes care of stock splits, stock dividends, bankruptcies, mergers, etc. so that the index is comparable over time. A capitalization-weighted index is calculated as $\sum_i N_i P_i/\beta$, where N_i is the number of outstanding shares of stock i and β is the base period value. Heavily capitalized companies thus carry more weights. A geometrically weighted index is calculated as

$$I(t) = \prod_{i=1}^{n} \left(\frac{P_i(t)}{P_i(t-1)} \right)^{1/n} I(t-1),$$

where n is the number of stocks in the index, $I(t)$ is the index value on day t, and $P_i(t)$ is the price of stock i on day t.

Stock indexes differ in the stock composition in addition to their weighting methods. For instance, the Dow Jones Industrial Average is an index of 30 blue-chip stocks, while the S&P 500 Index is an index of 500 listed stocks from three exchanges. Nevertheless, the correlations between their returns are extremely high (see Fig. 11.8). More stock indexes can be found in [88, 302, 422]. See Fig. 11.9 for sample quotations on stock indexes.

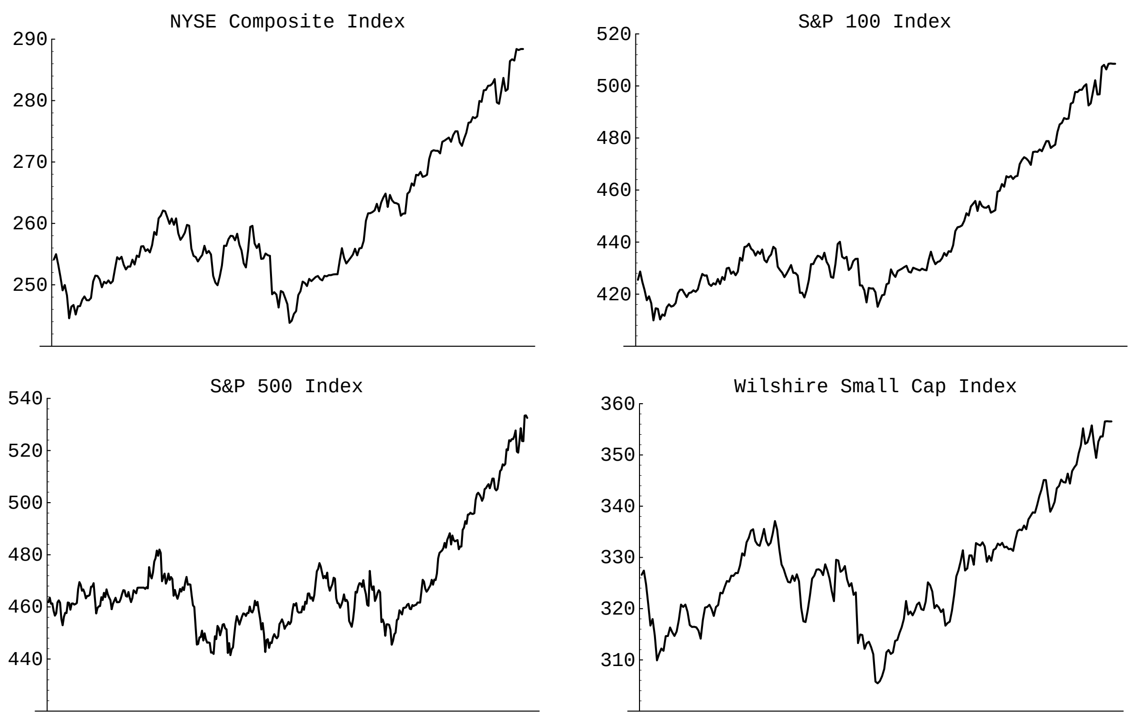

Figure 11.8: Four daily stock indexes from June 15, 1994 to June 5, 1995. Non-trading dates have been deleted.

An index option is an option on an index value. It can thus be used to speculate on the price movement of a diversified portfolio of stocks. Stock index options are settled in cash; only the cash *difference* between the index's current market value and the strike price is exchanged when the option is exercised. Some stock index options have maturities of up to two years, such as the SPX option. Options on the SPX, XMI and DJIA are European, while options on the OEX and NYA are American. See Fig. 11.10 for stock index option quotations. Options on stock market portfolios were first offered by insurance companies in 1977. Exchange-traded index options started in March 1983 with the trading of the OEX option at the Chicago Board Options Exchange [302].

