Chapter 13

Stochastic Processes and Brownian Motion

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Of all the intellectual hurdles which the human mind has confronted and has overcome in the last fifteen hundred years, the one which seems to me to have been the most amazing in character and the most stupendous in the scope of its consequences is the one relating to the problem of motion.

—Herbert Butterfield,
The Origins of Modern Science [125, p. 15]

This chapter introduces basic ideas in stochastic processes and Brownian motion. Brownian motion underlies all the continuous-time price models in this book.[1] From time to time, we go back to earlier discrete-time binomial models to mark the linkage; the transition to continuous time is as natural as it is inescapable.

[1]After losing money in warrants, Merton created an option pricing model based on jump processes [73].

13.1 Stochastic Processes

A **stochastic process** $X = \{ X(t) \}$ is a time series of random variables. In other words, $X(t)$ is a random variable for each time t and is usually called the **state** of the process at time t. For clarity, $X(t)$ is often written as X_t. Other terms for stochastic process are **random process** and **random function**. Any **realization** of X is called a **sample path** or **trajectory**. Note that a sample path defines an ordinary function of t. If the times t form a countable set, X is called a **discrete-time** stochastic process, a **stochastic sequence**, or a **time series**. In such a case, X is usually denoted by $\{ X_n \}$. If the times form a continuum, X is called a **continuous-time** stochastic process.

A continuous-time stochastic process $\{ X(t) \}$ is said to have **independent increments** if, for all $t_0 < t_1 < \cdots < t_n$, the random variables

$$X(t_1) - X(t_0),\ X(t_2) - X(t_1), \ldots,\ X(t_n) - X(t_{n-1})$$

are independent. It is said to possess **stationary** increments if $X(t+s) - X(t)$ have the same distribution for all t. That is, the distribution depends only on s. We have the following useful lemma.

Lemma 13.1.1

$$\begin{aligned} E[\, X(t) - X(0) \,] &= t\, E[\, X(1) - X(0) \,] \\ \mathrm{Var}[\, X(t) \,] - \mathrm{Var}[\, X(0) \,] &= t\ (\mathrm{Var}[\, X(1) \,] - \mathrm{Var}[\, X(0) \,]) \end{aligned}$$

if $\{ X(t), t \geq 0 \}$ *is a stationary independent increment process.* □

The **covariance function** of a stochastic process $X = \{ X(t) \}$ is defined as

$$K_X(s,t) \equiv \mathrm{Cov}[\, X(s), X(t) \,].$$

Note that $K_X(s,t) = K_X(t,s)$. The **mean function** is defined as

$$m_X(t) \equiv E[\, X(t) \,].$$

A stochastic process $\{ X(t) \}$ is (**covariance**) **stationary** if the random variable sets

$$\{ X(t_1), \ldots, X(t_n) \} \text{ and } \{ X(t_1 + h), \ldots, X(t_n + h) \}$$

have the same joint probability distribution for any $n > 0$, time points $t_1 < \cdots < t_n$, and h. From this definition,

$$m_X(t) = E[\, X(t) \,] = E[\, X(t+h) \,] = m_X(t+h)$$

for any h. In other words, the mean function becomes a constant. Similarly,

$$K_X(s, s+t) = E[\, (X(s) - m_X)\, (X(s+t) - m_X) \,] = E[\, (X(0) - m_X)\, (X(t) - m_X) \,].$$

In other words, the covariance function $K_X(s,t)$ depends only on the **lag**, $|\, s - t \,|$. A more generous definition of stationary process, which shall be adopted here, is the following.

Definition 13.1.2 *A stochastic process* $\{X(t)\}$ *is* **stationary** *if* $E[X(t)^2] < \infty$, *the mean function is a constant, and the covariance function depends only on the lag.* □

A **Markov process** is a stochastic process for which everything that we know about its future is summarized by its current value. Formally, a continuous-time stochastic process $X = \{X(t), t \geq 0\}$ is called a Markov process if

$$\text{Prob}[X(t) \leq x \mid X(u), 0 \leq u \leq s] = \text{Prob}[X(t) \leq x \mid X(s)]$$

for $s < t$. When $\text{Prob}[X(t+s) = j \mid X(s) = i]$ is independent of s, the Markov process is said to have **stationary transition probabilities**.

13.1.1 Random walks

Random walks of various kinds are the foundation of discrete-time probabilistic models of asset prices [294]. In fact, the binomial model of stock prices is a random walk in disguise. This subsection introduces some random walks whose importance will become clear in connection with continuous-time models.

Example 13.1.3 Consider a particle on the integer line, $0, \pm 1, \pm 2, \ldots$. At each time step, this particle can make one move to the right with probability p or one move to the left with probability $1 - p$. See Fig. 13.1. Let $P_{i,j}$ represent the probability that the particle will make a transition to point j when currently in point i. Then

$$P_{i,i+1} = p = 1 - P_{i,i-1}$$

for $i = 0, \pm 1, \pm 2, \ldots$. This random walk is **symmetric** if $p = 1/2$. The probability that a symmetric random walk will return to its original position after $2n$ steps is roughly $1/\sqrt{\pi n}$. Do you see the connection with the binomial option pricing model? □

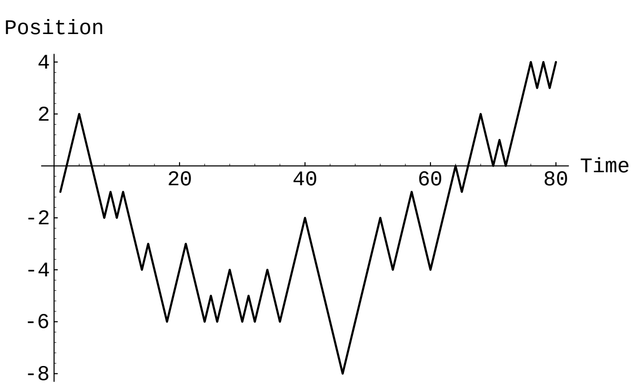

Figure 13.1: Random walk. The particle in each step can move up or down with equal probability.

Example 13.1.4 The **random walk with drift** is the following discrete-time process,

$$X_n = \mu + X_{n-1} + \xi_n, \tag{13.1}$$

where ξ_n are independent and identically distributed with zero mean. The drift μ is the expected change per period. The random walk with drift is an example of a Markov process. An alternative characterization of random walks is to consider $\{ S_n, n \geq 1 \}$ with $S_n \equiv \sum_{i=1}^n X_i$, where X_i are independent, identically distributed random variables with $E[X_i] = \mu$. □

Example 13.1.5 The **classical ruin problem** is a special kind of random walk [298]. Let z represent the capital, q_z the probability of the gambler's ultimate ruin when the wealth becomes zero, and $1-q_z$ the probability of winning when the wealth becomes c. After one trial, the gambler's fortune is either $z-1$ or $z+1$. Hence, for $1 < z < c-1$, we have

$$q_z = pq_{z+1} + qq_{z-1}, \tag{13.2}$$

where p is the probability of winning one dollar and $q \equiv p - 1$ is the probability of losing one dollar for each play. The boundary conditions are

$$q_1 = pq_2 + q \text{ and } q_{c-1} = qq_{c-2}.$$

If we adopt the convention that $q_0 = 1$ and $q_c = 0$, then (13.2) alone takes care of the boundary conditions as it is now valid for $1 \leq z \leq c-1$. □

13.2 Martingales ("Fair Games")

A stochastic process $\{ X(t), t \geq 0 \}$ is a **martingale** if $E[\,|X(t)|\,] < \infty$ for $t \geq 0$ and

$$E[X(t) \,|\, X(u), 0 \leq u \leq s] = X(s).$$

In the discrete-time setting, a martingale means

$$E[X_{n+1} \,|\, X_1, \dots, X_n] = X_n. \tag{13.3}$$

If X_n is interpreted as a gambler's fortune after the nth gamble, the above identity says that the expected fortune after the $(n+1)$st gamble equals the fortune after the nth gamble regardless of what may have occurred before. A martingale is therefore a generalized version of a fair game. The law of iterated conditional expectations applied to both sides of (13.3) implies

$$E[X_n] = E[X_1] \tag{13.4}$$

for all n and, similarly, $E[X(t)] = E[X(0)]$.

Example 13.2.1 Consider the stochastic process $\{ Z_n, n \geq 1 \}$, where $Z_n \equiv \sum_{i=1}^n X_i$, and X_i are independent random variables with zero mean. This process is a martingale because

$$\begin{aligned} E[Z_{n+1} \,|\, Z_1, \dots, Z_n] &= E[Z_n + X_{n+1} \,|\, Z_1, \dots, Z_n] \\ &= E[Z_n \,|\, Z_1, \dots, Z_n] + E[X_{n+1} \,|\, Z_1, \dots, Z_n] \\ &= Z_n + E[X_{n+1}] = Z_n. \end{aligned}$$

Note that $\{ Z_n \}$ subsumes the random walk in Example 13.1.4. □

Example 13.2.2 Let $\{X(t), t \geq 0\}$ be a stochastic process with independent increments. It is not hard to show that $\{X(t), t \geq 0\}$ is a martingale if, furthermore, $E[X(t) - X(s)] = 0$ for any $t, s \geq 0$ and $\text{Prob}[X(0) = 0] = 1$; in fact,

$$\begin{aligned} E[X(t_n) \mid X(t_{n-1}), \ldots, X(t_1)] &= E[X(t_n) - X(t_{n-1}) \mid X(t_{n-1}), \ldots, X(t_1)] + X(t_{n-1}) \\ &= X(t_{n-1}). \end{aligned}$$

The last equality is true because

$$\begin{aligned} & E[X(t_n) - X(t_{n-1}) \mid X(t_{n-1}), \ldots, X(t_1)] \\ = \; & E[X(t_n) - X(t_{n-1}) \mid X(t_{n-1}) - X(t_{n-2}), \ldots, X(t_2) - X(t_1), X(t_1) - X(0)] \\ = \; & E[X(t_n) - X(t_{n-1})] = 0. \end{aligned}$$

□

13.2.1 The binomial option pricing model and martingales

We learned back in Lemma 9.2.1 that the price of a European option is the expected discounted future payoff at expiration in a risk-neutral economy. This important principle can be generalized using the language of martingale. Recall the recursive valuation of European option via (9.5), repeated below,

$$C = \frac{pC_u + (1-p)\, C_d}{R},$$

in a risk-neutral economy, where p is the risk-neutral probability and \$1 grows to \$$R$ in a period. Let $C(i)$ denote the value of the call option at time i and consider the **discount process** $\{C(i)/R^i, i = 0, 1, \ldots, n\}$. From the above formula,

$$E\left[\frac{C(i+1)}{R^{i+1}} \,\middle|\, C(i) = C\right] = \frac{pC_u + (1-p)\, C_d}{R^{i+1}} = \frac{C}{R^i}.$$

The above result can be easily generalized to

$$E\left[\frac{C(k)}{R^k} \,\middle|\, C(i) = C\right] = \frac{C}{R^i}. \tag{13.5}$$

Hence, the discount process is a martingale as

$$\frac{C(i)}{R^i} = E_i^{\pi}\left[\frac{C(k)}{R^k}\right] \quad \text{for } k \geq i, \tag{13.6}$$

where E_i^{π} means the expectation is taken over the risk-neutral probability given the information available at time i. For this reason, the risk-neutral probability is also called the **equivalent martingale probability** [457].

In fact, under discrete-time models, (13.6) holds for any asset not just options and derivatives even if interest rates are stochastic [628]. In this general case, the equation becomes

$$\frac{C(i)}{M(i)} = E_i^{\pi}\left[\frac{C(k)}{M(k)}\right] \quad \text{for } k \geq i. \tag{13.7}$$

Above, $M(j)$ denotes the balance in the money market account at time j using the rollover strategy with an initial investment of \$1. For this reason, it will be called the **bank account process**. If interest rates are stochastic, $M(j)$ is a random variable. Note that $M(0) = 1$, and we assume $M(j)$ is known at time $j-1$. Note also that $M(j)$ is *not* the reciprocal of the market discount function, which is known today for sure.

This identity (13.7) is the general formulation of risk-neutral valuation. In plain English, it says the discount process is a martingale under π. The above fact and Comment 9.2.4 result in the following fundamental theorem for asset pricing.

Theorem 13.2.3 *There is no arbitrage in a discrete-time model if and only if there exists a probability measure such that the discount process is a martingale. (This probability measure is called the risk-neutral probability.)* □

13.2.2 Martingale and futures price under the binomial model

Futures prices form a martingale under the risk-neutral probability. This is because the expected futures price in the next period is

$$p_f F u + (1 - p_f)\, F d = F \left(\frac{1-d}{u-d}\, u + \frac{u-1}{u-d}\, d \right) = F.$$

(Review §12.4.6.) The above thesis can be generalized to

$$F = E^{\pi}[\, F_i \,], \tag{13.8}$$

where F_i is the futures price at time i. This identity holds under stochastic interest rates as well (see Exercise 13.2.11).

13.2.3 Martingales with respect to a process

More generally, a stochastic process $\{ X(t), t \geq 0 \}$ is called a **martingale with respect to process** $\{ Y(t) \}$ if $E[\, | X(t) | \,] < \infty$ for $t \geq 0$ and

$$E[\, X(t) \mid Y(u), 0 \leq u \leq s \,] = X(s) \tag{13.9}$$

for $s < t$. Intuitively, $\{ Y(u), 0 \leq u \leq s \}$ can be treated as the information up to time s.

Example 13.2.4 Consider the stochastic process $\{ S_n - n\mu, n \geq 1 \}$, where $S_n \equiv \sum_{i=1}^{n} X_i$, and $X_1, X_2, \ldots$ are independent random variables with mean μ. Check that

$$\begin{aligned} E[\, S_{n+1} - (n+1)\,\mu \mid X_1, \ldots, X_n \,] &= E[\, S_{n+1} \mid X_1, \ldots, X_n \,] - (n+1)\,\mu \\ &= S_n + \mu - (n+1)\,\mu \\ &= S_n - n\mu. \end{aligned}$$

Hence, $\{ S_n - n\mu, n \geq 1 \}$ is a martingale with respect to $\{ X_n, n \geq 1 \}$, generalizing the results in Example 13.2.1. □

Example 13.2.5 Let $S_n \equiv \sum_{k=1}^{n} Y_k$, where $Y_n \sim N(0,1)$ are independent. Lemma 6.1.1 says $S_n \sim N(0,n)$. Fix an $\alpha \in \boldsymbol{R}$ and define $X_n \equiv e^{\alpha S_n - n\alpha^2/2}$. We shall show that $\{X_n, n \geq 1\}$ is a martingale with respect to $\{Y_n, n \geq 1\}$. First,

$$E[\,|X_n|\,] = E[X_n] = e^{-n\alpha^2/2} E\left[e^{\alpha S_n}\right] = e^{-n\alpha^2/2} e^{n\alpha^2/2} = 1 < \infty.$$

The next-to-last equality above is due to (6.10) and Lemma 6.1.1. Now,

$$\begin{aligned} E[X_{n+1} \mid Y_1, \ldots, Y_n] &= E\left[e^{\alpha S_{n+1} - (n+1)\alpha^2/2} \,\middle|\, Y_1, \ldots, Y_n \right] \\ &= E\left[e^{\alpha S_n - n\alpha^2/2} e^{\alpha Y_{n+1} - \alpha^2/2} \,\middle|\, Y_1, \ldots, Y_n \right] \\ &= e^{\alpha S_n - n\alpha^2/2} E\left[e^{\alpha Y_{n+1} - \alpha^2/2} \right] \\ &= X_n E\left[e^{\alpha Y_{n+1} - \alpha^2/2} \right] = X_n \end{aligned}$$

since $E\left[e^{\alpha Y_{n+1} - \alpha^2/2}\right] = e^{-\alpha^2/2}\, \theta_{Y_{n+1}}(\alpha) = e^{-\alpha^2/2}\, e^{\alpha^2/2} = 1.$ □

13.3 Brownian Motion

A **Brownian motion process** is a stochastic process $\{X(t), t \geq 0\}$ satisfying the following three conditions.

1. $X(0) = 0$, unless stated otherwise;

2. for any $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $X(t_k) - X(t_{k-1})$ for $1 \leq k \leq n$ are independent;

3. for $0 \leq s < t$, $X(t) - X(s)$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$, where μ and $\sigma \neq 0$ are real numbers. The constant μ is called the **drift** and σ^2 the **variance**.

Such a process will be called a (μ, σ) Brownian motion. Figure 13.2 plots a realization of a Brownian motion process.

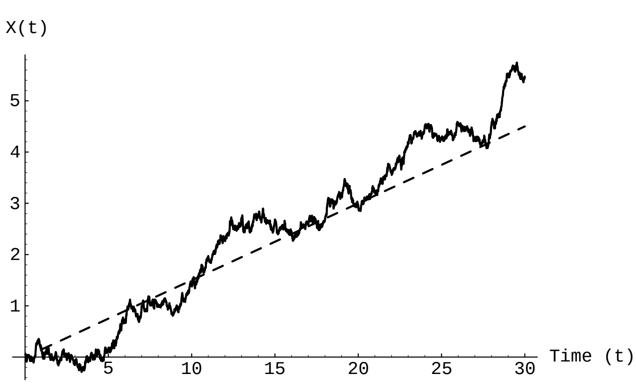

Figure 13.2: Sample path of a Brownian motion process. The stochastic process has volatility, as testified by the jittery of the path. Also plotted for reference is the related deterministic process with the randomness removed.

