Chapter 12

Forwards, Futures, Futures Options, and Swaps

Contents

It does not matter if I speak;
the future has already been determined.
—Sophocles (496 B.C.–406 B.C.),
Oedipus Tyrannus [717]

This chapter covers more derivatives, financial contracts whose value depends on the value of the underlying asset or some index. Derivatives are essential to risk management, speculation, efficient portfolio adjustment, and arbitrage. Interest rate-sensitive derivative securities, being more complex, require a separate chapter, Chapter 21.

12.1 Introduction

Four types of derivatives stand out: futures contracts, forward contracts, single- and multi-period options, and swaps. **Futures contracts** and **forward contracts** are contracts for deferred delivery of the underlying assets. The underlying assets can be physical commodities such as corn, oil, live cattle, pork bellies, precious metals, and so on or financial instruments such as bonds, currencies, stock indexes, mortgage securities, other derivatives, and so on [88, 422]. Futures contracts are *standardized* instruments that trade on futures exchanges such as the Chicago Board of Trade (CBT) and the Chicago Mercantile Exchange (CME). Futures and forward contracts can be used for speculation, hedging, or to arbitrage between the spot and the deferred-delivery markets.

Futures and forward contracts are obligations on both the buyers and the sellers. Options, in contrast, are only binding on the sellers; the buyers, in contrast, have the right, but not the obligation, to take a position in the underlying asset. Such a right commands a premium. Options can be used to hedge downside risk, speculation, or arbitrage markets.

Swaps are relatively new derivative instruments. Like the forward contracts, swaps are traded outside of organized exchanges by financial institutions and their corporate clients. A swap is a contract between two parties to exchange cash flows in the future based on a preset formula. Typically, one party pays a *fixed* price to the other party in exchange for a market determined *floating* price. Swaps are used to reduce financing costs and to hedge risks. Interest rate swaps and foreign exchange forward contracts make up banks' major derivative holdings [50].

Take the foreign exchange market as an example. Three types of trades take place in that market: spot, forward, and swap. Spot trades involve an agreement on the spot exchange rate today for settlement in two days. Forward trades involve an agreement on forward exchange rates today for future settlement, usually one to 52 weeks. A swap is the sale (purchase) of a foreign currency with a simultaneous agreement to repurchase (resell) it sometime in the future. The difference between the sale price and the repurchase price is called the **swap rate** [666].

Trading derivatives can be risky. Unmonitored, it may run up huge losses—US$1.6 billion in the case of Orange County, California that led to its bankruptcy in December 1994 [464] and the security firms involved paying US$739 million in subsequent settlements. It was reported in 1997 that the value of an interest rate swap contract held by Sears was a minus US$382 million based on a notional principal of US$996 million [786]. J.P. Morgan in 1997 declared it had US$659 million in nonperforming assets, 90% of which were defaults from Asian derivative counterparties. Chase Manhattan held US$7.6 trillion in derivatives (notional amount) as of early 1998 [49]. UBS wrote off US$699 million due to investment in the almost-failed hedge fund, Long-Term Capital.

Throughout this chapter, r will denote the riskless interest rate. Other notations include, unless stated otherwise, the current time t, the maturity date of the derivative T, the remaining time to maturity $\tau \equiv T - t$ (all measured in years), the spot price S, the spot price at maturity S_T, the delivery price X, the forward or futures price F for a newly written contract, and the value of the contract f. A price with a subscript t usually refers to the price at time t. Continuous compounding will be assumed throughout this chapter. Transactions costs, margin requirements, and taxes will be ignored.

The relevant riskless interest rate for many arbitrageurs operating in the futures market is the repo rate. A **repo** (**sale and repurchase agreement** or **RP**) is an agreement where the owner of securities ("seller") agrees to sell them to a counterparty ("buyer") and buy them back at a slightly higher price later. The counterparty hence provides a loan. This loan has little risk since the lender keeps the securities if the seller fails to buy them back; the lender essentially runs a pawnshop [269]. From the lender's perspective, this agreement is called a **reverse repo**. Overnight repo rates are lower than the federal funds rate. A loan of more than one day is called a **term repo** [283]. The dollar interest is determined by

$$\text{dollar principal} \times \text{repo rate} \times \frac{\text{repo term}}{360}.$$

The Bank of England was the first central bank to introduce repos in 1830. The Federal Reserve uses the the repo market to influence short-term interest rates. When the Fed is doing repo, it is actually lending money, not borrowing it [730].

12.2 Forward Contracts

Forward contracts are for delivery of the underlying asset for a certain **delivery price** on a specific time in the future. As a concrete example, consider a corn farmer who enters into a forward contract with a food processor to deliver 100,000 bushels of corn for \$2.5 per bushel on September 27, 1995. Assume the cost of growing corn is \$2.0 per bushel. Such a contract benefits both sides, the farmer, because he is assured of a buyer at an acceptable price, and the processor, because knowing the cost of corn in advance helps reduce uncertainty in planning. If the spot price of corn rises on the delivery date, the farmer misses the opportunity of extra profits. On the other hand, if the price declines, the processor will be paying more than it would. Hence, a forward agreement limits the risk and potential rewards, ideal for hedging purposes.

Forward contracts involve a specific seller delivering an asset to a specific buyer at a fixed future price on a specific future date. Problems may arise if one of participants fails to perform. The food processor may go bankrupt, the farmer can go bust, the farmer might not be able to harvest 100,000 bushels of corn because of bad weather, or the cost of growing corn jumps. More importantly, whichever way the corn price moves, either the food processor or the farmer has an incentive to default. Even corporate giants like Coca-Cola and Westinghouse defaulted on their forward contracts [666].

12.2.1 Forward exchange rate

Figure 12.1 shows the spot exchange rate and forward exchange rates for the German mark. The forward German marks are at a premium to the spot rate in terms of \$/DEM because the forward exchange rates exceed the spot exchange rate. This may be due to lower inflation in Germany.

EXCHANGE RATES Monday, March 20, 1995				
	U.S. \$ equiv.		Currency per U.S. \$	
Country	Mon.	Fri.	Mon.	Fri.
Germany (Mark)..	.7126	.7215	1.4033	1.3860
30-Day Forward.	.7133	.7226	1.4019	1.3839
90-Day Forward.	.7147	.7242	1.3991	1.3808
180-Day Forward	.7171	.7265	1.3945	1.3765

Figure 12.1: GERMAN MARK EXCHANGE RATE QUOTATIONS FROM *The Wall Street Journal*, MARCH 21, 1995.

Along with foreign exchange options, forward contracts provide an avenue to hedging currency risk. Consider a U.S. company expecting to receive DEM10 million in three-months' time. By using a forward sale at the three month forward exchange rate of \$.7147/DEM1, the firm is certain to receive exactly US\$7,147,000 in three months' time. Compared to hedging using foreign exchange options, the forward hedge insulates the firm from any movement in exchange rates whether that movement is favorable or unfavorable.[1]

Spot and forward exchange rates

The following arbitrage argument was first formulated by Keynes in 1923 to show that there is definite relationship between the spot and forward exchange rates [263]. Let S denote the spot exchange rate of the local currency in terms of the foreign currency (domestic/foreign) and F denote the forward exchange rate one year from now. As usual, no money changes hand in entering into a forward contract. Use r_f and r_l to denote the annual interest rates of the foreign currency and the local currency, respectively. Arbitrage opportunities will arise unless these four quantities satisfy a particular equation known as the (covered) **interest rate parity**.