RANGES FOR UNDERLYING INDEXES Monday, March 20, 1995	High	Low	Close	Net Chg.	From Dec. 31	%Chg.
S&P 100 (OEX)	465.88	464.31	465.42	+0.57	+36.79	+8.6
S&P 500 -A.M.(SPX)	496.61	495.27	496.14	+0.62	+36.87	+8.0
		...				
Russell 2000 (RUT).	257.87	257.28	257.83	+0.51	+7.47	+3.0
		...				
Major Mkt (XMI)...	435.14	432.82	434.90	+1.93	+34.95	+8.7
		...				
NYSE (NYA).......	268.36	267.68	268.05	+0.21	+17.11	+6.8
Wilshire S-C (WSX)	332.50	331.69	332.11	+0.16	+11.05	+3.4
		...				
Value Line (VLE)...	474.41	473.46	474.12	+0.58	+21.59	+4.8
		...				

Figure 11.9: INDEX QUOTATIONS FROM *The Wall Street Journal*, MARCH 21, 1995. They can be found under *Index Options Trading.* The stock market's spectacular rise in 1995–1999 can be seen by comparing this table with the one in Fig. 8.1.

The **size** of a stock index option contract is the dollar amount equal to \$100 times the index. A May OEX put with a strike price of \$380 costs $100 \times (5/16) = 31.25$ dollars from the data in Fig. 11.10. This particular put is out of the money.

One of the primary uses of stock index options is hedging large diversified portfolios. NYA puts, for example, can be used to protect a portfolio composed primarily of NYSE-listed securities against market declines. The alternative approach of buying puts for individual stocks would be much more expensive by Theorem 8.8.1 and cumbersome. Stock index options can also be used to create a long position in a portfolio of equities by the put-call parity. This synthetic security is easier to implement than buying individual stocks.

Valuation of stock index options typically relies on the Black-Scholes option pricing model with continuous dividend yields. This model actually approximates a broad-based stock market index better than individual stocks.[1] The formulae for European options on stock indexes thus appear in (9.28).

The cash settlement feature may lead to some risks to American option holders and writers. Since the exact amount to be paid when an option is exercised is determined by the closing price, there is uncertainty called the **exercise risk**. Consider an investor who is short a call on the OEX and long the appropriate amount of common stocks that comprise the index. If the call is exercised prematurely, the writer will be notified the *next* business day and pay cash based on today's closing price. Since the writer will still be holding the stocks and the index may open at a price different from today's closing price, those stocks may not fetch the same value as today's closing index value. Were the option settled in stocks, the writer would simply deliver the stocks the following business day without

[1] Technically speaking, it is incorrect to assume both the stock index and its individual stock prices satisfy the Black-Scholes option pricing model, because the sum of lognormal random variables is not lognormally distributed. But it works reasonably well in practice.

	Strike	Vol.	Last	Net Chg.	Open Int.
CHICAGO					
...					
S&P 100 INDEX(OEX)					
May	380p	8	5/16	...	897
Apr	385p	50	1/8	...	1,672
Apr	390p	335	1/8	...	4,406
...					
Apr	410c	12	57	+12	37
...					
Jun	490c	42	13/4	−1/4	719
Call vol. ...		75,513	Open Int. ...		415,627
Put vol. ..		104,773	Open Int. ...		447,094
S&P 500 INDEX-AM(SPX)					
Jun	350p	15	1/8	−1/8	2,027
Jun	375p	8	1/4	−1/8	5,307
Apr	400p	50	1/16	...	5,949
...					
Apr	405c	2	913/4	...	...
...					
Apr	490c	470	11	+3/8	23,903
Apr	490p	1,142	3	−3/8	14,476
May	490c	103	137/8	−5/8	3,086
May	490p	136	53/8	...	13,869
Jun	490c	2,560	167/8	+3/4	9,464
Jun	490p	333	67/8	−3/8	8,288
...					
Call vol. ...		34,079	Open Int. ...		656,653
Put vol. ...		59,582	Open Int. ...		806,961
AMERICAN					
...					
MAJOR MARKET(XMI)					
Jun	325p	20	1/16	−3/16	45
Apr	350p	300	1/16	...	400
...					
Call vol.		1,913	Open Int.		6,011
Put vol.		4,071	Open Int.		27,791
...					