The existence and uniqueness of such a process is guaranteed by Wiener's theorem [67]. Although Brownian motion is a continuous function of t with probability one, it is almost nowhere differentiable. Note that Property 2 implies $X(t) - X(s)$ is independent of $X(\lambda)$ for $\lambda \leq s$.

Any continuous-time process with stationary independent increments can be proved to be a Brownian motion process [371]. This fact explains the significance of Brownian motion in stochastic modeling. Brownian motion also demonstrates **statistical self-similarity** in that $X(rx)/\sqrt{r}$ remains normalized Brownian motion if X is such. This means, if we sample the process 100 times faster and then shrink the result 10 times, the path will look statistically the same as the original one. This property naturally links Brownian motion to fractals [207, 683]. Finally, Brownian motion is Markovian as

$$\begin{aligned} & \text{Prob}[\, X(t+s) \leq a \mid X(s) = x, X(u), 0 \leq u < s\,] \\ = \; & \text{Prob}[\, X(t+s) - X(s) \leq a - x \mid X(s) = x, X(u), 0 \leq u < s\,] \\ = \; & \text{Prob}[\, X(t+s) - X(s) \leq a - x\,] \\ = \; & \text{Prob}[\, X(t+s) \leq a \mid X(s) = x\,]. \end{aligned}$$

Brownian motion, named after Robert Brown (1773–1858), was first discussed mathematically by Bachelier and received rigorous treatments by Wiener (1894–1964), including the above concise definition. Therefore, it is also called **Wiener process**, **generalized Wiener process**, or **Wiener-Bachelier process** [299, 473]. We shall reserve the term Wiener process only for the $(0,1)$ Brownian motion, which is also called **normalized** or **standard** Brownian motion.

Example 13.3.1 Suppose the total value of a company, measured in millions of dollars, follows a $(20, 30)$ Brownian motion (i.e., with a drift of 20 per annum and a variance of 900 per annum). The starting total value is 50. At the end of one year, the total value will have a normal distribution with a mean of 70 and a standard deviation of 30. At the end of six months, as another example, it will have a normal distribution with a mean of 60 and a standard deviation of $\sqrt{450} = 21.21$. □

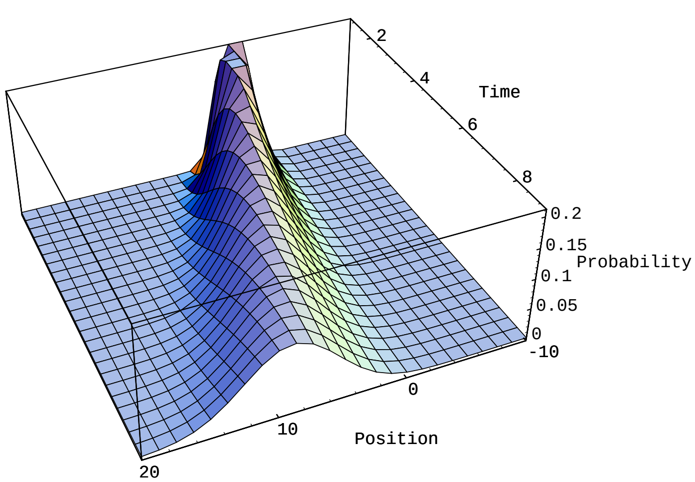

Figure 13.3: Uncertainty about Brownian motion. The future position of a Brownian motion process is normally distributed with a standard deviation (uncertainty) proportional to the square root of how far we look into the future.

From the definition, $X(t) \sim N(0,t)$ for normalized Brownian motion $\{ X(t), t \geq 0 \}$. More generally, if $\{ X(t), t \geq 0 \}$ is a (μ, σ) Brownian motion, then

$$X(t) - X(t_0) \sim N\left(\mu\,(t - t_0), \sigma^2(t - t_0)\right). \tag{13.10}$$

A (μ, σ) Brownian motion $Y = \{ Y(t), t \geq 0 \}$ can be expressed in terms of normalized Brownian motion via

$$Y(t) = \mu t + \sigma X(t). \tag{13.11}$$

It follows that $Y(t+s) - Y(t) \sim N\left(\mu s, \sigma^2 s\right)$. The implication is that, looking into the future, our uncertainty about the future value of Y as measured by the standard deviation grows as the square root of how far we look into the future. See Figs. 13.3 and 13.4.

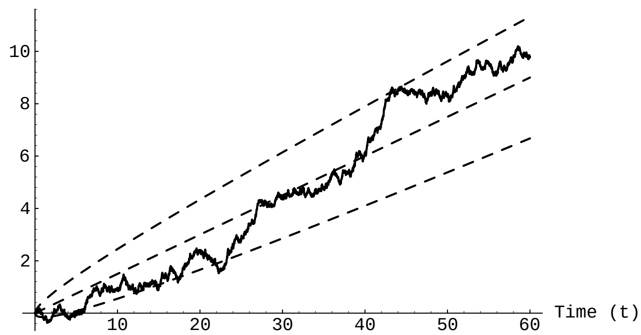

Figure 13.4: Drift and variance of Brownian motion. Depicted is a sample path of a $(0.15, 0.3)$ Brownian motion. The envelope is for one standard deviation, or $0.3\sqrt{t}$, around the mean.

13.3.1 Brownian motion as the limiting case of random walk

Brownian motion without drift is the limiting case of symmetric random walk. Suppose a particle moves Δx either to the left or to the right with equal probability in each Δt time units. Without loss of generality, assume $t/\Delta t$ is an integer. Its position at time t is then

$$Y(t) \equiv \Delta x \left(X_1 + X_2 + \cdots + X_{\frac{t}{\Delta t}} \right), \tag{13.12}$$

where

$$X_i \equiv \begin{cases} +1 & \text{if the } i\text{th move is to the right} \\ -1 & \text{if the } i\text{th move is to the left} \end{cases}$$

and X_i are independent with $\text{Prob}[X_i = 1] = \text{Prob}[X_i = -1] = 1/2$. Note that $E[X_i] = 0$ and $\text{Var}[X_i] = 1$. Hence, $E[Y(t)] = 0$ and $\text{Var}[Y(t)] = (\Delta x)^2 (t/\Delta t)$. Letting $\Delta x \equiv \sigma\sqrt{\Delta t}$ and $\Delta x \to 0$, we get

$$E[Y(t)] = 0 \quad \text{and} \quad \text{Var}[Y(t)] = \sigma^2 t.$$

Thus $\{ Y(t), t \geq 0 \}$ converges to a $(0, \sigma)$ Brownian motion by the Central Limit Theorem. It is interesting to note that the above heuristic argument would *not* work for $\Delta x \equiv \sigma\,\Delta t$. Also note that

$$Y(t + \Delta t) - Y(t) = \Delta x\, X_{\frac{t}{\Delta t}+1}$$

from (13.12). Hence, $E[\,|\,Y(t+\Delta t) - Y(t)\,|\,] = \Delta x\, E\left[\,\left|\,X_{\frac{t}{\Delta t}+1}\right|\,\right] = \sigma\sqrt{\Delta t}\,$.

The more general Brownian motion with drift is also the limiting case of random walk. Suppose a particle moves Δx either to the left with probability $1-p$ or to the right with probability p in each Δt time units. Its position at time t is also described by (13.12) except that, now, $\text{Prob}[\,X_i = 1\,] = p = 1 - \text{Prob}[\,X_i = -1\,]$. It is not hard to see that

$$E[\,Y(t)\,] = \Delta x\, \frac{t}{\Delta t}\,(2p-1) \quad\text{and}\quad \text{Var}[\,Y(t)\,] = (\Delta x)^2\, \frac{t}{\Delta t}\,\left(1-(2p-1)^2\right).$$

Letting $\Delta x \equiv \sigma\sqrt{\Delta t}$, $p \equiv \left(1 + (\mu/\sigma)\sqrt{\Delta t}\right)/2$, and $\Delta t \to 0$, we have

$$\begin{aligned} E[\,Y(t)\,] &= \sigma\sqrt{\Delta t}\,\frac{t}{\Delta t}\,\frac{\mu}{\sigma}\,\sqrt{\Delta t} = \mu t \\ \text{Var}[\,Y(t)\,] &= \sigma^2\Delta t\,\frac{t}{\Delta t}\,\left(1-\left(\frac{\mu}{\sigma}\right)^2\Delta t\right) \to \sigma^2 t \end{aligned}$$

Thus, $\{\,Y(t), t \geq 0\,\}$ converges to a (μ,σ) Brownian motion by the Central Limit Theorem. The choice for p is identical to the choice in calibrating the binomial option pricing model in (9.19)!

13.3.2 Geometric Brownian motion

If $X \equiv \{\,X(t), t \geq 0\,\}$ is a Brownian motion process, then the process

$$Y(t) \equiv e^{X(t)}, t \geq 0$$

is called **geometric Brownian motion**. Its other names include **exponential Brownian motion** and **lognormal diffusion**. See Fig. 13.5 for illustration. For the case where X is normalized, we have $X(t) \sim N(0,t)$ with the moment generating function

$$E\left[\,e^{sX(t)}\,\right] = e^{s^2 t/2} = E\,[\,Y(t)^s\,]$$

from (6.10). Thus, $E[\,Y(t)\,] = e^{t/2}$ and

$$\text{Var}[\,Y(t)\,] = E\left[\,Y(t)^2\,\right] - E[\,Y(t)\,]^2 = e^{2t} - e^{t}.$$

In the general case where X is a (μ,σ) Brownian motion, then

$$E[\,Y(t)\,] = \exp\left[\left(\mu + \frac{\sigma^2}{2}\right)t\right] \tag{13.13}$$

$$\text{Var}[\,Y(t)\,] = E[\,Y(t)\,]^2\left(e^{\sigma^2 t} - 1\right) \tag{13.13$'$}$$

Geometric Brownian motion models situations in which the *percentage* changes are independent and identically distributed. For example, suppose Y_n denotes the stock price at time n and assume relative returns $X_n \equiv Y_n/Y_{n-1}$ with $Y_0 = 1$ are independent and identically distributed. Then $Y_n = X_1 \cdots X_n$, and

$$\ln Y_n = \sum_{i=1}^{n} \ln X_i$$

is a sum of n independent, identically distributed random variables. Thus, $\{ \ln Y_n, n \geq 0 \}$ is approximately Brownian motion, and $\{ Y_n, n \geq 0 \}$ approaches geometric Brownian motion. (We knew from §9.3 that Y_n approaches the lognormal distribution.) Note that $\ln X_i$ is the continuously compounded rate of return between prices Y_{i-1} and Y_i.

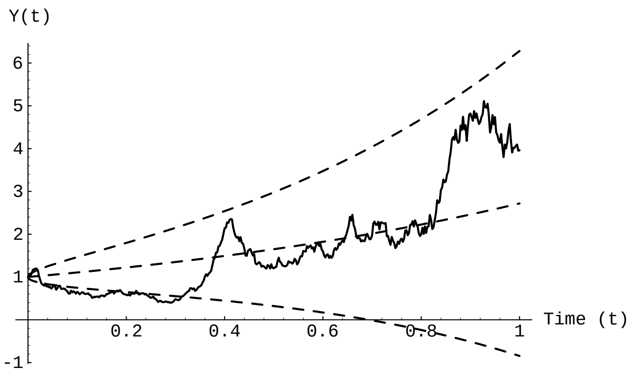

Figure 13.5: SAMPLE PATH OF GEOMETRIC BROWNIAN MOTION. The process is $Y(t) = e^{X(t)}$, where X is a $(0.5, 1)$ Brownian motion. The envelope is for one standard deviation, which is $\sqrt{(e^t - 1)\, e^{2t}}$, around the mean. Can you tell the qualitative difference between this plot and the stock price charts in Fig. 6.4?

13.3.3 Stationarity

Normalized Brownian motion $\{ X(t), t \geq 0 \}$ is *not* stationary (see Exercise 13.3.2). However, it can be transformed into one via

$$Y(t) \equiv e^{-t} X\left(e^{2t}\right). \tag{13.14}$$

This claim can be verified as follows. Since $Y(t) \sim N(0, 1)$, the mean function is zero, a constant. Furthermore,

$$E\left[\, Y(t)^2 \,\right] = E\left[e^{-2t}\, X\left(e^{2t}\right)^2 \right] = e^{-2t}\, e^{2t} = 1 < \infty.$$

Finally, the covariance function is, for $s < t$,

$$E\left[\, e^{-t} X\left(e^{2t}\right)\, e^{-s} X\left(e^{2s}\right) \right] = e^{-s-t} E\left[\, X\left(e^{2t}\right) X\left(e^{2s}\right) \right] = e^{-s-t} e^{2s} = e^{s-t},$$

where the next to last equality is due to Exercise 13.3.2. Therefore, $\{ Y(t), t \geq 0 \}$ is stationary. The process Y is called the **Ornstein-Uhlenbeck process** [199, 222, 472].

Another way to create stationary process from normalized Brownian motion is by

$$Y(t) = \int_t^{t+1} (X(s) - X(t))\, ds.$$

Note also that, trivially, $X(t+h) - X(t)$ is stationary for a (μ, σ) Brownian motion $\{ X(t) \}$ and any fixed $h \geq 0$.

13.3.4 Brownian martingales

Let $\{ X(t), t \geq 0 \}$ be a $(0, \sigma)$ Brownian motion. The following three processes are martingales with respect to $\{ X(t), t \geq 0 \}$: (1) $X(t)$; (2) $X(t)^2 - \sigma^2 t$; (3) $\exp\left[\, \alpha X(t) - \alpha^2 \sigma^2 t / 2 \,\right]$, where $\alpha \in \boldsymbol{R}$ (**Wald's martingale**). For instance, $\{\, X(t)^2 - \sigma^2 t, t \geq 0 \,\}$ is a martingale,

$$\begin{aligned}
& E\left[\, X(t)^2 - \sigma^2 t \mid X(u), 0 \leq u \leq s \,\right] \\
= \; & E\left[\, X(t)^2 \mid X(s) \,\right] - \sigma^2 t \\
= \; & E\left[\, (X(t) - X(s))^2 \mid X(s) \,\right] + 2\, E\left[\, X(s)(X(t) - X(s)) \mid X(s) \right] + E\left[\, X(s)^2 \mid X(s) \,\right] - \sigma^2 t \\
= \; & \sigma^2 (t - s) + 2 \times 0 + X(s)^2 - \sigma^2 t \\
= \; & X(s)^2 - \sigma^2 s.
\end{aligned}$$

13.3.5 Variations

Many formulae in standard calculus do not carry over to Brownian motion. Take the **quadratic variation** of any function $f : [\,0, \infty) \rightarrow R$ defined by

$$\sum_{k=0}^{2^n - 1} \left[f\left(\frac{(k+1)\, t}{2^n} \right) - f\left(\frac{kt}{2^n} \right) \right]^2 .$$

(For technical reasons, the partition of the time interval $[\,0, t\,]$ is **dyadic**, i.e., at points $kt/2^n$ for $0 < k < 2^n$.) It is not hard to see that the quadratic variation vanishes as $n \rightarrow \infty$ if f is differentiable.

The above conclusion no longer holds if f is Brownian motion: A theorem says

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{2^n - 1} \left[X\left(\frac{(k+1)\, t}{2^n} \right) - X\left(\frac{kt}{2^n} \right) \right]^2 = \sigma^2 t \tag{13.15}$$

with probability one, where $\{\, X(t), t \geq 0 \,\}$ is a (μ, σ) Brownian motion [473]. This result informally says $\int_0^t (dX(s))^2 = \sigma^2 t$, frequently written as

$$(dX)^2 = \sigma^2 \, dt. \tag{13.16}$$

The above differential formula does not make sense in standard calculus, but it becomes true in stochastic calculus. It can furthermore be shown that

$$(dX)^n = 0 \text{ for } n > 2 \tag{13.17}$$

and $dX\, dt = 0$.

With (13.15), the **total variation** of a Brownian path is infinite with probability one, that is,

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{2^n - 1} \left| X\left(\frac{(k+1)\, t}{2^n} \right) - X\left(\frac{kt}{2^n} \right) \right| = \infty. \tag{13.18}$$

Brownian motion is thus continuous but with highly irregular sample paths.

13.3.6 Brownian bridge

Brownian bridge is "tied down" Brownian motion [474]. It is defined as normalized Brownian motion $\{ X(t), 0 \leq t \geq 1 \}$ plus the constraint $X(0) = X(1) = 0$. An alternative formulation of Brownian bridge is

$$X(t) - tX(1), \quad 0 \leq t < 1.$$

For a general time period $[0, T]$, Brownian bridge can be written as [164]

$$Z(t) \equiv W(t) - \frac{t}{T} W(T), \quad 0 \leq t \leq T,$$

where $W(0) = 0$ and $W(T)$ are known at time zero. Observe that $Z(t)$ is pinned to zero at both endpoints, zero and T.