A holder of the local currency can either (a) lend the money in the domestic market for the annual rate of return r_l and receive e^{r_l} one year from now or (b) convert the local currency in the spot market for the foreign currency, lend for one year in the foreign market, and convert the foreign currency into the local currency at the fixed forward exchange rate in the future, F, by selling forward the foreign currency now. One unit of local currency will hence become Fe^{r_f}/S one year from now. If it exceeds e^{r_l}, an arbitrage profit can result from borrowing money in the domestic market and lending it in the foreign market.

[1]The same mechanism applies to speculation. Interested reader may consult [707, Chapter 2] for Keynes's disastrous currency speculations in 1920.

Conversely, if it is exceeded by e^{r_l}, then an arbitrage profit can result from borrowing money in the foreign market and lending it in the domestic market. Thus, we have established

$$\frac{F}{S} = \frac{e^{r_l}}{e^{r_f}} = e^{r_l - r_f} \tag{12.1}$$

called the **interest rate parity theorem**. It is a straightforward exercise to check that

$$\frac{F}{S} = \frac{1 + r_l}{1 + r_f} \tag{12.2}$$

under periodic compounding. The simultaneous buying the foreign currency in the spot market and selling forward the foreign currency in strategy (b) is called **currency swap**. The interest rate parity theorem says, if the domestic interest rate is higher than the foreign rate, the foreign country's currency will be selling at a premium in the forward market. Conversely, if the domestic interest rate is lower, the foreign currency will be selling at a discount in the forward market.

A close connection exists between the forward exchange rate and the expected future spot exchange rate. A trader's decisions in today's forward market reflect her expectations of the future spot exchange rate. In fact, if traders were indifferent to risk, the forward exchange rate would depend solely on the expectations about the future spot exchange rate thus [666],

$$\text{one-period forward exchange rate} = E[\,\text{spot exchange rate one period from now}\,].$$

This is called the **theory of uncovered interest rate parity** [497].

12.2.2 Forward price

The payoff from holding a forward contract at maturity is $S_T - X$ per unit of an asset (see Fig. 12.2). Contrast this with the case of call, $\max(S_T - X, 0)$. Forward contracts do not involve any initial cash flow, and the **forward price** is the delivery price which makes the forward contract worth zero initially. Formally, $f = 0$ when $F = X$. The delivery price cannot change, as it is written in the contract, but the forward price may change after the contract comes into existence. In other words, the value of a forward contract, f, is zero at the outset, and, thereafter, it will fluctuate with the spot price. Apparently, this value will be enhanced when the spot price climbs and depressed when the spot price declines. The forward price also varies with the maturity of the contract (see Fig. 12.1).

For example, a repo is a forward contract on a Treasury security. It has a zero initial value because the Treasury security is exchanged for its fair value in cash and the repurchase price is set to the forward price [456].

The underlying asset pays no income

Lemma 12.2.1 *For a forward contract on an underlying asset providing no income,*

$$F = Se^{r\tau}. \tag{12.3}$$

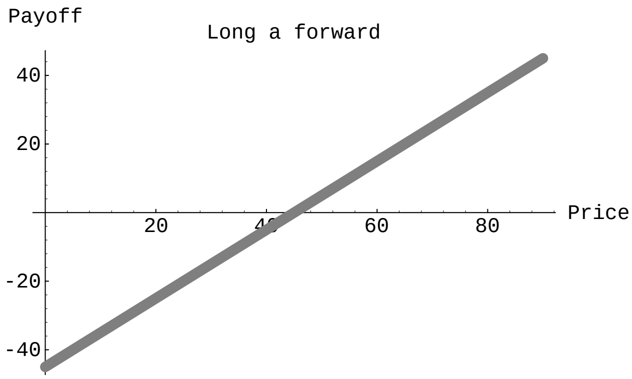

Figure 12.2: PAYOFF OF FORWARD CONTRACT. Shown is the payoff of a long forward contract with a delivery price of $45 at maturity.

PROOF: If $F > Se^{r\tau}$, an investor can borrow S dollars for τ years, buy the underlying asset, and short the forward contract with delivery price F. At maturity, the asset is sold for F, and $Se^{r\tau}$ is used to repay the loan, leaving an arbitrage profit of $F - Se^{r\tau} > 0$.

If $F < Se^{r\tau}$ instead, an investor can short the underlying asset, invest the proceeds for τ years, and take a long position in the forward contract with delivery price F. At maturity, the asset is bought for F to close out the short position, leaving a profit of $Se^{r\tau} - F$. □

Example 12.2.2 A new three-month forward contract on a six-month zero-coupon bond should command a delivery price of $Se^{r/4}$, where r is the annualized three-month riskless interest rate and S is the spot price of the bond. For instance, if $r = 6\%$ and $S = 970.87$, then the delivery price is $970.87 \times e^{0.06/4} = 985.54$. □

Example 12.2.3 A forward contract on a non-dividend-paying stock should satisfy (12.3) with F being the forward price and S the stock price. The delivery price in a forward contract initiated now and maturing in τ years would then be $Se^{r\tau}$. □

The forward price, as we mentioned before, may not maintain equality to the delivery price as time passes. In fact, the value of a forward contract providing no income at any time prior to maturity should be

$$f = S - Xe^{-r\tau}. \tag{12.4}$$

This can be verified by considering a portfolio of one long forward contract, cash amount $Xe^{-r\tau}$, and one short position in the underlying asset. The cash will grow to X at maturity, which can be used to take delivery of the forward contract. The delivered asset will then close out the short position. Since the value of the portfolio is zero at maturity, its present value must be zero.

Lemma 12.2.1 can be proved alternatively by by above identity because X must equal $Se^{r\tau}$ in order that $f = 0$. The same identity also implies that the value of a forward contract at maturity is $S - X$.

The underlying asset pays predictable income

Lemma 12.2.4 *For a forward contract on an underlying asset providing a predictable income with a present value of I,*

$$F = (S - I)\, e^{r\tau}. \tag{12.5}$$

PROOF: If $F > (S-I)\, e^{r\tau}$, an investor can borrow S dollars for τ years, buy the underlying asset, and short the forward contract with delivery price F. At maturity, the asset is sold for F, and $(S-I)\, e^{r\tau}$ is used to repay the loan, leaving an arbitrage profit of $F-(S-I)\, e^{r\tau} > 0$.

If $F < (S-I)\, e^{r\tau}$, an investor can short the underlying asset, invest the proceeds for τ years, and take a long position in the forward contract with delivery price F. At maturity, the asset is bought for F to close out the short position, and a profit of $(S-I)\, e^{r\tau} - F > 0$ is realized. □

Example 12.2.5 A ten-month forward contract on a \$50 stock which pays a dividend of \$1 every 3 months should be priced at

$$\left(50 - e^{-r_3/4} - e^{-r_6/2} - e^{-3\times r_9/4}\right) e^{r_{10}\times(10/12)},$$

where r_i is the annualized i-month interest rate. The forward price, for instance, is \$49.50 when $r_3 = r_6 = r_9 = r_{10} = 6\%$. This example illustrates how to generalize the results to non-flat yield curves. □

The value of a forward contract providing a predictable income with a present value of I equals

$$f = (S - I) - Xe^{-r\tau}. \tag{12.6}$$

This can be confirmed by considering a portfolio of one long forward contract, cash amount $Xe^{-r\tau} + I$, and one short position in the underlying asset. The cash will grow to X at maturity, after paying the dividends to the original stockholder. There is sufficient fund to take delivery of the forward contract, which then offsets the short position. Since the value of the portfolio is zero at maturity, its value must be zero at present.