Figure 11.10: Index option quotations from *The Wall Street Journal*, March 21, 1995. They can be found under *Index Options Trading*.

worrying about the change in the index value.

11.6 Foreign Exchange Options

In the **spot market** (or **cash market**) where prices are for immediate payment and delivery, exchange rates between the U.S. dollar and foreign currencies are generally quoted using the **European terms**. This method measures the amount of foreign currency needed to buy one U.S. dollar, i.e., foreign currency units per dollar. The **reciprocal of European terms** measures the U.S. dollar value of one foreign currency unit [88]. For example, if the European-terms quote is .63 British pounds per $1 (£.63/$1), then the reciprocal-of-European-terms quote is $1.587 per British pound ($1/£.63 or $1.587/£1). The reciprocal of European terms is also called the **American terms** [666]. A trader who buys foreign currency turns dollars into foreign currency. A trader who sells foreign currency turns foreign currency into dollars. Figure 11.11 provides a snapshot of spot exchange rates as of January 7, 1999. The **spot exchange rate** is the rate at which one currency can be exchanged for another, typically for settlement in two days [497]. Foreign exchange options are settled via delivery of the underlying currency.

EXCHANGE RATES
Thursday, January 7, 1999

	U.S. $ equiv.		Currency per U.S. $	
Country	Thu.	Wed.	Thu.	Wed.
	...			
Britain (Pound).....	1.6508	1.6548	.6058	.6043
1-month Forward	1.6493	1.6533	.6063	.6049
3-months Forward	1.6473	1.6344	.6071	.6119
6-months Forward	1.6461	1.6489	.6075	.6065
	...			
Germany (Mark)....	.5989	.5944	1.6698	1.6823
1-month Forward	.5998	.5962	1.6673	1.6771
3-months Forward	.6016	.5972	1.6623	1.6746
6-months Forward	.6044	.6000	1.6544	1.6666
	...			
Japan (Yen)........	.009007	.008853	111.03	112.95
1-month Forward	.009007	.008889	111.03	112.50
3-months Forward	.009008	.008958	111.02	111.63
6-months Forward	.009009	.009062	111.00	110.34
	...			
SDR...............	1.4106	1.4137	0.7089	0.7074
Euro...............	1.1713	1.1626	0.8538	0.8601

Figure 11.11: EXCHANGE RATE QUOTATIONS FROM *The Wall Street Journal*, JANUARY 8, 1999. They can be found under *Currency Trading*.

In Fig. 11.11, the German mark is a **premium currency** because the three-month forward exchange rate $.6016/DEM1 exceeds the spot exchange rate $.5989/DEM1. The German mark is therefore more valuable in the forward market than the spot market. In contrast, the British pound is a **discount currency**. **Forward exchange rates**, unlike spot exchange rates, are exchange rates for *deferred* delivery of a currency.

One of the primary uses of foreign exchange—or forex—options is to hedge **currency risk**. Consider a U.S. company expecting to receive 100 million Japanese yen in March. Since this company wants U.S. dollars, not Japanese yen, those 100 million Japanese yen will be exchanged for U.S. dollars. Although 100 million Japanese yen are worth 0.9007 million U.S. dollars as of January 7, 1999, they may be worth less or more than that in March. The company decides to use options to hedge against the depreciation of the yen against the dollar. From Fig. 11.12, since the **contract size** for the Japanese yen option is JPY6,250,000, the company decides to purchase $100,000,000/6,250,000 = 16$ puts on the Japanese yen with a strike price of \$.0088 and an exercise month in March. This gives the company the right to sell 100,000,000 Japanese yen for $100,000,000 \times .0088 = 880,000$ U.S. dollars. The options command $100,000,000 \times 0.000214 = 21,400$ U.S. dollars. The net proceeds per Japanese yen are hence $.88 - .0214 = 0.8586$ cent at the minimum.