Additional Reading

The idea of martingale is due to Lévy (1886–1974) and received thorough developments by Doob [174, 242, 472, 780]. See [719] for a complete treatment of random walks. Consult [543] for a history of Brownian motion from the physicists' point of view and [238] for Bachelier's contribution. The book [94] collects results and formulae in connection with Brownian motion. Books such as [179, 199, 319, 473] contain advanced materials. The heuristic arguments in §13.3.1 showing Brownian motion as the limiting case of random walk can be made rigorous by **Donsker's theorem** [67, 250, 472, 502].

Chapter 14

Continuous-Time Financial Mathematics

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The pursuit of mathematics is a
divine madness of the human spirit.
—Alfred North Whitehead (1861–1947),
Science and the Modern World [778, p. 20]

In after years I have deeply regretted that
I did not proceed far enough at least to understand
something of the great leading principles of mathematics.
—Charles Darwin (1809–1882),
Autobiography [215, p. 58]

This chapter introduces the mathematics behind continuous-time models. This approach was initiated by Merton [251]. Formidable as the mathematics seems to be, it can be made accessible at some expense of rigor and generality. The theory will be applied to a few fundamental financial problems.

14.1 Stochastic Integrals

We saw that classical calculus cannot be applied to Brownian motion in §13.3.5. One reason is that its sample path, regarded as a function, has unbounded total variation. Stochastic integral therefore cannot be defined in the conventional Riemann-Stieltjes sense. From now on, we shall use $W = \{ W(t), t \geq 0 \}$ to denote exclusively the Wiener process, that is, normalized Brownian motion. The purpose of this section is to develop stochastic integrals with respect to Brownian motion,

$$I_t(X) \equiv \int_0^t X\, dW, \quad t \geq 0, \tag{14.1}$$

for X from a class of stochastic processes.

Note that $I_t(X)$ is a random variable. These random variables are called **stochastic integrals** of X with respect to W, and the entire stochastic process $\{ I_t(X), t \geq 0 \}$ will be denoted by $\int X\, dW$. Typical requirements for X are: (1) Prob $\left[\int_0^t X^2(s)\, ds < \infty \right] = 1$ for all $t \geq 0$ or the stronger $\int_0^t E\left[X^2(s) \right] ds < \infty$ and (2) information at time t includes the history of X and W up to that point in time but nothing about the future evolution of X or W after t (**nonanticipating**, so to speak). The unknown future therefore does not influence the present. Hence, $\{ X(s), 0 \leq s \leq t \}$ is independent of $\{ W(t+u) - W(t), u > 0 \}$.

14.1.1 The Ito integral

The Ito integral is a theory of integration with respect to Brownian motion. As with calculus, we start with step functions. A stochastic process $\{ X(t) \}$ is called **simple** if there exist times $0 = t_0 < t_1 < t_2 < \cdots$ such that

$$X(t) = X(t_{k-1}) \text{ for } t \in [\, t_{k-1}, t_k) \text{ and } k = 1, 2, \ldots$$

for any realization. See Fig. 14.1 for illustration. The **Ito integral** of such a simple process is defined in the Riemann-Stieltjes sense as

$$I_t(X) \equiv \sum_{k=0}^{n-1} X(t_k)(W(t_{k+1}) - W(t_k)), \tag{14.2}$$

where $t_n = t$. We emphasize that the integrand X is evaluated at t_k, not t_{k+1}.

The natural step to follow is to define the Ito integral as a limiting random variable of the Ito integral of simple stochastic processes. Indeed, for a general stochastic process $X = \{ X(t), t \geq 0 \}$, there exists a random variable $I_t(X)$, unique almost certainly, such

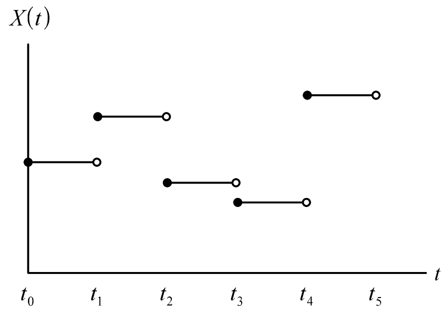

Figure 14.1: A SIMPLE STOCHASTIC PROCESS.

that $I_t(X_n) \to I_t(X)$ for each sequence of simple stochastic processes $X_1, X_2, \ldots$ such that $X_n \to X$. In particular, if X is continuous with probability one, then $I_t(X_n)$ converges in probability to $I_t(X)$ as $\delta_n \equiv \max_{1 \leq k \leq n}(t_k - t_{k-1})$ goes to zero, that is,

$$\int_0^t X\, dW = \text{st-lim}_{\delta_n \to 0} \sum_{k=0}^{n-1} X(t_k)\,(W(t_{k+1}) - W(t_k)). \tag{14.3}$$

It is a fundamental fact that $\int X\, dW$ is continuous almost surely [371, 494]. The following theorem says the Ito integral is a martingale. (Its discrete analogue appeared in Exercise 13.2.13.) A simple corollary is the mean value formula, $E\left[\int_a^b X\, dW\right] = 0$.

Theorem 14.1.1 *The Ito integral $\int X\, dW$ is a martingale.* □

Let us inspect (14.3) again. It says the simple stochastic process below can be used in place of X to approximate the stochastic integral $\int_0^t X\, dW$,

$$\widehat{X}(s) \equiv X(t_{k-1}) \quad \text{for } s \in [\,t_{k-1}, t_k) \text{ and } k = 1, 2, \ldots, n.$$

Note the nonanticipating feature of $\widehat{X}$: The information up to time s,

$$\left\{\, \widehat{X}(t), W(t), 0 \leq t \leq s \,\right\},$$

cannot determine the future evolution of either X or W. Had we defined the stochastic integral as

$$\sum_{k=0}^{n-1} X(t_{k+1})\,(W(t_{k+1}) - W(t_k)),$$

we would have been using a different simple stochastic process,

$$\tilde{X}(s) \equiv X(t_k) \quad \text{for } s \in [\,t_{k-1}, t_k) \text{ and } k = 1, 2, \ldots, n,$$

which clearly anticipates the future evolution of X. See Fig. 14.2.

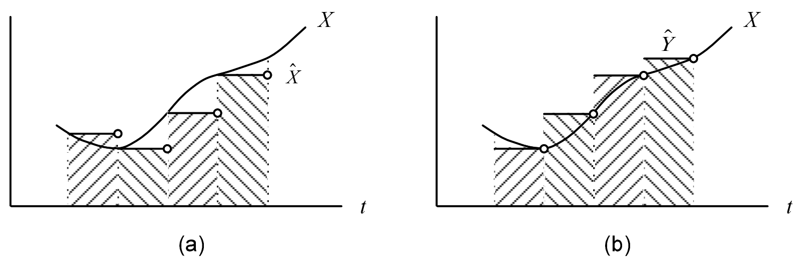

Figure 14.2: STOCHASTIC INTEGRALS. The simple process $\widehat{X}$ in (a) does not anticipate X, while the simple process $\tilde{X}$ in (b) does. They correspond to different ways of defining the stochastic integral.

As an illustration, we approximate $\int W\,dW$ as follows,

$$\begin{aligned}&\sum_{k=0}^{n-1} W(t_k)(W(t_{k+1}) - W(t_k))\\ =\ &\frac{1}{2}\sum_{k=0}^{n-1}\left[W(t_{k+1})^2 - W(t_k)^2\right] - \frac{1}{2}\sum_{k=0}^{n-1}(W(t_{k+1}) - W(t_k))^2\\ =\ &\frac{W(t)^2}{2} - \frac{1}{2}\sum_{k=0}^{n-1}(W(t_{k+1}) - W(t_k))^2.\end{aligned}$$

Since the second term above converges to $t/2$ by (13.15),

$$\int_0^t W\,dW = \frac{W(t)^2}{2} - \frac{t}{2}. \tag{14.4}$$

In calculus, we expect $\int_0^t W\,dW = W(t)^2/2$. (In fact, ordinary calculus formulae hold for processes with finite total variation.) So the extra $t/2$ term is surprising. This phenomenon can be traced to the infinite total variation of Brownian motion. Another way to see the mistake of $\int_0^t W\,dW = W(t)^2/2$ is through Theorem 14.1.1: $W(t)^2/2$ is *not* a martingale (see Exercise 14.1.3), but $\left(W(t)^2 - t\right)/2$ is (see §13.3.4).

14.2 Ito Processes

An **Ito process** is the stochastic process $X = \{X_t, t \geq 0\}$ satisfying

$$X_t = X_0 + \int_0^t a_s\,ds + \int_0^t b_s\,dW_s, \quad t \geq 0, \tag{14.5}$$

where X_0 is a scalar starting point, and $\{a_t : t \geq 0\}$ and $\{b_t : t \geq 0\}$ are stochastic processes satisfying $\int_0^t |a_s|\,ds < \infty$ and $\int_0^t |b_s|^2\,ds < \infty$, respectively, almost surely for all $t \geq 0$. The term a_t is called the **drift** and b_t the **diffusion** of the Ito process. Following Langevin's work in 1904 [24, 339], a shorthand for (14.5) is the following stochastic differential equation or **Ito differential**,

$$dX_t = a_t\,dt + b_t\,dW_t. \tag{14.6}$$

This is Brownian motion with an instantaneous drift of a_t and an instantaneous variance of b_t^2. The process becomes deterministic if $b_t \equiv 0$, in which case it is called a **Liouville process**.

An equivalent form to (14.6) is the so-called **first form of the Langevin equation**,

$$dX_t = a_t\, dt + b_t \sqrt{dt}\, \xi, \tag{14.7}$$

where $\xi \sim N(0,1)$, while (14.6) is in the **second form**. Note that dW is normally distributed with mean zero and variance dt. This formulation makes it easy to derive Monte Carlo simulation algorithms. Although $dt \ll \sqrt{dt}$, the deterministic term a_t still matters because the random variable ξ makes sure the fluctuation term b_t over successive intervals tends to cancel each other out.

14.2.1 Discrete approximations

The following finite difference approximation arises naturally from (14.7),

$$\widehat{X}(t_{n+1}) = \widehat{X}(t_n) + a\left(\widehat{X}(t_n)\right) \Delta t + b\left(\widehat{X}(t_n)\right) \Delta W(t_n), \tag{14.8}$$

where $t_n \equiv n\Delta t$. Note that $\Delta W(t_n)$ should be interpreted as $W(t_{n+1}) - W(t_n)$, not $W(t_n) - W(t_{n-1})$, because a and b are required to be nonanticipating. Under mild conditions, $\widehat{X}(t_n)$ indeed converges to $X(t_n)$ [501]. This method is called **Euler's method** or the **Euler-Maruyama method** [484]. To make $\widehat{X}$ well-defined for the whole time interval $[0, T]$, just use the linear interpolation scheme between two discrete time points. The more advanced Milsh'stein scheme adds $(1/2)\, bb' \left((\Delta W)^2 - \Delta t\right)$ to Euler's method to provide better approximations [577, 578]. For geometric Brownian motion, Euler's scheme says

$$\widehat{X}(t_{n+1}) = \widehat{X}(t_n) + \mu \widehat{X}(t_n)\, \Delta t + \sigma \widehat{X}(t_n)\, \Delta W(t_n),$$

whereas Milsh'stein scheme adds $\frac{1}{2}\sigma^2 \widehat{X}(t_n) \left((\Delta W(t_n))^2 - \Delta t\right)$ to the right.

Under fairly loose regularity conditions, the discrete approximation (14.8) can be replaced by

$$\widehat{X}(t_{n+1}) = \widehat{X}(t_n) + a\left(\widehat{X}(t_n)\right) \Delta t + b\left(\widehat{X}(t_n)\right) \sqrt{\Delta t}\, Y(t_n),$$

where $Y(t_0), Y(t_1), \dots$ are independent and identically distributed with zero mean and unit variance. This general result is guaranteed by Donsker's theorem [11].

Another discrete approximation scheme is by way of Bernoulli random variables,

$$\widehat{X}(t_{n+1}) = \widehat{X}(t_n) + a\left(\widehat{X}(t_n)\right) \Delta t + b\left(\widehat{X}(t_n)\right) \sqrt{\Delta t}\, \xi, \tag{14.9}$$

where $\text{Prob}[\xi = 1] = \text{Prob}[\xi = -1] = 1/2$. Note that $E[\xi] = 0$ and $\text{Var}[\xi] = 1$. This clearly defines a binomial model. As Δt goes to zero, $\widehat{X}$ converges to X [255, 388, 485]. All the above-mentioned schemes work for *non-stationary* stochastic differential equations, $dX = a(X,t)\, dt + b(X,t)\, dW$.

The quality of a discrete-time approximation must be judged based on some criterion reflecting the goal of simulations. The simulation of solutions of stochastic differential equations falls into two basic types. The first type, such as direct simulation, demands a good **pathwise** approximation, meaning the sample paths of the approximation should be close to those of the Ito process. The second type is concerned with approximating expectations of some function of the value of the Ito process or its probability distribution at a given time. This is useful when the analytic solution proves elusive. The requirements here are not as stringent as for pathwise approximations.

14.2.2 The Ito integral and trading strategies

Consider an Ito process $d\boldsymbol{S}_t = \mu_t\, dt + \sigma_t\, dW_t$. Interpret $\boldsymbol{S}_t$ as the vector of security prices at time t. Let $\boldsymbol{\phi}_t$ be a trading strategy denoting the quantity of each type of security held at time t. Clearly, the stochastic process $\boldsymbol{\phi}_t \boldsymbol{S}_t$ is the *value* of the portfolio $\boldsymbol{\phi}_t$ at time t. The stochastic integral $\boldsymbol{\phi}_t\, d\boldsymbol{S}_t \equiv \boldsymbol{\phi}_t\, (\mu_t\, dt + \sigma_t\, dW_t)$ then represents the change in the value from security price changes occurring at time t. The equivalent Ito integral,

$$G_T(\boldsymbol{\phi}) \equiv \int_0^T \boldsymbol{\phi}_t\, d\boldsymbol{S}_t = \int_0^T \boldsymbol{\phi}_t \mu_t\, dt + \int_0^T \boldsymbol{\phi}_t \sigma_t\, dW_t,$$

corresponds to the capital gains realized by the trading strategy over the period $[\,0, T\,]$. Finally, a strategy is self-financing if

$$\boldsymbol{\phi}_t \boldsymbol{S}_t = \boldsymbol{\phi}_0 \boldsymbol{S}_0 + G_t(\boldsymbol{\phi}) \quad 0 \le t < T. \tag{14.10}$$

These concepts can be captured clearly under discrete-time models. Let $t_0 < t_1 < \cdots < t_n$ denote the trading points. As before, $\boldsymbol{S}_k$ is the price vector at time t_k, and the vector $\boldsymbol{\phi}_k$ denotes the quantity of each security held during $[\,t_k, t_{k+1})$. Then $\boldsymbol{\phi}_k \boldsymbol{S}_k$ stands for the value of portfolio $\boldsymbol{\phi}_k$ right after its establishment at time t_k, and $\boldsymbol{\phi}_k \boldsymbol{S}_{k+1}$ stands for the value of portfolio $\boldsymbol{\phi}_k$ at time t_{k+1}, before any transactions are made. The nonanticipation requirement of the Ito integral means $\boldsymbol{\phi}_k$ must be established before $\boldsymbol{S}_{k+1}$ is known. The quantity $\boldsymbol{\phi}_k \Delta \boldsymbol{S}_k \equiv \boldsymbol{\phi}_k\, (\boldsymbol{S}_{k+1} - \boldsymbol{S}_k)$ represents the capital gains between times t_k and t_{k+1}, and the summation

$$G(n) \equiv \sum_{k=0}^{n-1} \boldsymbol{\phi}_k \Delta \boldsymbol{S}_k$$

is the total capital gains through time t_n. The above summation is consistent with the definition of the Ito integral (14.2). A trading strategy is self-financing if

$$\boldsymbol{\phi}_k \boldsymbol{S}_k = \boldsymbol{\phi}_{k-1} \boldsymbol{S}_k$$

for all $0 < k \le n$, that is, if there is no injection or withdrawal of funds at any time. This condition implies (14.10) under the discrete-time context (see Exercise 14.2.1). The transition to continuous time is now clear: The Ito integral is the limit of the Ito integrals of simple processes.

Consider the Ito process $dX_t = a_t\, dt + b_t\, dW_t$ again. The nonanticipating requirement says that a_t and b_t cannot embody future values of dW, and the future evolution of X depends solely upon its current value. In other words, X is a Markov process. Ito process is hence ideal for modeling asset price dynamics under the weak form of efficient markets.

14.2.3 Ito's lemma

The central tool in the Ito integral is Ito's lemma, which reduces many stochastic problems to deterministic differential equations. Ito's lemma basically says a smooth function of an Ito process is itself an Ito process.

Theorem 14.2.1 (Ito's lemma: one-dimensional version) *Suppose $f : R \to R$ is twice continuously differentiable,*[1] *and $dX = a_t\,dt + b_t\,dW$. Then $f(X)$ is the Ito process,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\; a_s\, ds + \int_0^t f'(X_s)\; b_s\, dW + \frac{1}{2}\int_0^t f''(X_s)\; b_s^2\, ds$$

for $t \geq 0$. □

In differential form, Ito's lemma becomes

$$df(X) = f'(X)\; a\, dt + f'(X)\; b\, dW + \frac{1}{2} f''(X)\; b^2\, dt. \tag{14.11}$$

Compared to calculus, the interesting part is the third term on the right-hand side. This can be traced to the positive quadratic variation of Brownian paths, making $(dW)^2$ non-negligible. A convenient formulation of Ito's lemma suitable for generalization to higher dimensions is

$$df(X) = f'(X)\, dX + \frac{1}{2} f''(X)(dX)^2. \tag{14.12}$$

Here, one is supposed to multiply out $(dX)^2 = (a\,dt + b\,dW)^2$ symbolically according to the following multiplication table:

$\times$	dW	dt
dW	dt	0
dt	0	0

Note that the $(dW)^2 = dt$ entry is justified by (13.16). This form is easy to remember because of the similarity to Taylor expansion.