The underlying asset pays continuous dividend yields

A continuous dividend yield means dividends are paid out continuously at an annual rate of q. The value of a forward contract at any time prior to maturity equals

$$f = Se^{-q\tau} - Xe^{-r\tau}. \tag{12.7}$$

This can be verified by considering a portfolio of one long forward contract, cash amount $Xe^{-r\tau}$, and a short position in $e^{-q\tau}$ units of the underlying asset. All dividends are paid for by shorting additional units of the underlying asset. Hence, the cash will grow to X at maturity, and the short position will grow to exactly one unit of the underlying asset. There is sufficient fund to take delivery of the forward contract, which then offsets the short position. Since the value of the portfolio is zero at maturity, its present value must be zero. One consequence of (12.7) is that the forward price is

$$F = Se^{(r-q)\,\tau}. \tag{12.8}$$

Forward price and value of forward contract

All the above cases satisfy the following relationship to prevent arbitrage profits,

$$f = (F - X)\, e^{-r\tau}. \tag{12.9}$$

Consider the case $f > (F - X)\, e^{-r\tau}$ first. An arbitrage opportunity can be created by buying one forward contract with delivery price F and shorting one forward contract with delivery price X, both maturing time τ from now. This generates an initial cash inflow of f since the first contract has zero value by the definition of F. The cash flow is

$$(S_T - F) + (X - S_T) = -(F - X)$$

at maturity. Hence, a cash flow with a present value of $f - (F - X)\, e^{-r\tau} > 0$ has been assured. The remaining case $f < (F - X)\, e^{-r\tau}$ is left as an exercise.

12.3 Futures Contracts

The Chicago Board of Trade developed the futures contracts in 1865. Futures contracts are different from forward contracts in several ways. First, they are usually traded on a central exchange rather than over-the-counter markets, leading to more efficient and accurate price determination. Secondly, the establishment of a clearinghouse means sellers and buyers no longer face each other; instead, as in the options market, the clearinghouse acts as a seller to all buyers and a buyer to all sellers. Hence, credit risk inherent in forward contracts is minimized, and investors can concentrate on the price movements. Thirdly, exchange-listed futures contracts are standardized instruments. They specify the delivery of a specific quantity of a specific commodity that meets quality standards at predetermined places and dates. This is in sharp contrast with forward contracts for which the only requirement is mutual agreedment. Fourthly, gains and losses of futures contracts are **marked to market** daily. Hence, the account is adjusted at the end of each trading day based on the **settlement price** to reflect the investor's gain or loss. The settlement price is the average of the prices at which the contract traded immediately before the bell signaling the end of trading for the day.

The **contract size**, or simply the **size**, of a futures contract is the amount of the underlying asset to be delivered under the contract. For instance, it is 5,000 bushels for the corn futures contracts on the CBT and one million U.S. dollars for the Eurodollar futures contracts on the CME.

Example 12.3.1 The Chicago Board of Trade July wheat contract specifies, among others, that the wheat delivered be 5,000 bushels of either no. 2 soft red wheat, no. 2 hard red winter wheat, no. 2 dark northern spring wheat, or no. 1 northern spring wheat on a date in the month of July chosen by the seller [699]. Consult [88] for more information. □

A position can be **closed out** or **offset** by entering into a reversing trade to the original one. An investor who is long one November soybeans futures contract can close out the position by shorting one November contract. The ease of offsetting a position is one benefit of having a clearinghouse, which simply cancels offsetting positions from its book. Most futures contracts are closed out in this way rather than having the underlying asset delivered. In contrast, forward contracts are meant for delivery.

Example 12.3.2 Consider a farmer who sold short corn futures and later realizes that the cost of growing the corn rises. He can offset his short futures position to reduce the losses. Consider another farmer who faces the problem that the crop is going to be different from the 100,000-bushel projection. Since corn futures trade in 5,000-bushel pieces, 20 contracts were sold to cover the anticipated 100,000-bushel crop. If the crop now appears to be only 80,000 bushels, the farmer can offset four of those contracts. This type of flexibility is not available to forward contracts. □

Now that price changes in the futures contract are settled daily, the spot price rather than the initial futures price is paid on the delivery date: The difference between them has been paid for in installments throughout the life of the contract. Since the commodity underlying a futures contract is delivered at the spot price upon maturity, there is no reason for the commodity producer to actually deliver on the contract. Rather, it is often easier to offset the open futures position and sell the commodity in the spot market.

Marking to market nullifies any financial incentives for not making delivery. Suppose a farmer enters into a contract to sell a food processor 100,000 bushels of corn at $2.00 per bushel in November. If the price of corn rises to $2.5 by November, the farmer has some incentives to sell his harvest in the spot market at $2.5 rather than to the processor at $2.00. With marking to market, however, the farmer has transferred $0.5 per bushel from his futures account to that of the food processor. The losses not accumulated, the farmer has little incentive to default. When the farmer makes delivery, he is paid the spot price, $2.5 per bushel. So the net price is $2.00 per bushel, the original delivery price.

The prospect of delivery ties the futures price to the spot price and makes hedging possible. In fact, changes in futures prices usually track those in spot (cash) prices. On the delivery date itself, the settlement price of the futures contract is determined by the spot price. Hence, when the delivery period is reached, the futures price should be very close to the spot price. Before the delivery date, the futures price could be above or below the spot price. If the futures price exceeds the spot price, we have **contango**. If the futures price is less than the spot price, we have **backwardation** [82].

Like options on stock index, some financial futures, such as stock or bond index futures contracts, are settled in cash rather than delivery of the underlying asset. (**Financial futures** are futures contracts based on a financial instrument or index.) When a contract is settled in cash, it is marked to market at the end of the trading day, and all positions are declared closed. The settlement price then is the closing spot price of the underlying asset, which ties the futures price to the spot price.

The futures contract on the SPX, for instance, is settled in cash rather than delivery of 500 stocks, and a contract is on $500 times the index (see Fig. 12.3). The related Mini S&P 500 futures is on $50 times the index. One futures contract on the Nikkei 225 Stock Average is on US$5 times the index. This amounts to fixing the dollar-yen exchange rate. The index is price-weighted and on a portfolio of 225 of the largest stocks trading on the

Tokyo Stock Exchange. Since the first financial futures were launched in 1972, the trading of financial futures has surpassed that of agricultural futures. The most popular equity financial futures today, the S&P 500 Index futures [768], were created in 1982.

12.3.1 Daily cash flows

Consider a futures contract with n days to maturity. Let F_i denote the futures price at the end of day i for $0 \leq i \leq n$. The contract's cash flow on day i is $F_i - F_{i-1}$ because of daily settlement. Hence, the net cash flow over the life of the contract is

$$(F_1 - F_0) + (F_2 - F_1) + \cdots + (F_n - F_{n-1}) = F_n - F_0 = S_T - F_0. \tag{12.10}$$

Recall that F_n equals the spot price S_T at maturity. Although a futures contract has the same payoff as a forward contract, it is much more difficult to price due to the reinvestment of daily cash flows, and how $S_T - F_0$ is distributed over the holding period is critical [205]. In contrast, no cash flows occur until settlement for forward contracts.