		—Call—		—Put—	
		Vol.	Last	Vol.	Last
		...			
German Mark					59.31
62,500 German Marks-European Style.					
581/2	Jan	...	0.01	27	0.06
59	Jan	...	0.01	210	0.13
61	Jan	27	0.07	...	0.01
611/2	Jan	210	0.02	...	0.01

		—Call—		—Put—	
		Vol.	Last	Vol.	Last
		...			
Japanese Yen					88.46
6,250,000 J. Yen-100ths of a cent per unit.					
661/2	Mar	...	0.01	1	2.53
73	Mar	...	...	10	0.04
75	Mar	...	0.01	137	0.06
76	Mar	...	...	9	0.09
77	Mar	...	...	17	0.09
78	Mar	...	...	185	0.18
79	Mar	...	...	10	0.16
80	Mar	...	...	77	0.40
81	Mar	...	...	60	0.36
86	Jan	...	0.01	5	0.14
88	Mar	...	...	10	2.14
89	Mar	...	...	10	2.51
90	Feb	...	0.01	12	2.30
91	Feb	...	0.01	5	2.50
100	Mar	2	0.86	...	0.01
		...			

Figure 11.12: FOREIGN EXCHANGE OPTION QUOTATIONS FROM *The Wall Street Journal*, JANUARY 8, 1999. They can be found under *Currency Trading*.

The above option can be seen in an alternative way: It gives the holder the right to buy $6,250,000 \times 0.0088 = 55,000$ U.S. dollars for 6,250,000 Japanese yen. This is a call on 55,000 U.S. dollars with a strike price of $1/0.0088 \approx 113.6$ yen per dollar. A call to buy a currency is an insurance against the relative appreciation of that currency, and a put on a currency is an insurance against the relative depreciation of that currency. Put and call options on a currency are therefore symmetrical: A put to sell X_A units of currency A for X_B units of currency B is the same as a call to buy X_B units of currency B for X_A units of currency A.

A DEM/\$ call (a call contract to buy German marks for U.S. dollars) and a \$/£ call (a call contract to buy U.S. dollars for British pounds) does not a DEM/£ call make. Consider a call option on DEM1 for \$.71, a call option on \$.71 for £.452, and a call on DEM1 for

£.452. Suppose the U.S. dollar falls to DEM1/\$.72 and £1/\$1.60. The first option will net a profit of \$.01, but the second option will expire worthless. Since $0.45 = 0.72/1.6$, we know DEM1/£.45. The DEM/£ option will therefore also expire worthless. Hence, the portfolio of DEM/\$ and \$/£ calls is worth more than the DEM/£ call. This conclusion can be shown to hold in general (see Exercise 15.4.18). Options such as the DEM/£ call are called **cross-currency options** (from dollar's point of view).

A large proportion of the forex options is done in the over-the-counter market. One possible reason is that the homogeneity and liquidity of the underlying asset make it easy to structure custom made deals [302]. Exchange-traded foreign exchange options are available on the Philadelphia Stock Exchange and the Chicago Mercantile Exchange (CME) as well as many other exchanges [302, Fig. 7.3]. Most exchange-traded foreign exchange options are denominated in the U.S. dollar.

11.6.1 The Black-Scholes option pricing model

Under the Black-Scholes option pricing model, foreign exchange options can be valuated as stock index options. Let S denote the spot exchange rate in domestic/foreign terms. Use σ for the volatility of the exchange rate and $\hat{r}$ for the riskless rate of interest in the foreign country. Both r, the domestic interest rate, and $\hat{r}$ are assumed to be constants.