Theorem 14.2.2 (Ito's lemma: higher-dimensional version) *Let $W_1, \ldots, W_n$ be n independent Wiener processes, and $X \equiv (X_1, \ldots, X_m)$ be a vector process. Suppose $f : R^m \to R$ is twice continuously differentiable and X_i is an Ito process with $dX_i = a_i\,dt + \sum_{j=1}^n b_{ij}\,dW_j$. Then, $df(X)$ is an Ito process with the differential,*

$$df(X) = \sum_{i=1}^m f_i(X)\, dX_i + \frac{1}{2}\sum_{i=1}^m \sum_{k=1}^m f_{ik}(X)\, dX_i\, dX_k. \tag{14.13}$$

Here, $f_i \equiv \partial f/\partial x_i$ and $f_{ik} \equiv \partial^2 f/\partial x_i \partial x_k$. □

The multiplication table for Theorem 14.2.2 is

$\times$	dW_i	dt
dW_k	$\delta_{ik}\,dt$	0
dt	0	0

[1] This means all first- and second-order partial derivatives exist and are continuous.

in which

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

In applying the higher-dimensional Ito's lemma, usually one of the variables, say X_1, is the time variable t and $dX_1 = dt$. Hence, $b_{1j} = 0$ for all j, and $a_1 = 1$. An alternative formulation incorporates the interdependence of the variables $X_1, \ldots, X_m$ into that between the Wiener processes.

Theorem 14.2.3 (Ito's lemma: alternative higher-dimensional version) *Let $W_1, \ldots, W_m$ be m Wiener processes and $X \equiv (X_1, \ldots, X_m)$ be a vector process. Suppose $f : R^m \to R$ is twice continuously differentiable, and X_i is an Ito process with $dX_i = a_i\, dt + b_i\, dW_i$. Then $df(X)$ is the following Ito process,*

$$df(X) = \sum_{i=1}^{m} f_i(X)\, dX_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} f_{ik}(X)\, dX_i\, dX_k$$

with the following multiplication table:

$\times$	dW_i	dt
dW_k	$\rho_{ik}\, dt$	0
dt	0	0

Here, ρ_{ik} denotes the correlation between dW_i and dW_k. □

In the above theorem, the correlation between $dW_i = \sqrt{dt}\, \xi_i$ and $dW_k = \sqrt{dt}\, \xi_k$ refers to that between the normally distributed random variables ξ_i and ξ_k.

14.2.4 Stochastic differential equations

For stochastic differential equations of the form $dX_t = a_t(X_t)\, dt + b_t(X_t)\, dW_t$, there are regularity conditions that guarantee the existence and uniqueness of solution [24, 327, 494]. The solution to a stochastic differential equation is also called a **diffusion process**. A stochastic differential equation is **linear** if both $b_t(x)$ and $a_t(x)$ are linear functions of x. In this case, the solution X_t is closely related to the solution of the corresponding deterministic differential equation. The mean and covariance functions of X_t as well as more general treatments of linear stochastic differential equations can be found in [24].

14.3 Applications

This section presents applications of the Ito process, some of which will be useful later.

Example 14.3.1 A (μ, σ) Brownian motion is $\mu\, dt + \sigma\, dW$ by Ito's lemma and (13.11). □

Example 14.3.2 Consider the Ito process $dX = \mu(t)\, dt + \sigma(t)\, dW$. It is identical to Brownian motion except that the drift $\mu(t)$ and diffusion $\sigma(t)$ are no longer constants. As with Brownian motion,

$$X(t) \sim N\left(X(0) + \int_0^t \mu(s)\, ds, \int_0^t \sigma^2(s)\, ds \right)$$

is normally distributed. □

Example 14.3.3 Consider the geometric Brownian motion process $Y(t) \equiv e^{X(t)}$, where $X(t)$ is a (μ, σ) Brownian motion. Ito's formula (14.11) implies

$$\frac{dY}{Y} = \left(\mu + \frac{1}{2}\sigma^2\right) dt + \sigma\, dW.$$

The instantaneous rate of return is $\mu + \sigma^2/2$ not μ, consistent with Exercise 13.3.5. □

Example 14.3.4 Consider the Ito process $U \equiv YZ$ with $dY = a\,dt + b\,dW$ and $dZ = f\,dt + g\,dW$. Processes Y and Z share the Wiener process W. Ito's lemma (Theorem 14.2.2) can be employed to show that

$$dU = Z\,dY + Y\,dZ + dY\,dZ.$$

This can be expanded using formal multiplication into

$$dU = Z\,dY + Y\,dZ + (a\,dt + b\,dW)(f\,dt + g\,dW) = Z\,dY + Y\,dZ + bg\,dt.$$

If either $b \equiv 0$ or $g \equiv 0$, then integration by parts holds. □

Example 14.3.5 (Geometric average of correlated geometric Brownian motion) Consider the Ito process $U \equiv YZ$, where $dY/Y = a\,dt + b\,dW_y$ and $dZ/Z = f\,dt + g\,dW_z$. The correlation between W_y and W_z is ρ. Apply Theorem 14.2.3 to show that

$$\begin{aligned} dU &= Z\,dY + Y\,dZ + dY\,dZ \\ &= ZY\,(a\,dt + b\,dW_y) + YZ\,(f\,dt + g\,dW_z) + YZ\,(a\,dt + b\,dW_y)(f\,dt + g\,dW_z) \\ &= U\,(a + f + bg\rho)\,dt + Ub\,dW_y + Ug\,dW_z. \end{aligned}$$

Note that dU/U has volatility $\sqrt{b^2 + 2bg\rho + g^2}$ by (6.13). The product of two (or more) correlated geometric Brownian motion processes thus remains geometric Brownian motion. This result has applications in **correlation options**, whose value depends on multiple assets [793]. Now, since

$$\begin{aligned} Y &= \exp\left[\left(a - \frac{b^2}{2}\right) dt + b\,dW_y\right] \\ Z &= \exp\left[\left(f - \frac{g^2}{2}\right) dt + g\,dW_z\right] \\ U &= \exp\left[\left(a + f - \frac{b^2 + g^2}{2}\right) dt + \sqrt{b\,dW_y + g\,dW_z}\right] \end{aligned}$$

$\ln U$ is Brownian motion with a mean equal to the sum of the means of $\ln Y$ and $\ln Z$. This holds even if Y and Z are correlated. Also, $\ln Y$ and $\ln Z$ have correlation ρ. □

Example 14.3.6 Assume S follows the geometric Brownian motion process,

$$\frac{dS}{S} = \mu\,dt + \sigma\,dW.$$

Then the process $F(S,t) \equiv Se^{y(T-t)}$ follows another geometric Brownian motion process,

$$\frac{dF}{F} = (\mu - y)\,dt + \sigma\,dW,$$

by Ito's lemma. This result has applications in pricing forward and futures contracts. □

14.3.1 Correlated geometric Brownian motion processes

Let $B \equiv [\, b_{ij}\,]_{1\leq i\leq n, 1\leq j\leq n}$ be a non-singular $n \times n$ matrix. (A square matrix is **non-singular** if its inverse exists.) Define $C \equiv [\, c_{ij}\,] = BB^{\mathrm{T}}$, that is, $c_{ij} = \sum_k b_{ik}\, b_{jk}$. Let $W_1, \ldots, W_n$ be n independent Wiener processes and

$$Y(t) \equiv [\, Y_1(t), \ldots, Y_n(t)\,] = \boldsymbol{\mu} t + B\boldsymbol{W}(t)$$

be vector Brownian motion, where $\boldsymbol{W} \equiv [\, W_1, \ldots, W_n\,]^{\mathrm{T}}$ and $\boldsymbol{\mu} \equiv [\, \mu_1, \ldots, \mu_n\,]^{\mathrm{T}}$. Thus, Y_i is Brownian motion with drift μ_i and variance $\sum_k b_{ik}^2$. In fact, the covariance matrix of $Y(t)$ is precisely C (see Exercise 14.3.11). Set $Z_i(t) \equiv Z_i(0)\, e^{Y_i(t)-(c_{ii}t/2)}$ for $i = 1, \ldots, n$ with constants $Z_i(0) > 0$. Ito's lemma can be applied here to obtain $dZ_i/Z_i = dY_i$ (see Exercise 14.3.12). Define the **correlated geometric Brownian motion** as $S_i(t) \equiv e^{rt} Z_i(t)$. Then $dS_i/S_i = r\, dt + dY_i$. This is one possible model for correlated stock prices.

14.3.2 The Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck process has the Ito differential,

$$dX = -\kappa X\, dt + \sigma\, dW, \tag{14.14}$$

where $\kappa, \sigma \geq 0$. See Fig. 14.3 for illustration. Given $X(t_0) = x_0$, it can be shown that

$$\begin{aligned} E[\, X(t)\,] &= e^{-\kappa\,(t-t_0)}\, E[\, x_0\,] \\ \mathrm{Var}[\, X(t)\,] &= \frac{\sigma^2}{2\kappa} \left(1 - e^{-2\kappa\,(t-t_0)}\right) + e^{-2\kappa\,(t-t_0)}\, \mathrm{Var}[\, x_0\,] \\ \mathrm{Cov}[\, X(s), X(t)\,] &= \frac{\sigma^2}{2\kappa}\, e^{-\kappa\,(t-s)} \left(1 - e^{-2\kappa\,(s-t_0)}\right) + e^{-\kappa\,(t+s-2t_0)}\, \mathrm{Var}[\, x_0\,] \end{aligned}$$

for $t_0 \leq s \leq t$. In fact, as with Brownian motion, $X(t)$ is normally distributed if x_0 is a constant or normally distributed [24]. For this reason, X is a **normal process**. Of course, $E[\, x_0\,] = x_0$ and $\mathrm{Var}[\, x_0\,] = 0$ if x_0 is a constant. If $x_0 \sim N\left(0, \frac{\sigma^2}{2\kappa}\right)$, then it is easy to see that X is stationary.

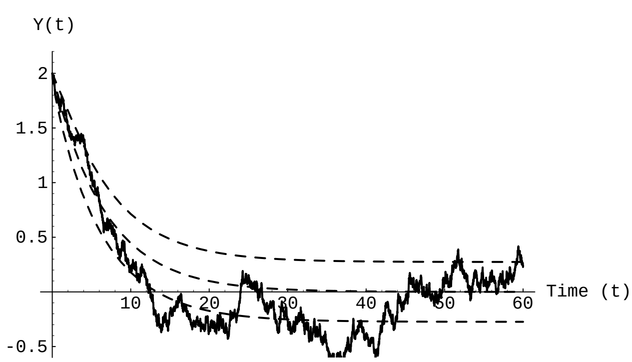

Figure 14.3: SAMPLE PATH OF ORNSTEIN-UHLENBECK PROCESS. Depicted is a sample path of the Ornstein-Uhlenbeck process $dY = -0.15\, Y\, dt + 0.15\, dW$, starting at $Y(0) = 2$. The envelope is for one standard deviation $\sqrt{\frac{(0.15)^2}{0.3}\,(1 - e^{-0.3\,t})}$ around the mean $2e^{-0.15\,t}$. In contrast to Brownian motion, which diverges to infinite values (see Fig. 13.3), the Ornstein-Uhlenbeck process converges to a stationary distribution.

The Ornstein-Uhlenbeck process describes the velocity of a tiny particle through a fluid in thermal equilibrium—in a word, Brownian motion in nature [339]. This process has the following **mean reversion** property. When $X > 0$, the dX term tends to be negative, pulling X toward zero, while if $X < 0$, the dX term tends to be positive, pulling X toward zero again. For this reason, the Ornstein-Uhlenbeck process is sometimes called **elastic random walk**. Other names for the Ornstein-Uhlenbeck process include **AR(1) process** and **mean-reverting Wiener process**.

Example 14.3.7 Suppose X is an Ornstein-Uhlenbeck process. Ito's lemma says $V \equiv X^2$ has the differential,

$$\begin{aligned} dV &= 2X\,dX + (dX)^2 = 2\sqrt{V}\left(-\kappa\sqrt{V}\,dt + \sigma\,dW\right) + \sigma^2\,dt \\ &= \left(-2\kappa V + \sigma^2\right)\,dt + 2\sigma\sqrt{V}\,dW, \end{aligned}$$

a **square-root process**. □

Consider the following process, also called the Ornstein-Uhlenbeck process,

$$dX = \kappa\,(\mu - X)\,dt + \sigma\,dW, \tag{14.15}$$

where $\sigma \geq 0$. Given $X(t_0) = x_0$, a constant, it can be shown that

$$E[\,X(t)\,] = \mu + (x_0 - \mu)\,e^{-\kappa\,(t-t_0)} \tag{14.16}$$

$$\text{Var}[\,X(t)\,] = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa\,(t-t_0)}\right) \tag{14.17}$$

for $t_0 \leq t$ [759]. Since the mean and standard deviation are roughly μ and $\sigma/\sqrt{2\kappa}$, respectively, for large t, the probability of X being negative is extremely unlikely in any finite time interval when $\mu > 0$ is relatively large compared to $\sigma/\sqrt{2\kappa}$ (say $\mu > 4\sigma/\sqrt{2\kappa}$).

The process in (14.15) has the salient mean-reverting feature that X tends to move toward μ. This property makes it useful for modeling term structure [759], stock price volatility [726], and stock price return [528].

14.3.3 The square-root process

The square-root process has the Ito differential,

$$dX = \kappa\,(\mu - X)\,dt + \sigma\sqrt{X}\,dW,$$

where $\kappa, \sigma \geq 0$, and the initial value of X is a non-negative constant. See Fig. 14.4 for illustration. Like the Ornstein-Uhlenbeck process, it has mean reversion in that X tends to move toward μ; the volatility is proportional to $\sqrt{X}$ instead of a constant, however. When X hits zero, the probability is one that it will not move below zero if $\mu \geq 0$; in other words, zero is a **reflecting boundary**. Hence, the square-root process is a good candidate for modeling interest rate movements [203]. The Ornstein-Uhlenbeck process, in contrast, would allow negative interest rates. The two processes are actually related (see Example 14.3.7).

Feller (1906–1970) showed that the random variable $2cX(t)$ follows the non-central Chi-square distribution [203],

$$\chi\left(\frac{4\kappa\mu}{\sigma^2}, 2cX(0)\, e^{-\kappa t}\right),$$

where $c \equiv \left(2\kappa/\sigma^2\right)\left(1 - e^{-\kappa t}\right)^{-1}$. Given $X(0) = x_0$, a constant, it can be proved that

$$\begin{aligned} E[\,X(t)\,] &= x_0 e^{-\kappa t} + \mu\left(1 - e^{-\kappa t}\right) \\ \mathrm{Var}[\,X(t)\,] &= x_0\,\frac{\sigma^2}{\kappa}\left(e^{-\kappa t} - e^{-2\kappa t}\right) + \mu\,\frac{\sigma^2}{2\kappa}\left(1 - e^{-\kappa t}\right)^2 \end{aligned}$$

for $t \geq 0$.

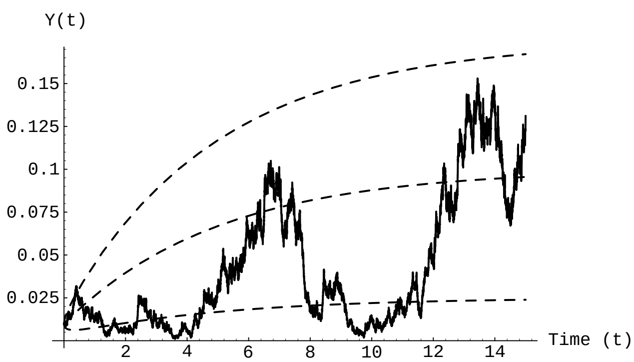

Figure 14.4: SAMPLE PATH OF SQUARE-ROOT PROCESS. Depicted is a sample path of the square-root process $dY = 0.2\,(0.1 - Y)\,dt + 0.15\,\sqrt{Y}\,dW$ with the initial condition $Y(0) = 0.01$. The envelope is for one standard deviation around the mean $0.01\,e^{-0.2\,t} + 0.1\,\left(1 - e^{-0.2\,t}\right)$.

14.4 Backward and Forward Equations

Let $p(x, y; t)$ denote the transition probability density function of a (μ, σ) Brownian motion starting at x. By (13.10),

$$p(x, y; t) = \frac{1}{\sqrt{2\pi t}\,\sigma}\exp\left[-\frac{(y - x - \mu t)^2}{2\sigma^2 t}\right],$$

which satisfies **Kolmogorov's backward equation**,

$$\frac{\partial p}{\partial t} = \frac{1}{2}\,\sigma^2\frac{\partial^2 p}{\partial x^2} + \mu\frac{\partial p}{\partial x},$$

and **Kolmogorov's forward equation** (also called the **Fokker-Planck equation**),

$$\frac{\partial p}{\partial t} = \frac{1}{2}\,\sigma^2\frac{\partial^2 p}{\partial y^2} - \mu\frac{\partial p}{\partial y}.$$

This generalizes Exercise 13.3.4.