12.3.2 Forward and futures prices

Somewhat surprisingly, Cox, Ingersoll, and Ross proved that futures price equals forward price if interest rates are non-stochastic [202]. This result often allows us to treat a futures contract as if it were a forward contract, ignoring its marking-to-market feature.

Consider forward and futures contracts on the same underlying asset with n days to maturity. Suppose the interest rate for day i is r_i; hence one dollar at the beginning of day i grows to $R_i \equiv e^{r_i}$ by day's end. Let F_i be the futures price at the end of day i. Note that one dollar invested in the n-day discount bonds at the end of day zero will be worth $R \equiv \prod_{j=1}^{n} R_j$ by the end of day n.

Starting from day zero, we maintain $\prod_{j=1}^{i} R_j$ long futures positions at the end of day $i-1$ and invest the (possibly negative) cash flow at the end of day i in riskless bonds maturing on day n, the delivery date. The cash flow from the position on day i is

$$(F_i - F_{i-1}) \prod_{j=1}^{i} R_j$$

because day i starts with $\prod_{j=1}^{i} R_j$ contracts. This amount will be compounded until the end of day n to become

$$(F_i - F_{i-1}) \prod_{j=1}^{i} R_j \prod_{j=i+1}^{n} R_j = (F_i - F_{i-1}) \prod_{j=1}^{n} R_j = (F_i - F_{i-1})\, R.$$

The value at the end of day n is therefore

$$\sum_{i=1}^{n} (F_i - F_{i-1})\, R = (F_n - F_0)\, R = (S_T - F_0)\, R.$$

Note that no investment is required for the strategy.

Monday, March 20, 1995 FUTURES PRICES								
						Lifetime		Open
	Open	High	Low	Settle	Change	High	Low	Interest
GRAINS AND OILSEEDS								
CORN (CBT) 5,000 bu.; cents per bu.								
Mar	$240\frac{1}{4}$	241	$239\frac{1}{2}$	$239\frac{3}{4}$	−1	$282\frac{1}{2}$	$220\frac{1}{2}$	2,432
May	$246\frac{3}{4}$	248	$246\frac{1}{4}$	$246\frac{1}{2}$	$-\frac{3}{4}$	285	228	113,058
...								
INTEREST RATE								
TREASURY BONDS (CBT)—$100,000; pts. 32nds of 100%								
Mar	104-31	105-05	104-18	104-20	−11	116-20	95-13	28,210
June	104-12	104-19	104-00	104-02	−11	113-15	94-27	328,566
...								
LIBOR-1MO. (CME)—$3,000,000; points of 100%								
Apr	93.84	93.84	93.82	93.83	...	6.17	...	27,961
...								
EURODOLLAR (CME)—$1 million; pts of 100%								
June	93.52	93.54	93.51	93.52	...	6.48	...	515,578
Sept	93.31	93.32	93.28	93.30	...	6.70	...	322,889
...								
CURRENCY								
JAPAN YEN (CME)—12.5 million yen; $ per yen (.00)								
June	1.1363	1.1390	1.1240	1.1301	−.0030	1.1390	.9915	56,525
Sept	1.1490	1.1491	1.1385	1.1430	−.0029	1.1491	1.0175	2,282
...								
DEUTSCHEMARK (CME)—125,000 marks; $ per mark								
June	.7249	.7255	.7124	.7147	−.0087	.7448	.5980	56,053
Sept	.7182	.7215	.7153	.7171	−.0087	.7415	.6290	1,776
...								
BRITISH POUND (CME)—62,500 pds; $ per pound								
June	1.5910	1.5936	1.5680	1.5734	−.0102	1.6530	1.5330	21,050
Sept	1.5770	1.5830	1.5680	1.5704	−.0102	1.6480	1.5410	149
...								
INDEX								
S&P 500 INDEX (CME)—$500 times index								
June	499.75	500.75	498.90	500.15	+.40	501.00	449.50	186,725
...								

Figure 12.3: FUTURES PRICE QUOTATIONS FROM *The Wall Street Journal*, MARCH 21, 1995.

Suppose the forward price f_0 exceeds the futures price F_0 on day zero. We can short R forward contracts, borrow $f_0 - F_0$, and carry out the above strategy. The initial cash flow is $f_0 - F_0 > 0$. On day n, the debt grows to $(f_0 - F_0)\, R$. So the net value is

$$f_0 R - S_T R - (f_0 - F_0)\, R + (S_T - F_0)\, R = 0.$$

Therefore, $f_0 - F_0 > 0$ is a pure arbitrage profit. The case of $f_0 < F_0$ is symmetrical. This completes the proof.

With stochastic interest rates, forward and futures prices are no longer theoretically identical because of daily settlements. In fact, this is the major reason for the price differences in the foreign exchange forward and futures markets [236]. For short-term contracts, however, the differences tend to be small. In fact, the differences are significant only for longer-term contracts on interest rate-sensitive assets. We shall treat forward and futures prices identically throughout this chapter.

12.3.3 Basis

The difference between the cash and futures prices of the same underlying asset, $S - F$, is called the **basis**. For example, if an elevator operator buys soybeans from a farmer at \$7.80 a bushel on October 10, and the November soybean futures contract is \$7.90 on the same day, the basis would be 10 cents under (−10 cents) the November contract. As another example, if the cash market price of a U.S. Treasury bond is 88-16 and the June adjusted futures price is 90-22, then the basis is 2-06 under the June contract. Note that T-bond futures are quoted in 32nds, and prices like 88 16/32 are written as 88-16.

Demand and supply affect basis. If the demand for cash grains is strong relative to the available supply, cash market prices could rise relative to futures prices. However, if the reverse holds, cash prices could fall relative to futures prices. Basis can be positive or negative, but it should converge eventually to zero. If the basis moves toward zero, it is said to be **narrowing**, while it is said to be **widening** if it moves away from zero. Although basis cannot be predicted precisely, it is generally less volatile than either the futures price or the cash price. As mentioned before, this makes hedging possible.

Example 12.3.3 A firm to be paid one million pounds in 60 days is worried that pound will weaken. Suppose a pound futures contract on the International Monetary Market (IMM) of the CME has a settlement date in 71 days. Since each contract controls 62,500 pounds, a hedge would consist of $1,000,000/62,500 = 16$ contracts. By selling these many contracts, the firm is replacing foreign currency risk by the hopefully smaller **basis risk**. □

Example 12.3.4 Suppose the spot price of wheat is \$4.225 per bushel and the July futures price of wheat is \$3.59 per bushel with a contract size of 5,000 bushels. The basis is $4.225 - 3.59 = 0.635$ per bushel. Imagine that the basis widens by \$.1 to \$.735 caused by, say, the futures price falling to \$3.54 and the spot price rising to \$4.275. A person with a short position in one futures contract and a long position in 5,000 bushels of wheat will make $5000 \times 0.1 = 500$ dollars in profit. If the basis narrows by \$.1 to \$0.535, the same investor will have a loss of equal amount. □

12.3.4 Stock index futures

Stock index futures started in 1982 when the Kansas City Board of Trade introduced the Value Line Stock Index futures. There are now stock index futures based on the S&P 500 Index, the Nikkei 225 Stock Average futures, the New York Stock Exchange Composite Index futures (traded on the New York Futures Exchange for $500 times the index), the Major Market Index (traded on the Chicago Board of Trade for $500 times the index), and the Dow Jones Industrial Average Index (traded on the Chicago Board of Trade for $10 times the index; ticker symbol DJ). Section 11.5 discussed stock indexes.