A foreign currency is analogous to a stock paying a known dividend yield because the owner of foreign currencies receives a "dividend yield" equal to $\hat{r}$ in the foreign currency. So the formulae derived in (9.28) apply with the role of dividend yield played by $\hat{r}$,

$$C = Se^{-\hat{r}\tau} N(x) - Xe^{-r\tau} N\left(x - \sigma\sqrt{\tau}\right) \tag{11.7}$$

$$P = Xe^{-r\tau} N\left(-x + \sigma\sqrt{\tau}\right) - Se^{-\hat{r}\tau} N(-x) \tag{11.7$'$}$$

where

$$x \equiv \frac{\ln(S/X) + \left(r - \hat{r} + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

The deltas of calls and puts are $e^{-\hat{r}\tau}N(x) \geq 0$ and $-e^{-\hat{r}\tau}N(-x) \leq 0$, respectively. Empirical evidence suggests that the Black-Scholes option pricing model produces acceptable results for major currencies such as the German mark, the Japanese yen, and the British pound [302].

Some pricing relationships

Many of the relationships in §8.2 continue to hold for foreign exchange options after the modifications required by the continuous dividend yield (the foreign interest rate). We first show that a European call satisfies

$$C \geq \max\left(Se^{-\hat{r}\tau} - Xe^{-r\tau}, 0\right). \tag{11.8}$$

Consider the following strategies:

	Initial investment	Value at expiration	
		$S_T > X$	$S_T \leq X$
Buy a call	C	$S_T - X$	0
Buy domestic bonds (face value \$$X$)	$Xe^{-r\tau}$	X	X
Total	$C + Xe^{-r\tau}$	S_T	X
Buy foreign bonds (face value 1 in foreign currency)	$Se^{-\hat{r}\tau}$	S_T	S_T

Hence, whichever case actually happens, the first portfolio is worth at least as much as the second and therefore cannot cost less. The above bound incidentally generalizes Exercise 8.3.3.

A European call's price may approach the lower bound in (11.8) as closely as may be desired (see Exercise 11.6.7). Hence, since the intrinsic value $S_t - X$ of an American call may exceed that lower bound, early exercise can be optimal.

Theorem 11.6.1 *American foreign exchange calls may be optimally exercised before expiration.* □

11.7 Compound Options

Compound options are options on options. There are naturally four basic types of compound options: a call on a call, a call on a put, a put on a call, and a put on a put. Formulae for compound options can be found in [422].

Section 11.2.1 showed that stock can be considered a call on the total value of the firm. Under such a framework, a stock option becomes a compound option. Renewable term life insurance can be considered a compound option, too: Paying a premium acquires the right to renew the contract for the next term. Thus the decision to pay a premium is an option on an option [557]. A **split-fee option** provides a window on the market at the end of which the buyer can decide whether to extend it up to the notification date or to let it expire worthless [46, 302]. If the spit-free option is extended up to the notification date, the second option can either expire or be exercised. The name comes from the fact that the user has to pay two fees to exercise the underlying asset [650].

Compound options are appropriate for situations where a bid, denominated in foreign currency, is submitted for the sale of equipment or plant. There are two levels of uncertainties: the winning of the bid and the currency risk (even if the bid is won, a depreciated foreign currency may make the deal unattractive). What is needed is an arrangement whereby the company can secure a foreign currency option against foreign currency depreciation *if* the bid is won. This is an example of **contingent foreign exchange option** [302]. A compound put option that grants the holder the right to purchase a put option in the future at prices that are agreed upon today solves the problem. Foreign exchange options are not ideal because they turn the bidder into a speculator if the bid is lost.

The binomial option pricing model can be applied with few complications. There are three conceptual levels of binomial trees here. The first level is the underlying asset's price process. The second level up is the price process of the option based on the underlying asset. This part was already covered. Finally, the top level is the price process of the compound option based on the price process of the underlying option; the computation is the same except that the underlying asset is no longer the stock, but the option.

11.8 Path-Dependent Derivatives

A European or American option's value depends only on the underlying asset's price regardless of how it gets there. It is **path-independent**. Some derivatives are **path-dependent** in that their terminal payoffs depend critically on the paths. As a consequence, the binomial tree may no longer combine (hence not computationally simple), and Monte Carlo methods become one of the few alternatives left.