Both the backward equation and the Fokker-Planck equation describe a large class of stochastic processes with continuous sample paths [327]. Consider the Ito process,

$$dX_t = a(X_t, t)\,dt + b(X_t, t)\,dW_t,$$

where $X_t \in \boldsymbol{R}$. It can be shown that, under some regularity conditions, the transition probability density function $p(x, s; y, t)$ of $X(t)$ given $X(s) = x$, i.e.,

$$\text{Prob}[\, a < X(t) < b \mid X(s) = x\,] = \int_a^b p(x, s; y, t)\, dy,$$

satisfies Kolmogorov's backward equation,

$$\frac{\partial p}{\partial s} = -\frac{1}{2}\, b(x, s)^2 \frac{\partial^2 p}{\partial x^2} - a(x, s)\, \frac{\partial p}{\partial x},$$

and the Fokker-Planck equation [24, 319],

$$\frac{\partial p}{\partial t} = \frac{1}{2}\, \frac{\partial^2 \left(b(y, t)^2 p\right)}{\partial y^2} - \frac{\partial\, (a(y, t)\, p)}{\partial y}.$$

For instance, the transition density function of the Ornstein-Uhlenbeck process $dX = -\kappa X\, dt + \sigma\, dW$ satisfies the following backward and Fokker-Planck equations,

$$\frac{\partial p}{\partial s} = -\frac{1}{2}\, \sigma^2 \frac{\partial^2 p}{\partial x^2} + \kappa x\, \frac{\partial p}{\partial x} \quad \text{and} \quad \frac{\partial p}{\partial t} = \frac{1}{2}\, \sigma^2\, \frac{\partial^2 p}{\partial y^2} + \kappa\, \frac{\partial\, (py)}{\partial y},$$

and that of the square-root process $dX = -\kappa X\, dt + \sigma \sqrt{X}\, dW$ satisfies

$$\frac{\partial p}{\partial s} = -\frac{1}{2}\, \sigma^2 x\, \frac{\partial^2 p}{\partial x^2} + \kappa x\, \frac{\partial p}{\partial x} \quad \text{and} \quad \frac{\partial p}{\partial t} = \frac{1}{2}\, \frac{\partial^2 \left(\sigma^2 y p\right)}{\partial y^2} + \frac{\partial\, (\kappa y p)}{\partial y}.$$

14.5 Applications in Finance

Some applications of continuous-time processes to finance are covered in this section. Many of them will be explored in more details later in the book.

14.5.1 Transactions costs

Transactions costs are a fact of life, never zero however negligible. Under the **proportional transactions cost model**, it is impossible to trade continuously. Here is the intuition. The transactions cost per trade is proportional to $|\, dW\,|$. But we already demonstrated that $\int_0^T |\, dW\,| = \infty$ almost surely in (13.18). As a consequence, a continuous trader would be bankrupt with probability one [571]. Even stronger claims can be made. For instance, the cheapest trading strategy to dominate the value of European call at maturity is covered call [715]; hence *any* strategy that replicates the European call must be trivial.

14.5.2 Stochastic interest rate models

Merton originated the following methodology to term structure modeling in 1970 [447]. Suppose the short rate r follows $dr = \mu(r, t)\, dt + \sigma(r, t)\, dW$. Let $P(r, t, T)$ denote the price at time t of a zero-coupon bond that pays one dollar at time T. Write its dynamics as

$$\frac{dP}{P} = \mu_p\, dt + \sigma_p\, dW$$

so that the expected instantaneous rate of return on a $(T-t)$-year zero-coupon bond is μ_p and the instantaneous variance is σ_p^2. Of course, $P(r,T,T)=1$ holds for any T. By Ito's lemma (Theorem 14.2.2),

$$\begin{aligned} dP &= \frac{\partial P}{\partial T}\,dT + \frac{\partial P}{\partial r}\,dr + \frac{1}{2}\,\frac{\partial^2 P}{\partial r^2}\,(dr)^2 \\ &= -\frac{\partial P}{\partial T}\,dt + \frac{\partial P}{\partial r}\,(\mu(r,t)\,dt + \sigma(r,t)\,dW) + \frac{1}{2}\,\frac{\partial^2 P}{\partial r^2}\,(\mu(r,t)\,dt + \sigma(r,t)\,dW)^2 \\ &= \left(-\frac{\partial P}{\partial T} + \mu(r,t)\,\frac{\partial P}{\partial r} + \frac{\sigma(r,t)^2}{2}\,\frac{\partial^2 P}{\partial r^2}\right)dt + \sigma(r,t)\,\frac{\partial P}{\partial r}\,dW, \end{aligned}$$

where $dt=-dT$ in the second equality. Hence,

$$-\frac{\partial P}{\partial T} + \mu(r,t)\,\frac{\partial P}{\partial r} + \frac{\sigma(r,t)^2}{2}\,\frac{\partial^2 P}{\partial r^2} = P\mu_p \quad\text{and}\quad \sigma(r,t)\,\frac{\partial P}{\partial r} = P\sigma_p. \tag{14.18}$$

Models in which the short rate is the only explanatory variable are called **short rate models**.

The Merton model

If we impose the local expectations theory, which means μ_p equals the prevailing spot rate $r(t)$ for all T, and assume μ and σ are constants, then the above partial differential equations have the following solution,

$$P(r,t,T) = \exp\left[-r\,(T-t) - \frac{\mu\,(T-t)^2}{2} + \frac{\sigma^2\,(T-t)^3}{6}\right]. \tag{14.19}$$

This model was first considered by Merton [571]. We make a few observations here. First, $\sigma_p = -\sigma\,(T-t)$, which says, sensibly, that bonds with longer maturity are more volatile. The dynamics of P is $dP/P = r\,dt - \sigma\,(T-t)\,dW$. Now, P goes to infinity as T does likewise, but this does not square with the reality at all. This results partially from the negative rates allowed by the model. We shall develop more elaborate interest rate models starting from Chapter 23.

Duration under parallel shifts

Define duration with respect to constant parallel shifts in the spot rate curve. For convenience, assume $t=0$. Recall that the spot rate curve is defined by

$$S(r,T) \equiv -\frac{\ln P(r,T)}{T}.$$

Parallel shift means $S(r+\Delta r,T) = S(r,T)+\Delta r$ for any Δr; so $\partial S(r,T)/\partial r = 1$. This implies $S(r,T) = r + g(T)$ for some g with $g(0)=0$ as $S(r,0)=r$. Consequently,

$$P(r,T) = e^{-(r+g(T))\,T}$$

Substitute this identity into the left-hand part of (14.18) and assume the local expectations theory to obtain

$$g'(T) + \frac{g(T)}{T} = \mu(r) - \frac{\sigma(r)^2}{2} T.$$

As the left-hand side is independent of r, so must the right-hand side. But the only way this can hold for all T is when both $\mu(r)$ and $\sigma(r)$ are constants—the Merton model. As mentioned before, this model is flawed, so must duration *as such* [450].

Immunization under parallel shifts revisited

A duration-matched portfolio under parallel shifts in the spot rate curve begets arbitrage opportunities in that the portfolio value exceeds the liability for any *instantaneous* rate change. This was shown in §5.10.2. However, this seeming inconsistency with equilibrium disappears if changes in portfolio value *through* time are considered. Indeed, we can show that, for some interest rate models, at any given time in the future, a portfolio value is a convex function of the prevailing interest rate, but a liability immunized by a duration-matched portfolio exceeds the minimum portfolio value. Thus, the claimed arbitrage profit evaporates because the portfolio value does not always cover the liability.

This point can be illustrated by the Merton model $dr = \mu\, dt + \sigma\, dW$, which results from parallel shifts in the spot rate curve and the local expectations theory. To immunize a \$1 liability due at time s, a two-bond portfolio is constructed now with maturity dates t_1 and t_2, where $t_1 < s < t_2$. Each bond is a zero-coupon bond with \$1 par value, and the portfolio contains $b_i > 0$ units of bond i for $i = 1, 2$. The portfolio matches the present value of the liability today, and its value relative to the present value of the liability is minimum among all such portfolios (review §4.2.2). Let the current time be zero and consider any future time t such that $0 < t < t_1$. With $A(t)$ denoting the portfolio value and $L(t)$ the liability value at time t, it can be shown that the asset/liability ratio $A(t)/L(t)$ is a convex function of the prevailing interest rate and $A(t) < L(t)$ (see Exercise 14.5.5). Consult [45] for tackling the same issue for other interest rate models such as the Ho-Lee model.

14.5.3 Modeling stock prices

The most popular stochastic model for stock prices has been geometric Brownian motion,

$$\frac{dS}{S} = \mu\, dt + \sigma\, dW. \tag{14.20}$$

From the analysis in Example 14.3.3, we know that $S/S(0)$ is built from e^X, where X is a $(\mu - \sigma^2/2, \sigma)$ Brownian motion. Equivalently, $\ln(S/S(0))$ is a $(\mu - \sigma^2/2, \sigma)$ Brownian motion. See Fig. 14.5 for illustration. This model best describes an equilibrium where expectations about the future returns have settled down [571].

To understand (14.20) better, look at its discrete analogue,

$$\frac{\Delta S}{S} = \mu\, \Delta t + \sigma \sqrt{\Delta t}\, \xi,$$

where $\xi \sim N(0, 1)$. Hence, $\Delta S/S \sim N(\mu\, \Delta t, \sigma^2 \Delta t)$. The percentage return for the next Δt time hence has mean $\mu\, \Delta t$ and variance $\sigma^2 \Delta t$. In other words, the percentage return per

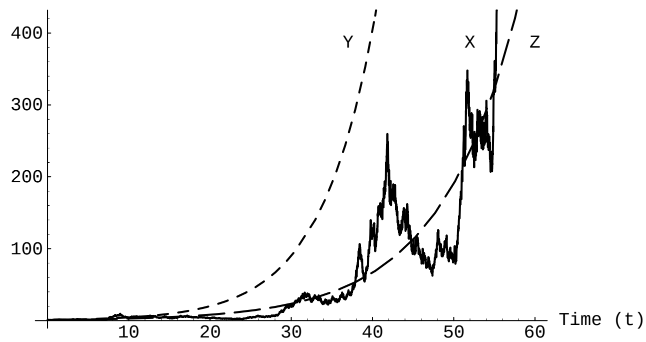

Figure 14.5: SAMPLE PATH OF GEOMETRIC BROWNIAN MOTION. The process X has the Ito differential $dX = 0.15\,X\,dt + 0.3\,X\,dW$ starting at $X(0) = 1$. Two related deterministic processes are plotted for reference: $dY = 0.15\,Y\,dt$ and $dZ = \left(0.15 - (0.3)^2/2\right) Z\,dt$. From the discussions in §14.5.3, we know that X is built by taking the exponentiation of a $\left(0.15 - (0.3)^2/2, 0.3\right)$ Brownian motion. Hence, it is Z, not Y, which traces the expected value of $X(t)$.

unit time has mean μ and variance σ^2, justifying calling μ the expected instantaneous rate of return and σ^2 (σ) the instantaneous variance (price volatility) of the rate of return. Note that, if there is no uncertainty about the stock price, that is, $\sigma \equiv 0$, then $S(t) = S(0)\,e^{\mu t}$, in which case the stock price grows at a continuously compounded rate of μ.

Comment 14.5.1 It may seem strange that the rate of return of a stock with $dS/S = \mu\,dt + \sigma\,dW$ is μ instead of $\mu - \sigma^2/2$. After all, we know that $S/S(0) = e^X$, where X is a $\left(\mu - \sigma^2/2, \sigma\right)$ Brownian motion, and therefore the continuously compounded rate of return over the time period $[0, T]$ is

$$\frac{\ln(S(T)/S(0))}{T} = \frac{X(T) - X(0)}{T} \sim N\left(\mu - \frac{\sigma^2}{2}, \sigma^2\right). \tag{14.21}$$

The expected continuously compounded rate of return is then $\mu - \sigma^2/2$! Well, they refer to alternative definitions of rates of return. Unless stated otherwise, it is the *former* (instantaneous rate of return μ) that we have in mind from now. It should be pointed out that the μ used in the binomial option pricing model, (9.17), referred to the latter rate of return. In summary,

$$\begin{aligned} \frac{E[\,(S(\Delta t) - S(0))/S(0)\,]}{\Delta t} &\to \mu \\ \frac{\ln E[\,S(T)/S(0)\,]}{T} &= \mu \\ \frac{E[\,\ln(S(T)/S(0))\,]}{T} &= \mu - \frac{\sigma^2}{2} \end{aligned}$$

by Comment 9.3.2, Corollary 9.3.4, Exercise 13.3.5, Example 14.3.3, and Exercise 14.5.6. □

Stock price and rate of return under the binomial model

What is the Ito process for the stock's rate of return in a risk-neutral economy to which the binomial model in §9.2 converges? A quick review first. The continuously compounded rate of return of the stock price over a period of length τ was shown to be a sum of n independent identically distributed random variables

$$X_i = \begin{cases} \ln u & \text{with probability } p \\ \ln d & \text{with probability } 1 - p \end{cases}$$

where $u \equiv e^{\sigma\sqrt{\tau/n}}$, $d \equiv e^{-\sigma\sqrt{\tau/n}}$, and $p \equiv \left(e^{r\tau/n} - d\right)/(u-d)$. The rate of return is hence the random walk $\sum_{i=1}^{n} X_i$.

It is straightforward to verify that

$$E\left[\sum_{i=1}^{n} X_i\right] \to \left(r - \frac{\sigma^2}{2}\right)\tau \quad \text{and} \quad \text{Var}\left[\sum_{i=1}^{n} X_i\right] \to \sigma^2\tau. \tag{14.22}$$

Employing the arguments in §13.3.1 to conclude that the continuously compounded rate of return converges to a $\left(r - \sigma^2/2, \sigma\right)$ Brownian motion. The stock price consequently follows

$$\frac{dS}{S} = r\,dt + \sigma\,dW \tag{14.23}$$

in a risk-neutral economy.

Additional Reading

We followed [472, 473, 666, 667] in the exposition of stochastic processes and [24, 250, 371, 474] in the discussion of stochastic integrals. Rigorous proofs of Ito's lemma can be found in [24, 371], whereas informal ones can be found in [421, 446, 571]. *Mathematica* programs for carrying out some of the manipulations are explained in [758]. Consult [484, 485, 486, 673] for numerical solutions of stochastic differential equations. See [528] for the multivariate Ornstein-Uhlenbeck process. Other useful references include [725] (diffusion), [480] (Ito integral), [242, 662] (stochastic processes), [180, 319, 676] (stochastic differential equations), [222] (stochastic convergence), [484, 485] (numerical techniques for stochastic differential equations), [475] (stochastic optimization in trading), [92, 154] (competitive trading without probabilistic assumptions), and [101, 223, 235, 588] (transactions costs).

Chapter 15

Continuous-Time Derivative Pricing

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This problem of time in the art of music
is of capital importance.
—Igor Stravinsky (1882–1971),
Poetics of Music [734, p. 31]

This chapter first presents the partial differential equation that an option value should satisfy. The general methodology is then extended to any derivatives, including options on stocks that pay continuous dividends, futures, futures options, correlation options, exchange options, path-dependent options, currency-related options, barrier options, convertible bonds with call provisions, and options under stochastic volatility.

15.1 Partial Differential Equations: a Brief Introduction

A two-dimensional (that is, two independent variables) second-order partial differential equation has the following form,

$$p \frac{\partial^2 \theta}{\partial x^2} + q \frac{\partial^2 \theta}{\partial x \, \partial y} + r \frac{\partial^2 \theta}{\partial y^2} + s \frac{\partial \theta}{\partial x} + t \frac{\partial \theta}{\partial y} + u\, \theta + v = 0, \tag{15.1}$$

where p, q, r, s, t, u, v may be functions of the independent variables x and y as well as the dependent variable θ and its derivatives. It is called **elliptic**, **parabolic**, or **hyperbolic** according as $q^2 < 4pr$, $q^2 = 4pr$, or $q^2 > 4pr$, respectively, over the domain of interest. For this reason, $q^2 - 4pr$ is called the **discriminant**. See [398, §3.1] for discussions on the significance of the above classification in solving these equations.

Partial differential equations can also be classified into **initial value** and **boundary value problems**. An initial value problem propagates the solution forward in time from the values given at the starting point. In contrast, a (two-point) boundary value problem has known values which must be satisfied at both ends of the interval [348]. If the conditions for some independent variables are given in the form of initial values and those for others as boundary conditions, we have an **initial value boundary problem**.