Example 12.3.5 The S&P 500 Index futures contract is based on the S&P 500 Index. The minimum fluctuation (tick size) is 0.05 point. Because the value of a contract is $500 times the Index, a change of 0.05 represents a $\$500 \times 0.05 = \$25$ tick. □

Indexes can be viewed as dividend-paying securities, the security being the basket of stocks comprising the index and the dividends being those paid by the stocks. If the index is broadly based, dividends can be assumed to be paid continuously. With q denoting the average annualized dividend yield during the life of the contract, the futures price is

$$F = Se^{(r-q)\tau}. \tag{12.11}$$

Example 12.3.6 Consider a three-month futures contract on the S&P 500 Index. Suppose the stocks underlying the index provide a dividend yield of 3% per annum, the current value of the index is 480, and the interest rate is 8%. In this case, $r = 0.08$, $S = 400$, $\tau = 0.25$, and $q = 0.03$. The theoretical futures price is $480 \times e^{0.05 \times 0.25} = 486.038$. □

When (12.11) fails to hold, arbitrage profits can be created by trading the stocks underlying the index and the index futures. For example, when $F > Se^{(r-q)\tau}$, profits can be made by buying the stocks underlying the index and shorting futures contracts. Do the reverse if $F < Se^{(r-q)\tau}$. These strategies are known as **index arbitrage** and are executed by computers, an activity known as **program trading**.

Comment 12.3.7 Equation (12.11) is not applicable to the Nikkei 225 futures. Recall that one such contract is on the *dollar* amount equal to five times the index, which is measured in *yen*. Although there are securities whose value is five (yen) times the index, and there are securities whose value is $5 times the security times the proper exchange rate, there exist no securities whose value is $5 times the index. Consequently, the arbitrage argument breaks down. □

For indexes that all stocks tend to pay dividends on the same date, we can estimate the dividends' dollar amount and timing. Then the index becomes a security providing known income, and (12.5) says the futures price is

$$F = (S - I)\, e^{r\tau}. \tag{12.12}$$

12.3.5 Forward and futures contracts on currencies

Let S denote the domestic/foreign exchange rate and X denote the delivery price of the forward contract. Use r_f to refer to the foreign riskless interest rate. A portfolio consisting of one long forward contract, cash amount $Xe^{-r\tau}$ (in domestic currency), and one short position in the amount of $e^{-r_f\tau}$ (in foreign currency) is clearly worth zero at time T. Hence, its current value must be zero, that is,

$$f + Xe^{-r\tau} - Se^{-r_f\tau} = 0.$$

The value X that makes $f = 0$ is the forward price (the forward exchange rate),

$$F = Se^{(r-r_f)\,\tau},$$

which is exactly the interest rate parity.

12.3.6 Futures on commodities

Some commodities are held solely for investment (such as gold and silver), while others are held primarily for consumption. Arbitrage arguments can be used to obtain futures prices in the former case, but they give only upper bounds in the latter.

For a commodity held for investment purposes and with zero storage costs, the futures price is $F = Se^{r\tau}$ from (12.3). In general, if U stands for the present value of the storage costs incurred during the life of a futures contract, then (12.5) says

$$F = (S + U)\,e^{r\tau}$$

as storage costs are negative income. Alternatively, if u denotes the storage costs per annum as a proportion of the spot price, then (12.8) implies

$$F = Se^{(r+u)\,\tau}$$

as storage costs provide a negative dividend yield.

For commodities held primarily for consumption, however, we can only show, corresponding to each of the two views of storage costs, that

$$F \leq (S + U)\,e^{r\tau} \quad \text{and} \quad F \leq Se^{(r+u)\,\tau}$$

because of the benefits of holding the physicals. These benefits are measured by the so-called **convenience yield** defined as the y such that

$$Fe^{y\tau} = (S + U)\,e^{r\tau} \quad \text{and} \quad Fe^{y\tau} = Se^{(r+u)\,\tau}, \tag{12.13}$$

respectively.

Cost of carry

We can frame the relationship between the futures and spot prices in terms of **cost of carry**, which is the storage cost plus the interest cost paid to carry the asset but less the income earned on the asset. For a stock paying no dividends, the cost of carry is r since it neither incurs storage costs nor generates any income; for a stock index, it is $r - q$ as income is earned at rate q; for a currency, it is $r - r_f$; for a commodity with storage costs, it is $r + u$.

Example 12.3.8 Let the cost of carry be c and the convenience yield be y. For an investment asset, the futures price is $F = Se^{c\tau}$, while for a consumption asset, it is $F = Se^{(c-y)\tau}$. □

Cost of carry is often cast in monetary terms, also called **carrying charge** or **carrying cost**. It measures the dollar cost of carrying the asset over a period and consists of interest expense I, storage costs U, minus cash flows generated by the asset D,

$$C \equiv I + U - D. \tag{12.14}$$

Cost of carry will be in dollar terms from now on unless stated otherwise. Similarly, the convenience yield can also be expressed in dollar terms,

$$\text{convenience yield} \equiv S + C - F.$$

As a consequence, the basis $S - F$ is simply the convenience yield minus the cost of carry,

$$\text{basis} = \text{convenience yield} - C. \tag{12.15}$$

Recall that convenience yield is negligible for financial instruments and commodities held primarily for investment purposes. For such assets, changes in basis are due entirely to changes in the cost of carry.

Example 12.3.9 A manufacturer needs to acquire gold in three months. The following options are open to her: (1) Buy the gold now or (2) go long one three-month gold futures contract and take delivery in three months. If she buys the gold now, the money that has tied up could have been invested in money market instruments. This is the opportunity cost of buying physical gold (see Exercise 12.3.6). □

Look up the futures prices of corn in Fig. 12.3. Since the prices for the near months are lower than the distant months, this market is said to be **normal**. The premium that the distant months command over near months is due to the greater carrying costs. In an **inverted** or **discount** market, the distant months sell at lower prices than near months. A strong demand for cash grains or the willingness of elevator owners to store grains at less than the full storage costs can create an inverted market.

When the forward price equals the sum of spot price and carrying charge (zero convenience yield), the forward price is said to be at **full carry**. Forward and futures prices should be set at full carry for a commodity that has zero storage cost and can be sold short or in ample supply (see Exercises 12.3.7 and 12.3.8). As mentioned before, commodities held for investment purposes should reflect full carry. For instance, if the total cost of storing corn were, say, four cents per bushel a month and if futures prices reflected the full carrying cost, the prices for the different delivery months might look like the following table.

December	March	May	July	September
\$2.00	\$2.12	\$2.20	\$2.28	\$2.36

12.4 Futures Options and Forward Options

12.4.1 Futures options

Futures options differ from stock options in that the underlying asset is a futures contract rather than a cash security. Upon exercise, the option holder takes a position in the futures contract with a futures price equal to the option's strike price. In particular, a futures call (put) option holder acquires a long (short) futures position. The option writer does the opposite: A futures call (put) option writer acquires a short (long) futures position. The futures contract is then marked to market immediately, and the futures position of the two parties will be at the prevailing futures price. The option holder of course can withdraw in cash the difference between the prevailing futures price and the strike price.