11.8.1 Options on a function of time series

An option's settlement price can be based on a statistic computed on a time series. Let $S_0, \ldots, S_n$ denote the prices of the underlying asset over the life of the option, where S_0 is the known price at time zero and S_n is the price at expiration. The traditional call option, for instance, has a terminal value depending only on the last price, $\max(S_n - X, 0)$. An (arithmetic) **average-rate call** has a terminal value given by

$$\max\left(\frac{1}{n+1}\sum_{i=0}^{n} S_i - X, 0\right).$$

An **average-rate put**'s terminal value is defined similarly,

$$\max\left(X - \frac{1}{n+1}\sum_{i=0}^{n} S_i, 0\right).$$

Average-rate options are useful hedging tools for traders who will make a stream of purchases over a time period since the costs are likely to be linked to the average price of the materials (oil, lumber, etc.). Upon initiation, average-rate options cannot be more expensive than standard European options under the Black-Scholes option pricing model [480]. They are notoriously hard to price [657]. Average-rate options are also called **Asian options**.

The binomial tree for pricing average-rate options no longer combines. Take the terminal price $Suud$. Different paths to it such as $S \to Su \to Suu \to Suud$ and $S \to Sd \to Sdu \to Sduu$ lead to different option payoffs: $\max((S + Su + Suu + Suud)/4 - X, 0)$ and $\max((S + Sd + Sdu + Sduu)/4 - X, 0)$, respectively. In general, if there are n periods, then 2^n paths have to be constructed. This exponential complexity makes the algorithm impractical save for trees of very shallow depths (see Fig. 11.13). Average-rate options also satisfy certain put-call parity identities [95, 519].

If averaging is done *geometrically*, then the payoffs become

$$\max\left(\left(\prod_{i=0}^{n} S_i\right)^{1/(n+1)} - X, 0\right) \quad \text{and} \quad \max\left(X - \left(\prod_{i=0}^{n} S_i\right)^{1/(n+1)}, 0\right).$$

Interestingly, this version of average-rate option becomes computationally tractable. In practice, average-rate options almost exclusively utilize arithmetic averages [794].

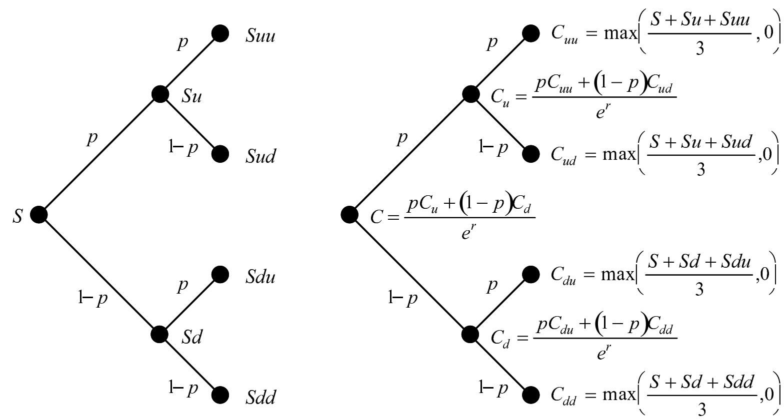

Figure 11.13: BINOMIAL TREE FOR AVERAGE-RATE (ASIAN) CALL. Here, p is the risk-neutral probability. Note that the tree grows exponentially in terms of the depth of the tree even though the tree for the stock prices can be drawn as a lattice.

Another class of options, **lookback options**, is similar except that the time series determines the **strike price** [192, 345, 738]. In particular, a lookback call option on the minimum has a terminal value of $\max(0, S_n - \min_{1 \leq i \leq n} S_i)$ and a lookback put option on the maximum has a terminal value of $\max(0, \max_{1 \leq i \leq n} S_i - S_n)$. One can also define lookback call and put options on the average. Such options are also called **average-strike options** [422].

Additional Reading

Consult [494] for alternative models for warrants, [446] for more theoretical analysis of callable convertible bonds, [326] for an analytical approach to pricing American barrier options, [302, Chapter 7] and [458, Chapter 11] for other kinds of foreign exchange options, [787] for the term structure of exchange rate volatility, [331, 422, 438, 650, 715] for more information on path-dependent derivatives, and [95, 333, 480, 518, 652, 657, 755] for average-rate option pricing. Finally, option pricing theory has applications in capital investment decisions [240, 584].