A standard elliptic equation is the two-dimensional **Poisson's equation**,

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = -\rho(x, y).$$

The **wave equation**,

$$\frac{\partial^2 \theta}{\partial t^2} - \frac{1}{v^2} \frac{\partial^2 \theta}{\partial x^2} = 0,$$

is hyperbolic. The most important parabolic equation is the **diffusion equation**,

$$\frac{1}{2} D \frac{\partial^2 \theta}{\partial x^2} - \frac{\partial \theta}{\partial t} = 0,$$

which is a special case of the Fokker-Planck equation. Given the initial condition $\theta(x, 0) = f(x)$ for $-\infty < x < \infty$, its unique bounded solution for $t > 0$ is a weighted average of $f(x)$,

$$\frac{1}{\sqrt{2\pi D t}} \int_{-\infty}^{\infty} f(z)\, e^{-(x-z)^2/(2Dt)}\, dz, \tag{15.2}$$

when $f(x)$ is bounded and piecewise continuous for all real x. The solution clearly depends on the entire initial condition (say, the temperature distribution). This is the diffusion equation on an infinite domain.

It is known that (15.1) can be reduced to generalized forms of Poisson's equation, the diffusion equation, or the wave equation according as it is elliptic, parabolic, or hyperbolic. variables for x and y.

15.2 The Black-Scholes Differential Equation

Black and Scholes in their path-breaking work [80] showed that the price of any derivative on non-dividend-paying stock must satisfy a partial differential equation. The crust of the argument lies in setting up a *riskless* portfolio of the stock and derivative. The trick is to recognize that the same random process drives both securities; it is **systematic**. Given that their prices are perfectly correlated, we can figure out the right proportion of stock such that the gain from one offsets the loss from the other exactly. This done, the portfolio value at the end of a short period of time is known for sure. This forces its return to be the riskless rate of interest in order to avoid arbitrage opportunities.

Several assumptions will be made. Chiefly among them are the following. The stock price follows the geometric Brownian motion $dS = \mu S\, dt + \sigma S\, dW$ with constant μ and σ. There are no dividends during the life of the derivative. Trading is continuous. Short selling is allowed, and there are no transactions costs or taxes. All securities are infinitely divisible. There are no riskless arbitrage opportunities. The term structure of riskless rates is flat at r, and there is unlimited riskless borrowing and lending. We remark that some of the assumptions can be relaxed; for instance, μ, σ, and r can be known functions of t instead of constants [421]. In the following, t denotes the current time (in years), and T denotes the expiration time with $\tau \equiv T - t$.

15.2.1 Merton's derivation

Let C be the price of a derivative on S. From Ito's lemma in Theorem 14.2.2, we have

$$dC = \left(\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \frac{\partial C}{\partial S}\, dW.$$

Note that the same W drives both C and S. The appropriate portfolio of the stock and the derivative that eliminates this random source is short one derivative and long $\partial C / \partial S$ shares of stock (see Fig. 15.1). Define Π as the value of the portfolio. By construction,

$$\Pi = -C + S \frac{\partial C}{\partial S}.$$

The change in the value of the portfolio in time dt is given by

$$d\Pi = -dC + \frac{\partial C}{\partial S}\, dS.$$

Substitute the formulae for dC and dS into the above equation to yield

$$d\Pi = \left(-\frac{\partial C}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt.$$

Since this equation does not involve dW, the portfolio is riskless during time dt and hence earns an instantaneous return rate of r, that is, $d\Pi = r\Pi\, dt$. This equation becomes

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\,\sigma^2 S^2\, \frac{\partial^2 C}{\partial S^2} \right) dt = r \left(C - S\, \frac{\partial C}{\partial S} \right) dt$$

after applying the formulae for $d\Pi$ and Π. Equate the terms to finally obtain

$$\frac{\partial C}{\partial t} + rS\, \frac{\partial C}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\, \frac{\partial^2 C}{\partial S^2} = rC. \tag{15.3}$$

This is the celebrated **Black-Scholes differential equation**.

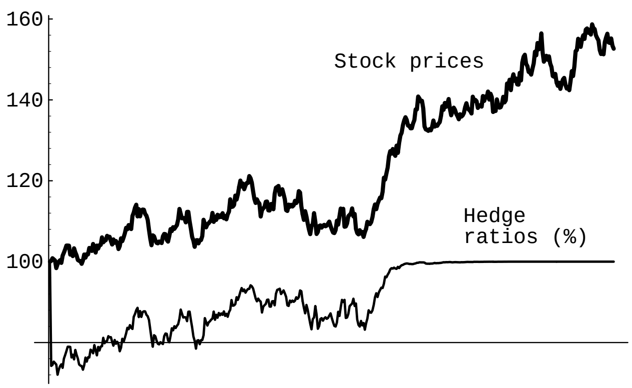

Figure 15.1: STOCK PRICE AND DELTA, $\partial C/\partial S$. Here, the current stock price is \$100, and the strike price is \$95.

The Black-Scholes differential equation can be expressed in terms of sensitivity numbers,

$$\Theta + rS\Delta + \frac{1}{2}\,\sigma^2 S^2\, \Gamma = rC. \tag{15.4}$$

(Review §10.1 for the definitions of sensitivity measures. Note that, there, differentiation was with respect to τ, not t.) Note that the above equation leads to an alternative way of computing theta Θ numerically from delta Δ and gamma Γ. In particular, if a portfolio is delta-neutral, then the above equation becomes

$$\Theta + \frac{1}{2}\,\sigma^2 S^2\, \Gamma = rC,$$

which shows a definite relationship between Γ and Θ.

15.2.2 Initial and boundary conditions

Solving the Black-Scholes differential equation depends on the initial and boundary conditions defining the particular derivative security with S as the underlying asset. These conditions spell out the values of the derivative at various values of S and t. In the case of European call, the key final condition is that the call value is $\max(S(T) - X, 0)$ at expiration. In the case of European put, it is that the put value is $\max(X - S(T), 0)$ at expiration. Note that a final condition becomes an initial condition by the change of variables $t' \equiv T - t$. There are also boundary conditions. The call value is zero when $S(t) = 0$, and the put is $Xe^{-r\,(T-t)}$ when $S(t) = 0$ (see Exercise 8.6.3). Furthermore, as S goes to infinity, the call value is S and the put is zero.

The boundary conditions above are more than what are mathematically necessary; however, they improve the accuracy of numerical methods [781]. The accuracy is even better if $S - Xe^{-r(T-t)}$ is used in place of S for the European call as $S \to \infty$.

The American put is more complicated because of the possibility of early exercise whose boundary $\overline{S}(t)$ is unknown *a priori*. The formulation that guarantees a unique solution is

$$\begin{aligned} \frac{\partial P}{\partial t} + rS\,\frac{\partial P}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 P}{\partial S^2} = rP \quad \text{and} \quad P > X - S \quad &\text{for} \quad \overline{S} < S < \infty \\ P = X - S \quad &\text{for} \quad 0 \leq S < \overline{S} \\ \frac{\partial P}{\partial S} = -1 \quad \text{and} \quad P = X - S \quad &\text{for} \quad S = \overline{S} \\ P = 0 \quad &\text{for} \quad S \to \infty \end{aligned}$$

plus the terminal condition $P = \max(X - S, 0)$ [135]. The region $0 \leq S < \overline{S}$ is where early exercise is optimal.

15.2.3 Remarks

Continuous adjustments

Note that the portfolio Π is riskless only for an infinitesimally short period of time. If $\partial C/\partial S$ changes as S and t do, the portfolio must be continuously adjusted to ensure that it remains riskless.

Number of random sources

There is no stopping at the single-factor random source. In the presence of two random sources, three securities suffice to eliminate uncertainty: Use two to eliminate the first source and add the third to eliminate the second source [213]. Of course, the factors must be traded. A **traded security** is an asset that is held solely for investment by a significant number of individuals. Generally speaking, a market is complete only if the number of traded securities is at least as large as the number of random sources [70].

Risk neutrality

Like the binomial option pricing model, the Black-Scholes differential equation does not depend directly on the risk preferences of investors. All the variables in the equation (current stock price, time to maturity, stock price volatility, and riskless rate) are independent of risk preferences. The one variable that depends on the risk preferences, the expected return on the stock, is not part of the equation. As a consequence, any risk preference can be used in pricing, including the risk-neutral one, and the resulting solutions will be valid in all worlds.

In a risk-neutral economy, the expected return on all securities is r. This is because risk-neutral investors do not require a premium for taking risks. Furthermore, present value can be obtained by discounting the expected value at the riskless rate. The risk-neutral assumption simplifies the analysis of derivatives. Lemma 9.2.1 says the same thing of the binomial option pricing model. In fact, the Black-Scholes formula can be derived along this line [421, 446, 793]. We emphasize that r is the *instantaneous* return rate of the stock (see Comment 14.5.1 for the subtleties).

15.3 Solving the Black-Scholes Differential Equation

The Black-Scholes differential equation can be solved directly for the European option. We shall follow Merton's steps [571]. Transform (15.3) with the change of variable, $C(S,\tau) \equiv B(S,\tau)\, e^{-r\tau} X$. The partial differential equation becomes, after simplification,

$$-\frac{\partial B}{\partial \tau} + rS\,\frac{\partial B}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 B}{\partial S^2} = 0,$$

where $B(0,\tau) = 0$ for $\tau > 0$ and $B(S,0) = \max(S/X - 1, 0)$ for $S > 0$. With $D(x,\tau) \equiv B(S,\tau)$ and $x \equiv (S/X)\, e^{r\tau}$, we end up with the diffusion equation,

$$-\frac{\partial D}{\partial \tau} + \frac{1}{2}\,(\sigma x)^2\,\frac{\partial^2 D}{\partial x^2} = 0,$$

where $D(0,\tau) = 0$ for $\tau > 0$ and $D(x,0) = \max(x-1, 0)$ for $x > 0$. After one more transformation, $u \equiv \sigma^2 \tau$, the function $H(x,u) \equiv D(x,\tau)$ satisfies

$$-\frac{\partial H}{\partial u} + \frac{1}{2}\,x^2\,\frac{\partial^2 H}{\partial x^2} = 0,$$

where $H(0,u) = 0$ for $u > 0$ and $H(x,0) = \max(x-1, 0)$ for $x > 0$. One final transformation, $\Theta(z,u)\, x \equiv H(x,u)$ with $z \equiv (u/2) + \ln x$, lands us at

$$-\frac{\partial \Theta}{\partial u} + \frac{1}{2}\,\frac{\partial^2 \Theta}{\partial z^2} = 0.$$

The boundary conditions are $|\,\Theta(z,u)\,| \leq 1$ for $u > 0$ and $\Theta(z,0) = \max(1 - e^{-z}, 0)$. By (15.2), the above diffusion equation has the following solution,

$$\begin{aligned}
\Theta(z,u) &= \frac{1}{\sqrt{2\pi u}} \int_0^\infty (1 - e^{-y})\, e^{-(z-y)^2/(2u)}\, dy \\
&= \frac{1}{\sqrt{2\pi u}} \int_0^\infty e^{-(z-y)^2/(2u)}\, dy - \frac{1}{\sqrt{2\pi u}} \int_0^\infty e^{-y}\, e^{-(z-y)^2/(2u)}\, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-z/\sqrt{u}}^\infty e^{-w_1^2/2}\, dw_1 - \frac{1}{\sqrt{2\pi}\, x} \int_{-(z-u)/\sqrt{u}}^\infty e^{-w_2^2/2}\, dw_2 \\
&= N\left(\frac{z}{\sqrt{u}}\right) - \frac{1}{x}\, N\left(\frac{z-u}{\sqrt{u}}\right)
\end{aligned}$$

with the change of variables $w_1 = (y-z)/\sqrt{u}$ and $w_2 = (y - z + u)/\sqrt{u}$. Hence,

$$H(x,u) = \Theta(\ln x + (u/2), u)\, x = x\, N\left(\frac{\ln x + (u/2)}{\sqrt{u}}\right) - N\left(\frac{\ln x - (u/2)}{\sqrt{u}}\right).$$

Put everything back to obtain

$$\begin{aligned}
C(S,\tau) &= H\left(\frac{S}{X}\, e^{r\tau}, \sigma^2 \tau\right) e^{-r\tau} X \\
&= SN\left(\frac{\ln(S/X) + r\tau + \sigma^2\tau/2}{\sqrt{\sigma^2 \tau}}\right) - e^{-r\tau} X\, N\left(\frac{\ln(S/X) + r\tau - \sigma^2\tau/2}{\sqrt{\sigma^2 \tau}}\right),
\end{aligned}$$

which is the Black-Scholes formula for the European call.

15.4 Variations and Applications

15.4.1 Continuous dividend yields

The price for a stock that continuously pays out dividends at an annualized rate of q follows

$$\frac{dS}{S} = (\mu - q)\, dt + \sigma\, dW, \tag{15.5}$$

where μ is the rate of return from stock ownership with the dividends included. This process was postulated for the stock index and exchange rate—price of foreign currency—before. In a risk-neutral economy, $\mu = r$ and the dynamics becomes

$$\frac{dS}{S} = (r - q)\, dt + \sigma\, dW.$$

In general, any derivative security whose value f depends on a stock paying a continuous dividend yield must satisfy a differential equation. From Ito's lemma in Theorem 14.2.2,

$$df = \left((\mu - q)\, S\, \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S\, \frac{\partial f}{\partial S}\, dW.$$

Set up a portfolio which is short one derivative security and long $\partial f/\partial S$ shares. The value of the portfolio is

$$\Pi = -f + S\, \frac{\partial f}{\partial S}.$$

The change in the value of the portfolio in time dt is given by

$$d\Pi = -df + \frac{\partial f}{\partial S}\, dS.$$

Substitute the formulae for df and dS into the above equation to yield

$$d\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 f}{\partial S^2} \right) dt.$$

The total wealth change is simply the above amount *plus* the dividends,

$$\left(-\frac{\partial f}{\partial t} - \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 f}{\partial S^2} \right) dt + qS\, \frac{\partial f}{\partial S}\, dt.$$

As its value is not stochastic, the portfolio must be instantaneously riskless; thus

$$\left(-\frac{\partial f}{\partial t} - \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 f}{\partial S^2} \right) dt + qS\, \frac{\partial f}{\partial S}\, dt = r\Pi\, dt.$$

Simplify to obtain

$$\frac{\partial f}{\partial t} + (r - q)\, S\, \frac{\partial f}{\partial S} + \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 f}{\partial S^2} = rf. \tag{15.6}$$

For the European call, the boundary conditions are identical to those of the standard option except that its value should be $Se^{-q(T-t)}$ as S goes to infinity. Review (9.28) for the solution. For the American call, the formulation that guarantees a unique solution is

$$\begin{aligned} \frac{\partial C}{\partial t} + (r-q)\, S\, \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC \ \text{ and } \ C > \max(S-X, 0) \quad &\text{for} \quad 0 \leq S < \overline{S} \\ C = S - X \quad &\text{for} \quad \overline{S} < S < \infty \\ \frac{\partial C}{\partial S} = 1 \ \text{ and } \ C = S - X \quad &\text{for} \quad S = \overline{S} \\ C = 0 \quad &\text{for} \quad S = 0 \end{aligned}$$

plus the terminal condition $C = \max(S - X, 0)$, of course [781].

15.4.2 Futures and futures options

Since the futures price is related to the spot price via $F = Se^{(r-q)(T-t)}$, we have

$$\frac{dF}{F} = \sigma\, dW$$

by Example 14.3.6. So, for pricing futures option, the futures price can be treated as a stock paying a continuous dividend yield equal to r. This is the rationale behind the Black model.

15.4.3 Barrier options

The value of a barrier option B satisfies the Black-Scholes differential equation. The initial and boundary conditions are different from those of the corresponding standard option only at the barrier H. Take the European down-and-out call on a non-dividend-paying stock for instance and assume $H < X$. The terminal value remains $B(S,T) = \max(S(T) - X, 0)$, and its value prior to expiration is approximately S as it goes to infinity. However, $B(H,t) = 0$ for $t < T$ because the option is worthless once S hits H.

For European down-and-in calls, the boundary conditions can be similarly derived. Suppose the barrier is yet to be crossed. Then the option's terminal value is zero, and its value prior to the expiration date is also zero as S goes to infinity because the stock price is unlikely to fall that low. The additional boundary condition is $B(H,t) = C(X,t)$, where $C(X,t)$ denotes the value of an otherwise identical standard European call. This clearly holds because the down-and-in call becomes a standard European call the moment the barrier is hit.

15.4.4 Path-dependent options

All the path-dependent options in §11.8 have continuous-time counterparts. To simplify the notation, assume the option was initiated at time zero. The average-rate call and put now have terminal values given by

$$\max\left(\frac{\int_0^T S(u)\, du}{T} - X, 0\right) \quad \text{and} \quad \max\left(X - \frac{\int_0^T S(u)\, du}{T}, 0\right),$$

respectively. This kind of averaging is called arithmetic average. Arithmetic average-rate options are notoriously hard to price [793]. Most average-rate options are European; however, the prices are sampled at *discrete* points in time [518]. If averaging is done geometrically, then the payoffs become

$$\max\left(\exp\left[\frac{\int_0^T \ln S(u)\,du}{T}\right]-X,0\right) \quad\text{and}\quad \max\left(X-\exp\left[\frac{\int_0^T \ln S(u)\,du}{T}\right],0\right),$$

respectively. Geometric averages are lognormally distributed when the underlying asset prices are so distributed (see Example 14.3.5). The lookback call and put on the average have values $\max\left(S(T)-\frac{1}{T}\int_0^T S(u)\,du, 0\right)$ and $\max\left(\frac{1}{T}\int_0^T S(u)\,du - S(T), 0\right)$ at expiration, respectively.