The whole process works as if the option writer delivered a futures contract to the option holder and paid the holder the prevailing futures price minus the strike price in the case of a call. In the case of a put, it works as if the option writer took delivery a futures contract from the option holder and paid the holder the strike price minus the prevailing futures price. Note that the amount of money that changes hands upon exercise is only the difference between the strike price and the prevailing futures price, not the whole strike price. Of course, the option's expiration date should precede the futures contract's delivery date. See Fig. 12.4 for sample quotations.

Example 12.4.1 Suppose a holder of a futures call with a strike price of \$35 exercises the call when the futures price is \$45. The call holder is given a long position in the futures contract at \$35, and the call writer is assigned the matching short position at \$35. The futures positions of both are immediately marked to market by the exchange. Because the prevailing futures price is \$45, the long futures position (the position of the call holder) realizes a gain of \$10, while the short futures position (the position of the call writer) realizes a loss of \$10. The call writer pays the exchange \$10 and the call holder receives from the exchange \$10. The call holder, who now has a long futures position at \$45, can either liquidate the futures position at \$45 without costs, or maintain it. □

Example 12.4.2 Consider an investor with a July futures call on 5,000 bushels of soybeans with a strike price of 600 cents per bushel. Suppose the current futures price of soybeans for delivery in July is 610 cents. The investor can exercise the option to receive 500 dollars ($5,000 \times 10$ cents) plus a long position in a futures contract to buy 5,000 bushels of soybeans in July. Similarly, consider an investor with a July futures put on 5,000 bushels of soybeans with a strike price of 620 cents per bushel. Suppose the current futures price of soybeans for delivery in July is 610 cents. The investor can exercise the option to receive 500 dollars

Monday, March 20, 1995

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INTEREST RATE

T-BONDS (CBT)

$100,000; points and 64ths of 100%

Strike	Calls—Settle			Puts—Settle		
Price	Apr	May	Jun	Apr	May	Jun
102	2-06	2-26	2-47	0-03	0-23	0-43
103	1-12	1-44	...	0-08	0-40	...
104	0-30	1-05	1-29	0-26	1-01	1-25
105	0-07	0-39	...	1-03	1-34	...
106	0-01	0-21	0-40	1-61	...	2-35
107	0-01	0-10	...	...	...	...

...

T-NOTES (CBT)

$100,000; points and 64ths of 100%

Strike	Calls—Settle			Puts—Settle		
Price	Apr	May	Jun	Apr	May	Jun
102	2-26	...	2-45	0-01	...	0-20
103	1-28	...	1-60	0-02	0-19	0-35
104	0-36	...	1-19	0-11	...	0-57
105	0-07	0-32	0-51	0-45	1-06	1-25
106	0-01	...	0-28	1-39	...	2-01
107	0-01	0-05	0-14	...	...	2-51

...

LIBOR — 1 Mo. (CME)

$3 million; pts. of 100%

Strike	Calls—Settle			Puts—Settle		
Price	Apr	May	Jun	Apr	May	Jun
9325	0.58	0.51	0.45	0.00	0.01	0.03
9350	0.34	0.29	0.24	0.01	0.04	0.07
9375	0.11	0.10	0.09	0.03	...	0.17
9400	0.01	...	0.03	...	...	...
9425	...	...	...	...	...	...
9450	0.00	0.00	...	...	...	...

...

INDEX

S&P 500 STOCK INDEX (CME)

$500 times premium

Strike	Calls—Settle			Puts—Settle		
Price	Apr	May	Jun	Apr	May	Jun
490	12.55	14.80	16.75	2.45	4.75	6.75
495	8.75	11.25	13.35	3.65	6.15	8.25
500	5.70	8.10	10.20	5.55	7.95	10.05
505	3.45	5.55	7.50	8.25	10.35	12.30
510	1.85	3.55	5.30	11.65	...	15.00
515	0.85	2.15	3.55	...	...	...

...

Figure 12.4: FUTURES OPTION QUOTATIONS FROM *The Wall Street Journal*, MARCH 21, 1995. They can be found under *Futures Options Prices*. Months refer to the expiration month of the underlying futures contract.

$(5,000 \times 10$ cents) plus a short position in a futures contract to buy 5,000 bushels of soybeans in July. □

Futures options were created in 1982 when the Chicago Board of Trade began trading options on Treasury bond futures. Futures options are preferred to options on the cash instrument in some markets on the following grounds [650]. In contrast to the cash markets, which are often fragmented and over-the-counter, futures trading takes place in competitive, centralized markets. Futures options have less the liquidity problems associated with shortages of the cash assets—selling a commodity short may be significantly more difficult than selling a futures contract. Futures options are also useful in implementing certain strategies. Finally, futures options are popular because of their limited capital requirements.

12.4.2 Forward options

Forward options are similar to futures options except that what is delivered is an underlying forward contract with a delivery price equal to the option's strike price. In particular, exercising a call (put) forward option results in a long (short) position in a forward contract. Note that exercising a forward option incurs no immediate costs. Unlike futures options, forward options are traded not on organized exchanges but in over-the-counter markets.

Example 12.4.3 Consider a call with strike \$100 and an expiration date in September. The underlying asset is a forward contract with a delivery date in December. Suppose the forward price in July is \$110. Upon exercise, the call holder receives a forward contract with a delivery price of \$100. If an offsetting position is then taken in the forward market, a \$10 profit in September will be assured. Were the contract a call on the futures, then the \$10 profit would be realized in July. □

12.4.3 Pricing relationships

Assume a constant, positive interest rate. This is acceptable for short-term contracts on most commodities. Under this assumption, forward prices equal futures prices. However, forward options do not have the same value as futures options. For simplicity, assume that delivery takes place at time T and the current time is zero. The option on the futures or forward contract has expiration date t ($t \leq T$). Note that there is no difference between forward and futures contracts except that the futures contract can be marked to market when the option is exercised.

Example 12.4.3 demonstrated the following identities for the futures options and forward options when they are exercised at time t,

$$\text{value of futures option} = \max(F_t - X, 0) \tag{12.16}$$

$$\text{value of forward option} = \max(F_t - X, 0)\, e^{-r(T-t)} \tag{12.17}$$

Furthermore, a European futures option is worth the same as the corresponding European option on the underlying asset if the futures contract has the same maturity as the option [111]. The reason is that futures price equals spot price at maturity. This conclusion is independent of the model for the spot price.

The put-call parity is slightly different from the one in Theorem 8.3.1. Whereas the undiscounted stock price was used in the case of stock options, it is the discounted futures/forward prices that should be used here.

Theorem 12.4.4 (Put-call parity) *For European options on futures contracts,*

$$C = P - (X - F)\, e^{-rt}.$$

For European options on forward contracts,

$$C = P - (X - F)\, e^{-rT}.$$

PROOF: Consider a portfolio of one short call, one long put, one long position in the futures contract, and a loan of $(X - F)\, e^{-rt}$. We have the following cash flow at time t.