Analytic formulae are available for geometric average-rate European options and lookback options on the average [421, 479]. In the case of geometric average-rate European option, for example, the formulae can be derived from the Black-Scholes formulae with its volatility replaced by $\sigma/\sqrt{3}$ and dividend yield replaced by $(1/2)\left(r+q+\sigma^2/6\right)$.

We proceed to derive the partial differential equation satisfied by the value V of a European arithmetic average-rate option. Introduce a new variable $A(t) \equiv \int_0^t S(u)\,du$. First, it is not hard to verify that $dA = S\,dt$ without any explicit stochastic term. Ito's lemma (Theorem 14.2.2) applied to V says

$$dV = \left(\mu S\,\frac{\partial V}{\partial S}+\frac{\partial V}{\partial t}+\frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2}+S\,\frac{\partial V}{\partial A}\right)dt+\sigma S\,\frac{\partial V}{\partial S}\,dW.$$

Consider the portfolio of short one derivative and long $\partial V/\partial S$ shares of stock. This portfolio must earn riskless returns because of the lack of randomness. Hence, the partial differential equation is very much like the Black-Scholes differential equation,

$$\frac{\partial V}{\partial t}+rS\,\frac{\partial V}{\partial S}+\frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2}+S\,\frac{\partial V}{\partial A}=rV.$$

15.4.5 Options on more than one asset

For a correlation option whose value depends on the prices of two other assets, both of which follow lognormal diffusions, the partial differential equation is

$$\frac{\partial C}{\partial t}+rS_1\,\frac{\partial C}{\partial S_1}+rS_2\,\frac{\partial C}{\partial S_2}+\frac{1}{2}\,\sigma_1^2S_1^2\,\frac{\partial^2 C}{\partial S_1^2}+\frac{1}{2}\,\sigma_2^2S_2^2\,\frac{\partial^2 C}{\partial S_2^2}+\rho\sigma_1\sigma_2S_1S_2\,\frac{\partial^2 C}{\partial S_1\partial S_2}=rC. \tag{15.7}$$

15.4.6 Exchange options

An **exchange option** is a correlation option that gives the holder the right to exchange one asset for another. The value of such an option at expiration is

$$\max(S_2(T)-S_1(T),0), \tag{15.8}$$

where $S_1(T)$ and $S_2(T)$ are the prices of the two assets at expiration. The above formula implies two ways to look at the option: as a call on asset 2 with a strike price equal to the future price of asset 1, or as a put on asset 1 with a strike price equal to the future value of asset 2. This is an option which can be exercised only at T when it will yield $S_2 - S_1$ if exercised or nothing if not.

Assume that the two assets do not pay dividends and their prices follow two geometric Brownian motion processes with correlation ρ:

$$\frac{dS_1}{S_1} = \mu_1\, dt + \sigma_1\, dW_1 \quad \text{and} \quad \frac{dS_2}{S_2} = \mu_2\, dt + \sigma_2\, dW_2$$

and ρ is the correlation between dW_1 and dW_2. The option then has value

$$V(S_1, S_2, t) = S_2 N(x) - S_1 N\left(x - \sigma\sqrt{T-t}\right) \tag{15.9}$$

at time t where

$$\begin{aligned} x &\equiv \frac{\ln(S_2/S_1) + \left(\sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}} \\ \sigma^2 &\equiv \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2 \end{aligned} \tag{15.10}$$

This is called **Margrabe's formula** [551].

Margrabe's formula can be derived as follows. First, $V(\lambda S_1, \lambda S_2, t) = \lambda V(S_1, S_2, t)$; in other words, $V(x, y, t)$ is **homogeneous of degree one** in x and y. It says that an exchange option based on λ times the prices of the two assets is equal in value to λ original exchange options. Intuitively, this is true because of

$$\max(\lambda S_2(T) - \lambda S_1(T), 0) = \lambda \times \max(S_2(T) - S_1(T), 0)$$

from (15.8) and the perfect market assumption [571, p. 264]. The price of asset 2 relative to asset 1 is $S \equiv S_2/S_1$. The option sells for $V(S_1, S_2, t)/S_1 = V(1, S_2/S_1, t)$ with asset 1 as the **numeraire**.[1] The interest rate on a riskless loan denominated in units of asset 1 is zero in a perfect market because a lender of one unit of asset 1 demands one unit of asset 1 back as repayment of principal. Since the option to exchange asset 1 for asset 2 is a call on asset 2 with a strike price equal to unity and the interest rate equal to zero, this is a special case of the Black-Scholes option pricing model. Thus,

$$\frac{V(S_1, S_2, t)}{S_1} = V(1, S, t) = S N(x) - 1 \times e^{-0\times(T-t)} N\left(x - \sigma\sqrt{T-t}\right),$$

where

$$x \equiv \frac{\ln(S/1) + \left(0 + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln(S_2/S_1) + \left(\sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}$$

from Theorem 9.3.3, and the proof is complete.

The partial differential equation satisfied by the exchange option is easy to derive. Suppose the option holder sells $V_1 \equiv \partial V/\partial S_1$ units of asset 1 short and buys $-V_2 \equiv -\partial V/\partial S_2$

[1] Walras (1834–1910) introduced numeraire in his equilibrium analysis, recognizing that only *relative* prices matter [25]. Walras was considered by Schumpeter as the greatest economist in his *History of Economic Analysis* [685].

units of asset 2. Since $V(\cdot)$ is homogeneous of degree one in S_1 and S_2, the hedger's investment will be

$$V - V_1 S_1 - V_2 S_2 = 0$$

by Euler's theorem (see Exercise 15.4.7). Thus, the value of the position is zero. The return on this investment over a short interval is also zero,

$$dV - V_1\, dS_1 - V_2\, dS_2 = 0. \tag{15.11}$$

From Ito's lemma (Theorem 14.2.2), the return on the option is

$$dV = V_1\, dS_1 + V_2\, dS_2 + V_3\, dt + \frac{V_{11}\sigma_1^2 S_1^2 + 2\, V_{12}\sigma_1\sigma_2\rho S_1 S_2 + V_{22}\sigma_2^2 S_2^2}{2}\, dt,$$

where $V_3 \equiv \partial V/\partial t$ and $V_{ij} \equiv \partial^2 V/\partial S_i \partial S_j$. The above two equations imply

$$V_3 + \frac{V_{11}\sigma_1^2 S_1^2 + 2\, V_{12}\sigma_1\sigma_2\rho S_1 S_2 + V_{22}\sigma_2^2 S_2^2}{2} = 0. \tag{15.12}$$

The initial and boundary conditions are

$$V(S_1, S_2, T) = \max(0, S_2 - S_1)$$
$$0 \leq V(S_1, S_2, t) \leq S_2 \text{ if } S_1, S_2 \geq 0$$

It is easy to verify that (15.12) is satisfied by Margrabe's formula.

Margrabe's formula is not much more complicated if S_i pays out a continuous dividend yield of q_i for $i = 1, 2$. Here, one simply replaces each occurrence of S_i with $S_i e^{-q_i\,(T-t)}$ to obtain

$$V(S_1, S_2, t) = S_2 e^{-q_2\,(T-t)} N(x) - S_1 e^{-q_1\,(T-t)} N\left(x - \sigma\sqrt{T-t}\right) \tag{15.13}$$

at time t where

$$x \equiv \frac{\ln(S_2/S_1) + \left(q_1 - q_2 + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}$$
$$\sigma^2 \equiv \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$$

See [669] for an alternative derivation based on the binomial model.

15.4.7 Options on foreign currency and assets

Foreign exchange options were introduced in §11.6. This subsection covers additional correlation options involving foreign currency and assets. Analysis of such options can either take place in the domestic market or the foreign market before being converted back into the domestic currency [639].

In the following, $S(t)$ denotes the spot exchange rate in terms of the domestic value of one unit of foreign currency and follows the geometric Brownian motion process,

$$\frac{dS}{S} = (r - r_f)\, dt + \sigma_s\, dW_s(t),$$

in a risk-neutral economy. We knew from §11.6.1 that foreign currency is analogous to a stock paying a continuous dividend yield equal to the foreign riskless interest rate r_f in foreign currency. The foreign asset will be assumed to pay a continuous dividend yield of q_f, and its price follows

$$\frac{dG_f}{G_f} = (\mu_f - q_f)\, dt + \sigma_f\, dW_f(t)$$

in foreign currency. The correlation between the rate of return of the exchange rate and that of the foreign asset price is denoted by ρ. More precisely, ρ is the correlation between dW_s and dW_f.

Foreign equity options

By (9.28), European options on the foreign asset G_f with the terminal payoffs $S(T) \times \max(G_f(T) - X_f, 0)$ and $S(T) \times \max(X_f - G_f(T), 0)$ are worth

$$\begin{aligned} C_f &= G_f e^{-q_f \tau} N(x) - X_f e^{-r_f \tau} N\left(x - \sigma_f \sqrt{\tau}\right) \\ P_f &= X_f e^{-r_f \tau} N\left(-x + \sigma_f \sqrt{\tau}\right) - G_f e^{-q_f \tau} N(-x) \end{aligned}$$

in foreign currency, where

$$x \equiv \frac{\ln(G_f/X_f) + \left(r_f - q_f + \sigma_f^2/2\right) \tau}{\sigma_f \sqrt{\tau}}$$

and X_f is the strike price in foreign currency. Hence, they will fetch SC_f and SP_f, respectively, in domestic currency. These options are called **foreign equity options** struck in foreign currency.

Foreign domestic options

Foreign equity options concern themselves with values in foreign currency. A foreign equity call, for instance, may allow the holder to participate in a foreign market rally, but the profits might be wiped out if the foreign currency depreciates against the domestic currency. Let the desired payoff be $\max(S(T)\, G_f(T) - X, 0)$, which is a call in *domestic* currency. Such an option is called a **foreign domestic option** or foreign equity option *struck in domestic currency*. Observe that SG_f is the value of the foreign asset in domestic currency.

To foreign investors, this call is an option to exchange X units of domestic currency (foreign currency to them) for one share of foreign asset (domestic asset to them)—an exchange option, in short. By (15.13), its price equals

$$G_f e^{-q_f \tau} N(x) - \frac{X}{S}\, e^{-r\tau} N\left(x - \sigma\sqrt{\tau}\right)$$

in foreign currency, where

$$\begin{aligned} x &\equiv \frac{\ln(G_f S/X) + \left(r - q_f + \sigma^2/2\right)\, \tau}{\sigma\sqrt{\tau}} \\ \sigma^2 &\equiv \sigma_s^2 + 2\rho\sigma_s\sigma_f + \sigma_f^2 \end{aligned}$$

(The sign in front of $2\rho\sigma_s\sigma_f$ in σ^2 is a plus rather than a minus because the correlation between G_f and $1/S$ is $-\rho$.) Its price in domestic currency is then

$$C = SG_f e^{-q_f \tau} N(x) - Xe^{-r\tau} N\left(x - \sigma\sqrt{\tau}\right).$$

Similarly, a put has a price of

$$P = Xe^{-r\tau} N\left(-x + \sigma\sqrt{\tau}\right) - SG_f e^{-q_f \tau} N(-x).$$

Cross-currency options

A cross-currency option, we recall, is an option in which the currency of the strike price is different from the currency in which the underlying asset is priced [674]. An option to buy 100 yen at a strike price of 1.18 Canadian dollars provides one example. It would be a conventional foreign exchange option if the price of yen were in Canadian dollars. More complicated examples are clearly possible. Usually, a third currency, usually the U.S. dollar, is involved in pricing because of the lack of relevant exchange-traded options for the two currencies in question (yen and Canadian dollars in the above example) in order to calculate the needed volatility. For this reason, the notation below will be slightly different.

Let S_A denote the price of the foreign asset and S_C the price of currency C that the strike price X is based on. Both S_A and S_C are in, say, U.S. dollars. If S is the price of the foreign asset as measured in currency C, then

$$S = \frac{S_A}{S_C} \tag{15.14}$$

to avoid arbitrage. Assume S_A and S_C follow the geometric Brownian motion processes $dS_A/S_A = \mu_A\, dt + \sigma_A\, dW_A$ and $dS_C/S_C = \mu_C\, dt + \sigma_C\, dW_C$, respectively. Parameters σ_A, σ_C, and ρ can be inferred from exchange-traded options. By Exercise 14.3.9,

$$\frac{dS}{S} = (\mu_A - \mu_C - \rho\sigma_A\sigma_C)\, dt + \sigma_A\, dW_A - \sigma_C\, dW_C,$$

where ρ is the correlation between dW_A and dW_C. Hence, $\left(\sigma_A^2 - 2\rho\sigma_A\sigma_C + \sigma_C^2\right)^{1/2}$ is the volatility of dS/S.

Quanto options

Consider a call with a terminal payoff $\widehat{S} \times \max(G_f(T) - X_f, 0)$ in domestic currency, where $\widehat{S}$ is a constant. In other words, the exchange rate is guaranteed to be $\widehat{S}$ independent of the market. For instance, a call on the Nikkei 225 futures, if it existed, could easily fit into this framework with $\widehat{S} = 5$ and G_f denoting the futures price (see Comment 12.3.7). The process $U \equiv \widehat{S}G_f$ in a risk-neutral economy can be shown to follow

$$\frac{dU}{U} = (r_f - q_f - \rho\sigma_s\sigma_f)\, dt + \sigma_f\, dW$$

in domestic currency [421]. Hence, it can be treated as a stock paying a continuous dividend yield of $q \equiv r - r_f + q_f + \rho \sigma_s \sigma_f$. Apply (9.28) to obtain

$$\begin{aligned} C &= \widehat{S}\left(G_f e^{-q\tau} N(x) - X_f e^{-r\tau} N\left(x - \sigma_f \sqrt{\tau}\right)\right) \\ P &= \widehat{S}\left(X_f e^{-r\tau} N\left(-x + \sigma_f \sqrt{\tau}\right) - G_f e^{-q\tau} N(-x)\right) \end{aligned}$$

where

$$x \equiv \frac{\ln(G_f/X_f) + \left(r - q + \sigma_f^2/2\right)\tau}{\sigma_f \sqrt{\tau}}.$$

This kind of guaranteed exchange rate option is called a **quanto option** or simply a **quanto**.

In general, a **quanto derivative** has nominal payments in the foreign currency, but these payments are converted into the domestic currency at a *fixed* exchange rate. A **cross-rate swap**, for example, is like a currency swap except that the foreign currency payments are converted into the domestic currency at a fixed exchange rate. Quanto derivatives form a rapidly growing segment of international financial markets [11].

15.4.8 Convertible bonds with call provisions

The holder of this kind of security has the right, if the bond is called, either to convert the bond or to redeem it at the call price. Bonds can be called instantaneously. Assume the firm and the investor pursue an optimal strategy whereby (a) the investor maximizes the value of the convertible bond at each instant in time through conversion and (b) the firm minimizes the value of the convertible bond at each instant in time through call.

Let the market value of the firm's securities, $V(t)$, be determined exogenously and independent of the call and conversion strategies, as justified by the Modigliani-Miller irrelevance theorem. The implication is that minimizing the value of the convertible bonds also maximizes the stockholder value. We shall assume the market value follows

$$\frac{dV}{V} = \mu \, dt + \sigma \, dW.$$

The stock may pay dividends, and the bond may pay coupon interests. Assume the firm in question has only two classes of obligations: n shares of common stock and m convertible bonds. Suppose the conversion ratio is k. The **conversion value** per bond is then

$$C(V,t) = \frac{V(t)\,k}{n + mk} \equiv zV(t). \tag{15.15}$$

Finally, T stands for the maturity date.

Let $W(V,t)$ denote the market value at time t of one convertible bond with a par value of \$1,000. From assumption (a), the bond never sells below the conversion value because

$$W(V,t) \geq C(V,t). \tag{15.16}$$

In fact, the uncalled bond can never sell at the conversion value except immediately prior to a dividend date. This is because, otherwise, its rate of return up to the next dividend will not fall below the stock's by (15.15) and (15.16); actually, it will be higher because of the higher priority of bondholders if the stock price declines sharply. Therefore, the bond will sell above the conversion value and the investor will not convert it. As a result, (15.16) holds with strict inequality between dividend dates, and conversion only needs to be considered at dividend or call dates.

We proceed to consider the implications of the call strategy. When a bond is called, the investor has the option either to redeem at the call price $P(t)$ or convert it for $C(V,t)$, called **forced conversion**. The value of the bond if called is hence given by

$$V_c(V,t) \equiv \max(P(t), C(V,t)).$$

There are two cases to consider.

1. $C(V,t) > P(t)$ **when the bond is callable:** The bond will be called immediately because, by an earlier argument, an uncalled bond would sell for at least the conversion value $C(V,t)$ which is the value if called. Hence,

$$W(V,t) = C(V,t). \tag{15.17}$$

2. $C(V,t) \leq P(t)$ **when the bond is callable:** Note that the call price equals the value if called, V_c. The bond should be called when its value if not called equals its value if called. This holds because, in accordance with assumption (b), the firm will call the bond when the value if not called exceeds $V_c(V,t)$ and will not call the bond when the value if not called is exceeded by $V_c(V,t)$. Hence,

$$W(V,t) \leq V_c(V,t) = P(t), \tag{15.18}$$

and the bond will be called when its value if not called equals the call price.