	$F_t \leq X$	$F_t > X$
A short call	0	$X - F_t$
A long put	$X - F_t$	0
A long futures	$F_t - F$	$F_t - F$
A loan of $(X - F)\, e^{-rt}$	$F - X$	$F - X$
Total	0	0

Since the net future cash flow is zero in both cases, the portfolio must have zero value today. This proves the theorem in the case of futures option.

The case of forward option is identical except that the loan amount is $(X - F)\, e^{-rT}$ instead. The reason is that the forward contract can only be settled at time T. □

American forward options should be worth the same as their European counterparts. In other words, the early exercise feature is not valuable.

Theorem 12.4.5 *American forward options should not be exercised before expiration as long as the probability of the option's ending up out of the money is not zero.*

PROOF: Consider a portfolio of one long forward call, one short position in the forward contract with delivery price F, and a loan of $(F - X)\, e^{-rT}$. If $F_t < X$ at t, then the wealth at t is

$$0 + (F - F_t)\, e^{-r\,(T-t)} - (F - X)\, e^{-r\,(T-t)} = (X - F_t)\, e^{-r\,(T-t)} > 0.$$

If $F_t \geq X$ at t, then the wealth at t is

$$(F_t - X)\, e^{-r\,(T-t)} + (F - F_t)\, e^{-r\,(T-t)} - (F - X)\, e^{-r\,(T-t)} = 0.$$

So the value of the call C satisfies

$$C - (F - X)\, e^{-rT} > 0.$$

On the other hand, if the call is exercised immediately, the present value at time zero is only $\max(F - X, 0)\, e^{-rT}$. The case for the put is left as an exercise. □

Early exercise may be optimal for American futures options. Hence, an American futures option is worth more than the corresponding European option on the underlying asset even if the underlying has no payouts [111].

Theorem 12.4.6 *American futures options may be exercised optimally before expiration.* □

12.4.4 The Black model

Black developed the following formulae for European futures options in 1976 [72],

$$\begin{aligned} C &= Fe^{-rt}N(x) - Xe^{-rt}N\left(x - \sigma\sqrt{t}\right) \\ P &= Xe^{-rt}N\left(-x + \sigma\sqrt{t}\right) - Fe^{-rt}N(-x) \end{aligned} \tag{12.18}$$

where

$$x \equiv \frac{\ln(F/X) + \left(\sigma^2/2\right) t}{\sigma\sqrt{t}}.$$

The above formulae are related to those for options on stocks paying continuous dividend yields. In fact, they are exactly (9.28) with the dividend yield q set to the interest rate r. This observation also proves Theorem 12.4.6 based on the discussions in §9.6.2. The above formulae no longer apply when interest rate-sensitive securities underlie the futures contract.

Black's formulae can be expressed in terms of S instead of F via $F = Se^{(r-q)T}$. The delta for the call is $\partial C/\partial F = e^{-rt}N(x)$ and that for the put is $\partial P/\partial F = e^{-rt}\left(N(x) - 1\right)$. The delta for the call can also be cast with respect to the spot price,

$$\frac{\partial C}{\partial S} = \frac{\partial C}{\partial F}\frac{\partial F}{\partial S} = e^{-rt}N(x)\, e^{(r-q)T} = e^{-r\,(t-T)-qT}N(x).$$

Other sensitivity measures can be easily derived [650, p. 345].

For European options on forward contracts,

$$\begin{aligned} C &= Fe^{-rT}N(x) - Xe^{-rT}N\left(x - \sigma\sqrt{t}\right) \\ P &= Xe^{-rT}N\left(-x + \sigma\sqrt{t}\right) - Fe^{-rT}N(-x) \end{aligned}$$

where

$$x \equiv \frac{\ln(F/X) + \left(\sigma^2/2\right) t}{\sigma\sqrt{t}}$$

since forward options differ from futures options by a factor of $e^{-r\,(T-t)}$ due to (12.16)–(12.17). The binomial tree algorithm for forward options needs one modification in the standard implementation: Use (12.17) for the payoff when the option is exercised.

12.4.5 Options on stock index futures

Three stock market indexes, DJIA, NYA, and SPX, have both options and futures options. The SPX and DJIA index options are European, while their index futures options are American. The NYA index option and futures option are both American. Black's formulae apply to *European* futures options. As for American futures options, the valuation problem is similar to the valuation of options on an index or a stock paying continuous dividend yields.

American futures options should be priced higher than their corresponding European options by Theorem 12.4.6. The result in Exercise 12.4.4 further implies that the SPX futures option should be priced higher than the SPX index option. This claim can be verified by comparing numbers in Figs. 12.4 and 11.10, almost. Exceptions occur between, for example, the June futures call and the June call, both at strike \$490: The former has a premium of \$16.75, while the latter has a higher premium of \$16 7/8. This may be due to the problem of simultaneity, especially with closing prices (futures options close after spot options [302]).

Note the difference between the index futures options and index options upon exercise. Options on an index futures contract are deliverable, the deliverable asset being the underlying index futures. Options on the index itself, on the other hand, are settled in cash for their fair value in the same way that index futures are settled.

12.4.6 Binomial model for futures/forward prices

The risk-neutral probability used in the binomial model for the futures price is

$$p_f = \frac{1-d}{u-d}$$

by (9.29) and treating the futures price as the price of a stock paying a continuous dividend yield of r. Here, u and d are the up and down moves of the underlying *futures* price. In other words, the futures price moves from F to Fu with probability p_f and to Fd with probability $1-p_f$. This holds for forward prices under non-stochastic interest rates as well.

Since futures price is related to spot price via $F = Se^{rT}$ if the underlying asset does not pay dividends, the above model means the stock price moves from S to $S_u \equiv Fue^{-r\,(T-\Delta t)} = Se^{r\Delta t}u$ with probability p_f and to $S_d \equiv Se^{r\Delta t}d$ with probability $1-p_f$. So the return rate of the stock is r because

$$p_f S_u + (1-p_f)\, S_d = Se^{r\Delta t}\left(\frac{1-d}{u-d}\,u + \frac{u-1}{u-d}\,d\right) = Se^{r\Delta t}.$$

Options can be replicated by a portfolio of futures contracts and bonds. This avenue may be preferred to replication with stocks because of the looser restrictions on shorting futures than stocks [458]. An equivalent portfolio of h_f futures contracts and $\$B$ in riskless bonds can be set up for the call that costs C_u if the stock price moves to S_u and C_d if the stock price moves to S_d. The futures price moves to Fu and Fd, respectively, we recall. To make the portfolio have the same payoff as the call option, we impose

$$h_f(Fu-F) + e^{r\Delta t}B = C_u \quad \text{and} \quad h_f(Fd-F) + e^{r\Delta t}B = C_d.$$

Solve the above equations to obtain

$$\begin{aligned} h_f &= \frac{C_u - C_d}{(u-d)\,F} \geq 0 \\ B &= \frac{(u-1)\,C_d - (d-1)\,C_u}{(u-d)\,e^{r\Delta t}} \end{aligned}$$

Compared with the delta using stocks and bonds in (9.2) repeated below,

$$h = \frac{C_u - C_d}{S_u - S_d} = \frac{C_u - C_d}{(u-d)\,Se^{r\Delta t}},$$

we conclude that

$$h_f = \frac{C_u - C_d}{(u-d)\, Se^{rT}} = he^{-r\,(T-\Delta t)} \le h.$$

Hence, delta with futures never exceeds that with stocks when there are no interest rate uncertainties.

As $0 < p_f < 1$, we have $0 < 1 - p_f < 1$ as well. This suggests the following method to solve the problem of negative risk-neutral probabilities mentioned in §9.3.1: Build the binomial tree for the futures price F of the futures contract expiring at the same time as the option, then calculate S from F at each node via $S = Fe^{-(r-q)(T-t)}$ [422].

12.5 Swaps

Swaps are agreements between two **counterparties** to exchange cash flows in the future according to a predetermined formula. There are two basic types of swaps: interest rate and currency. An **interest rate swap** occurs when two parties exchange interest payments periodically. Currency swaps are agreements to deliver one currency against another [666]. Currency swaps made their debut in 1979, and interest rate swaps followed suit in 1981 [557]. In the following decade, the growth of their notional volume was so spectacular as to dwarf that of any other market; for instance, interest rate swaps alone stood at over US$2 trillion in 1993. Swaps also spurred the growth of related instruments such as multi-period options and forward rate agreements.

Currency and interest rate swaps are collectively called **rate swaps**. Swaps on commodities are also available. For example, a company that consumes 200,000 barrels of oil per annum may pay $2 million per year for the next five years and in return receive $200,000 \times S$, where S is the prevailing market price of oil per barrel. This transaction locks in the price for its oil at $10 per barrel. We will concentrate on currency swaps here.

12.5.1 Currency swaps

A currency swap involves two parties to exchange cash flows in different currencies. As an example, consider the following fixed rates available to party A and party B in U.S. dollars and Japanese yen.

	Dollars	Yen
A	D_A%	Y_A%
B	D_B%	Y_B%

Suppose A wants to take out a fixed-rate loan in Japanese yen, and B wants to take out a fixed-rate loan in U.S. dollars. A straightforward scenario is for A to borrow yen at Y_A% and B to borrow dollars at D_B%.

Now, assume A is *relatively* more competitive in the dollar market than the yen market, and vice versa for B; $Y_B - Y_A < D_B - D_A$, in other words. Consider this alternative: A borrows dollars, B borrows yen, and they enter into a currency swap, perhaps with a bank as the financial intermediary. With a swap, the counterparties exchange principal at the beginning and the end of the life of the swap. This act transforms A's loan into a yen loan and B's yen loan into a dollar loan. The total gain to all parties is $((D_B - D_A) - (Y_B - Y_A))\%$. The reason is that, originally, the total interest rate is $(Y_A + D_B)\%$, while the new arrangement pays a total rate of $(D_A + Y_B)\%$, which is smaller. Of course, this arrangement will happen only if the total gain is distributed in such a way that the cost to *each* party is less than the original scenario.

Example 12.5.1 Two parties A and B face the following borrowing rates.

	Dollars	Yen
A	9%	10%
B	12%	11%

Assume A wants to borrow yen, and B wants to borrow dollars. A can borrow yen directly at 10%, and B can borrow dollars directly at 12%. As the rate differential in dollars (3%) is different from that in yen (1%), a currency swap with a total saving of $3 - 1 = 2\%$ is possible. Note that A is relatively more competitive in the dollar market, and B is relatively more competitive in the yen market. Figure 12.5 demonstrates one arrangement which is beneficial to all parties involved, in which A effectively borrows yen at 9.5% and B borrows dollars at 11.5%. The gain is 0.5% for A, 0.5% for B, and, if we treat dollars and yen identically, 1% for the bank. □

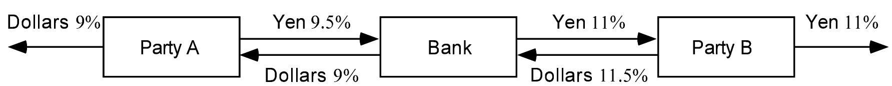

Figure 12.5: CURRENCY SWAP. It turns a dollar liability into a yen liability and vice versa.

With the arrangement in Fig. 12.5 and principal amounts of US\$1 million and 100 million yen, the bank makes an annual gain of 0.025 million dollars and an annual loss of 1.5 million yen. The bank hence bears some currency risk, and hedging may be required. Neither A nor B bears any currency risk. It is not hard to see that currency risk can be redistributed but not eliminated.

12.5.2 Valuation of currency swaps

As a package of cash market instruments

In the absence of default risk, the valuation of currency swap is rather straightforward. Take B in Fig. 12.5 as an example. The swap is equivalent to a long position in a yen bond paying 11% annual interest and a short position in a dollar bond paying 11.5% annual interest. The general pricing formula is thus

$$SP_Y - P_D,$$

where P_D is the dollar bond's value in dollars, P_Y is the yen bond's value in yen, and S is the \$/yen spot exchange rate. Note that the swap has zero value when the principal amounts are identical, or $SP_Y = P_D$. The value of a currency swap therefore depends on the term structures of interest rates in the currencies involved and the spot exchange rate.

Example 12.5.2 Take a two-year swap in Fig. 12.5 with principal amounts of US$1 million and 100 million yen. The payments are made once a year. Assume the spot exchange rate is 90 yen/$ and, to simplify the calculation, the term structures are flat in both nations—8% in the U.S. and 9% in Japan (generalization is straightforward). The value of the swap is

$$\frac{1}{90} \times \left(11 \times e^{-0.09} + 11 \times e^{-0.09\times 2} + 111 \times e^{-0.09\times 3}\right)$$

$$-\left(0.115 \times e^{-0.08} + 0.115 \times e^{-0.08\times 2} + 1.115 \times e^{-0.08\times 3}\right) \quad = \quad 0.074$$

million dollars for B. □

As a package of forward contracts

Alternatively, swaps can be viewed as a package of forward contracts. From (12.7), the forward contract maturing i years from now has a dollar value of

$$f_i = (SY_i)\, e^{-qi} - D_i e^{-ri}, \tag{12.19}$$

where Y_i is the yen inflow at year i, S is the $/yen spot exchange rate, q is the yen interest rate, D_i is the dollar outflow at year i, and r is the dollar interest rate. This formulation may be preferred to the cash market approach in cases involving costs of carry and convenience yields because forward prices already incorporate them [458]. For simplicity, flat term structures were assumed, but generalization is straightforward.

Take the swap in Example 12.5.2. Every year, B receives 11 million yen and pays 0.115 million dollars. In addition, at the end of the third year, B receives 100 million yen and pays 1 million dollars. Each of these transactions represents a forward contract. In particular, $Y_1 = Y_2 = 11$, $Y_3 = 111$, $S = 1/90$, $D_1 = D_2 = 0.115$, $D_3 = 1.115$, $q = 0.09$, and $r = 0.08$. Plug in these numbers to get exactly $f_1 + f_2 + f_3 = 0.074$ million dollars again.

Equation (12.19) can be equivalently cast in terms of forward exchange rates thus,

$$f_i = (F_i Y_i - D_i)\, e^{-ri},$$

where F_i is the i-year forward exchange rate. Even though the swap may have zero value (equivalently, $\sum_i f_i = 0$), it does not imply that each of the forward contracts has zero value.

Additional Reading

The richness of derivative products makes their exhaustive listing impossible. Consult [82, 88, 302, 325, 422, 458, 557, 603, 650, 782] for more information. See [794] for a thorough treatment of "exotic" options. Pointers to empirical studies on the relationship between futures and forward prices can be found in [458].