Finally, the Black-Scholes differential equation implies

$$\frac{\partial W}{\partial t} + rV\frac{\partial W}{\partial V} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 W}{\partial V^2} = rW.$$

We summarize the boundary conditions for the above differential equation below.

- They include (15.16), (15.17), (15.18) (the latter two when the bond is callable and under their respective conditions), and the maturity value condition

$$W(V,T) = \begin{cases} zV(T) & \text{if } zV(T) \geq 1000 \\ 1000 & 1000 \times m \leq V(T) \leq 1000/z \\ V(T)/m & V(T) \leq 1000 \times m \end{cases}$$

  These three conditions above correspond to the cases when the firm's total value (1) is greater than the total conversion value and the bonds are converted, (2) is greater than the par value but less than the total conversion value and the bonds are redeemed at par, and (3) is less than the par value and the bondholders take control of the firm.

- It must hold that
$$0 \leq m\,W(V,t) \leq V(t)$$
since the total value of the bonds cannot exceed that of the firm; in particular,
$$W(0,t) = 0$$

- Since a convertible bond is dominated by a portfolio of an otherwise identical straight bond with value $B(V,t)$ and shares with total value equal to the conversion value, we have
$$W(V,t) \leq B(V,t) + zV(t).$$
Note that $B(V,t)$ is easy to calculate under constant interest rates.

- When the bond is not callable and $V(t)$ is high enough to make negligible the possibility of default, it behaves like an option to buy a fraction z of the firm. Hence,
$$\lim_{V\to\infty} \frac{\partial W(V,t)}{\partial V} = z.$$

- On a dividend date,
$$W(V,t^-) = \max(W(V-D,t^+), z\,V(t)),$$
where t^- denotes the instant before the event and t^+ the following instant. This condition takes into account conversion just before the dividend date.

- On a coupon date and when the bond is not callable,
$$W(V,t^-) = W(V-mc,t^+) + c,$$
where c is the amount of the coupon.

- On a coupon date and when the bond is callable,
$$W(V,t^-) = \min(W(V-mc,t^+) + c, V_c(V,t)).$$

The resulting partial differential equation has to be solved numerically. Section 18.1 will discuss such numerical methods. We have followed [106].

15.5 A General Approach to Derivative Pricing

This section generalizes the continuous-time approach to derivative pricing. It no longer require that securities be traded.

15.5.1 The simple case: single source of randomness

We begin with the case of a single source of randomness. Let S follow the Ito process,

$$\frac{dS}{S} = \boldsymbol{\mu}\, dt + \boldsymbol{\sigma}\, dW,$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ may depend only on S and t. Let $f_1(S,t)$ and $f_2(S,t)$ be the prices of two derivative securities with the dynamics,

$$\frac{df_i}{f_i} = \mu_i\, dt + \sigma_i\, dW, \quad i = 1, 2.$$

Note that they share the same Wiener process as S.

A portfolio consisting of $\sigma_2 f_2$ units of the first derivative and $-\sigma_1 f_1$ units of the second derivative is instantaneously riskless. This is because

$$\begin{aligned}\sigma_2 f_2\, df_1 - \sigma_1 f_1\, df_2 &= \sigma_2 f_2 f_1 (\mu_1\, dt + \sigma_1\, dW) - \sigma_1 f_1 f_2 (\mu_2\, dt + \sigma_2\, dW) \\ &= (\sigma_2 f_2 f_1 \mu_1 - \sigma_1 f_1 f_2 \mu_2)\, dt,\end{aligned}$$

which is devoid of volatilities. It must therefore hold that

$$(\sigma_2 f_2 f_1 \mu_1 - \sigma_1 f_1 f_2 \mu_2)\, dt = r\, (\sigma_2 f_2 f_1 - \sigma_1 f_1 f_2)\, dt,$$

i.e., $\sigma_2 \mu_1 - \sigma_1 \mu_2 = r\, (\sigma_2 - \sigma_1)$. After rearranging the terms, we conclude

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} \equiv \lambda \ \text{ for smoe } \lambda.$$

It follows that any derivative whose value depends only on S and t and which follows the Ito process $df/f = \mu\, dt + \sigma\, dW$ must satisfy

$$\frac{\mu - r}{\sigma} = \lambda \ \text{ or } \ \mu - r = \sigma \lambda. \tag{15.19}$$

We call λ the **market price of risk**, which is independent of the specifics of the derivative.

Ito's lemma can be used to derive the formulae for μ and σ as

$$\mu f = \frac{\partial f}{\partial t} + \boldsymbol{\mu} S \frac{\partial f}{\partial S} + \frac{1}{2} \boldsymbol{\sigma}^2 S^2 \frac{\partial^2 f}{\partial S^2} \ \text{ and } \ \sigma f = \boldsymbol{\sigma} S \frac{\partial f}{\partial S}.$$

Substitute the above into (15.19) and get

$$\frac{\partial f}{\partial t} + (\boldsymbol{\mu} - \lambda \boldsymbol{\sigma})\, S \frac{\partial f}{\partial S} + \frac{1}{2} \boldsymbol{\sigma}^2 S^2 \frac{\partial^2 f}{\partial S^2} = r f.$$

Note how similar the above equation is to the Black-Scholes differential equation; in fact, it reduces to the latter when $\boldsymbol{\mu} - \lambda \boldsymbol{\sigma} = r$. This suggests the following risk-neutral valuation scheme: Discount the expected payoff of f at the riskless interest rate with the revised process,

$$\frac{dS}{S} = (\boldsymbol{\mu} - \lambda \boldsymbol{\sigma})\, dt + \boldsymbol{\sigma}\, dW.$$

If S is a non-dividend-paying stock, then the above process becomes $dS/S = r\, dt + \boldsymbol{\sigma}\, dW$ by (15.19).

15.5.2 The general case: multiple sources of randomness

We generalize the above method to the case where there is more than one state variable. Suppose $S_1, \ldots, S_n$ pay no dividends and follow the processes

$$\frac{dS_i}{S_i} = \boldsymbol{\mu}_i \, dt + \boldsymbol{\sigma}_i \, dW_i.$$

Let $\boldsymbol{\rho}_{jk}$ be the correlation between dW_j and dW_k, r the instantaneous riskless interest rate (short rate), and $f_1, \ldots, f_{n+1}$ the prices of securities whose values depend on $S_1, \ldots, S_n$ and t.

By Ito's lemma (Theorem 14.2.2),

$$\begin{aligned} df_j &= \left(\frac{\partial f_j}{\partial t} + \sum_i \boldsymbol{\mu}_i S_i \frac{\partial f_j}{\partial S_i} + \sum_{i,k} \frac{1}{2} \boldsymbol{\rho}_{ik} \boldsymbol{\sigma}_i \boldsymbol{\sigma}_k S_i S_k \frac{\partial^2 f_j}{\partial S_i \partial S_k} \right) dt + \sum_i \boldsymbol{\sigma}_i S_i \frac{\partial f_j}{\partial S_i} \, dW_i \\ &\equiv \mu_j f_j \, dt + \sum_i \sigma_{ij} f_j \, dW_i \end{aligned} \tag{15.20}$$

Now, maintain a portfolio of k_j units of f_j such that

$$\sum_j k_j \sigma_{ij} f_j = 0 \quad \text{for } i = 1, \ldots, n. \tag{15.21}$$

Since

$$\begin{aligned} \sum_j k_j \, df_j &= \left(\sum_j k_j \mu_j f_j \right) dt + \sum_j k_j \sum_i \sigma_{ij} f_j \, dW_i \\ &= \left(\sum_j k_j \mu_j f_j \right) dt + \sum_i \left(\sum_j k_j \sigma_{ij} f_j \right) dW_i = \left(\sum_j k_j \mu_j f_j \right) dt \end{aligned}$$

without any randomness, the portfolio is instantaneously riskless. Hence, it must hold that its return equals the short rate, $\sum_j k_j \mu_j f_j = r \sum_j k_j f_j$; in other words,

$$\sum_j k_j f_j \left(\mu_j - r \right) = 0. \tag{15.22}$$

Equations (15.21) and (15.22) coupled with the insistence that not all k_j be zero imply

$$f_j \left(\mu_j - r \right) = \sum_i \lambda_i \sigma_{ij} f_j \quad \text{for } j = 1, \ldots, n+1 \tag{15.23}$$

for some $\lambda_1, \ldots, \lambda_n$ which depend only on $S_1, \ldots, S_n$ and t (see Exercise 15.5.1). Simplify it to

$$\mu_j - r = \sum_i \lambda_i \sigma_{ij}.$$

It follows that any derivative whose value depends only on $S_1, \ldots, S_n$ and t with the Ito process

$$\frac{df}{f} = \mu \, dt + \sum_i \sigma_i \, dW_i \tag{15.24}$$

must have

$$\mu - r = \sum_i \lambda_i \sigma_i, \tag{15.25}$$

where λ_i is the market price of risk for S_i.

The term $\lambda_i \sigma_i$ measures the extent to which the required return on a security is affected by its dependence on S_i. The above equation links the excess expected return and risk, a topic to be returned to when we discuss the Capital Asset Pricing Model. Furthermore, $\lambda_i \sigma_i < 0$ would mean the addition of S_i has the effect of reducing the risk in the portfolio, causing the investor to require a lower return than would otherwise be.

Risk-neutral valuation

The μ and σ_i's in (15.24) equal

$$\frac{\partial f}{\partial t} + \sum_i \boldsymbol{\mu}_i S_i \frac{\partial f}{\partial S_i} + \frac{1}{2} \sum_{i,k} \boldsymbol{\rho}_{ik} \boldsymbol{\sigma}_i \boldsymbol{\sigma}_k S_i S_k \frac{\partial^2 f}{\partial S_i \partial S_k}$$

and $\sum_i \boldsymbol{\sigma}_i S_i \frac{\partial f}{\partial S_i}$, respectively, by virtue of (15.20). Plugging them into (15.25) and rearranging the terms, we obtain

$$\frac{\partial f}{\partial t} + \sum_i (\boldsymbol{\mu}_i - \lambda_i \boldsymbol{\sigma}_i) S_i \frac{\partial f}{\partial S_i} + \frac{1}{2} \sum_{i,k} \boldsymbol{\rho}_{ik} \boldsymbol{\sigma}_i \boldsymbol{\sigma}_k S_i S_k \frac{\partial^2 f}{\partial S_i \partial S_k} = rf. \tag{15.26}$$

The following risk-neutral valuation scheme is hence applicable: Discount the expected payoff of f at the riskless interest rate assuming that the S_i's follow

$$\frac{dS_i}{S_i} = (\boldsymbol{\mu}_i - \lambda_i \boldsymbol{\sigma}_i)\, dt + \boldsymbol{\sigma}_i\, dW_i.$$

The above equations *define* the risk-neutral economy with the correlation between the dW_i's unchanged.

For instance, a derivative security with a payoff f_T at time T and nothing before T has value $e^{-r(T-t)} E_t^{\pi}[\, f_T\,]$, where E_t^{π}, we recall, is the expected value taken in a risk-neutral economy given the information at time t. As an application, consider futures price F under constant interest rate r. Note that forward price equals futures price when interest rates are constant. With a delivery price of X, a futures contract has value $f = e^{-r(T-t)} E_t^{\pi}[\, S_T - X\,]$. Since F is the X that makes f zero, we have

$$0 = E_t^{\pi}[\, S_T - F\,] = E_t^{\pi}[\, S_T\,] - F.$$

In other words, $F = E_t^{\pi}[\, S_T\,]$. Therefore, as in the binomial model (see Exercise 13.2.10), the futures price is an unbiased estimator of the expected spot price *in a risk-neutral economy.* When the interest rate is one of the stochastic state variables, a derivative security with a payoff of f_T at time T and nothing before has value $E_t^{\pi}\left[\, e^{-\overline{r}(T-t)} f_T\,\right]$, where $\overline{r}$ is the average rate over the time interval $[\, t, T\,]$.

Comment 15.5.1 The presence of $\boldsymbol{\mu}_i$ in (15.26) shows that the investor's risk preference is *not* irrelevant, and we can no longer assume that the derivative security is independent of the underlying assets' growth rates and their market prices of risk. Only when the underlying variables are the prices of a traded security can we assume $\boldsymbol{\mu}_i = r$ in pricing derivative securities [421]. Interest rate, for instance, is not a traded security, while stocks and bonds are. □

15.6 Stochastic Volatility

The Black-Scholes model assumes that the volatility is constant. The resulting Black-Scholes formula is known to display some bias in practice. Besides the "smile" implied volatility curve mentioned earlier, volatility seems to change greatly from month to month. Volatility also tends to be mean-reverting in that extreme volatilities tend to return to average values over time. Volatility furthermore seems to fall as the price of the underlying asset rises [302, 726]. Finally, out-of-the-money options and options on low-volatility assets are underpriced.

These facts led people to consider stochastic volatility. That this achieves the first-order pricing improvement has been empirically documented [37]. It must be emphasized, however, that the Black-Scholes model has been reasonably supported by empirical research, and gains from complicated models may be rather limited [462].

Stochastic volatility injects an extra source of randomness if this uncertainty is not perfectly correlated with the one driving the stock price process. In this case, another traded security besides stocks and bonds is needed in the replicating portfolio. In fact, if volatility were the price of a traded security, then there would exist a dynamic self-financing portfolio strategy consisting of stocks, bonds, and the volatility security that replicates the option. (See the argument on Page 244 under **Number of random sources**.) Without this additional security, pricing would have to resort to a dynamic equilibrium model [129].

15.6.1 Uncorrelated volatility

Hull and White considered the following model,

$$\begin{aligned} \frac{dS}{S} &= \mu\, dt + \sigma\, dW_1 \\ \frac{dV}{V} &= \mu_v\, dt + \sigma_v\, dW_2 \end{aligned}$$

where the instantaneous variance is $V \equiv \sigma^2$ [422]. Assume that μ depends on S, σ, and t, while μ_v depends on σ and t (but not S). The Wiener processes dW_1 and dW_2 have correlation ρ. The riskless rate r is constant or at least deterministic.

From (15.26), we have

$$\frac{\partial f}{\partial t} + (\mu - \lambda\sigma)\, S\, \frac{\partial f}{\partial S} + (\mu_v - \lambda_v \sigma_v)\, V\, \frac{\partial f}{\partial V} + \frac{1}{2}\left(\sigma^2 S^2\, \frac{\partial^2 f}{\partial S^2} + 2\rho\sigma\sigma_v SV\, \frac{\partial^2 f}{\partial S \partial V} + \sigma_v^2 V^2\, \frac{\partial^2 f}{\partial V^2}\right) = rf.$$

Since the stock is a traded security (but volatility is not), Comment 15.5.1 says the above equation becomes

$$\frac{\partial f}{\partial t} + rS\,\frac{\partial f}{\partial S} + (\mu_v - \lambda_v \sigma_v)\,V\,\frac{\partial f}{\partial V} + \frac{1}{2}\left(\sigma^2 S^2\,\frac{\partial^2 f}{\partial S^2} + 2\rho\sigma\sigma_v SV\,\frac{\partial^2 f}{\partial S \partial V} + \sigma_v^2 V^2\,\frac{\partial^2 f}{\partial V^2}\right) = rf.$$

After two more assumptions: $\rho = 0$ (volatility is uncorrelated with the stock price) and $\lambda_v \sigma_v = 0$ (volatility has zero systematic risk), the equation now boils down to

$$\frac{\partial f}{\partial t} + rS\,\frac{\partial f}{\partial S} + \mu_v V\,\frac{\partial f}{\partial V} + \frac{1}{2}\left(\sigma^2 S^2\,\frac{\partial^2 f}{\partial S^2} + \sigma_v^2 V^2\,\frac{\partial^2 f}{\partial V^2}\right) = rf.$$

See [422] for a series solution.

The above assumes that the volatility risk is not priced [506]. To assume otherwise, one needs to model risk premium on the variance process as in [394]. If the volatility follows the Ornstein-Uhlenbeck process with an uncorrelated Wiener process, then closed-form solutions exist [726].

Johnson and Shanno [459] assumed there exists a traded security with a price instantaneously perfectly correlated with the stochastic variance. Suppose the stock price and volatility obey the following stochastic processes,

$$\frac{dS}{S} = \mu\,dt + \sigma\,dW \quad \text{and} \quad \frac{d\sigma}{\sigma} = \mu_v\,dt + \sigma_v\,dW_v,$$

where dW and dW_v have correlation ρ. Suppose further that there is a third security with price P that follows the stochastic process,

$$\frac{dP}{P} = \mu_p\,dt + \sigma_p\,dW_v.$$

Note that it has the same random term as the volatility of the stock. Then a second-order partial differential equation can be derived.

Additional Reading

See [181, 362, 398, 763, 784] for more information on partial differential equations, [647, 649, 793] for more information on currency-related options, and [108, 445, 601] for more information on pricing convertible bonds. Consult [375, 421, 529] for discussions on the bias of the Black-Scholes option pricing model. Many papers pursue the idea of stochastic volatility [40, 394, 396, 401, 422, 459, 506, 691, 726]. See [743] for a review of the literature and [79] for an early empirical work by Black and Scholes. The volatility process does not need to follow an Ito process; **jump processes**, for instance, have been proposed [152]. See [247] for models based on the GARCH (generalized autoregressive conditional heteroskedastic) process (see §20.2.4).