Separators

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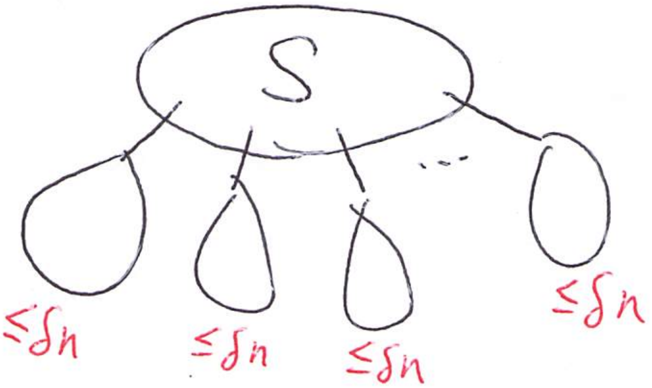
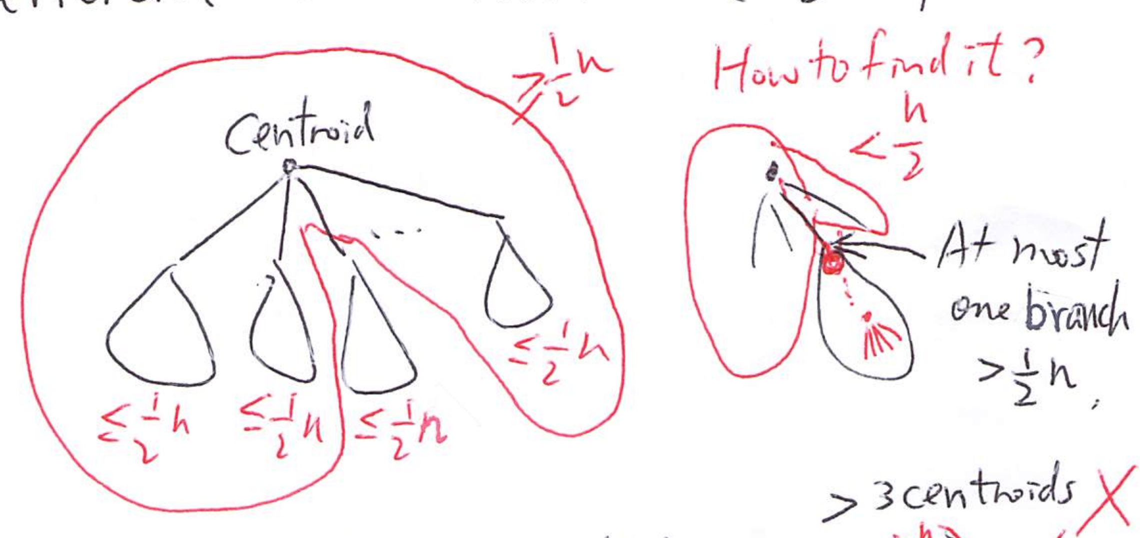
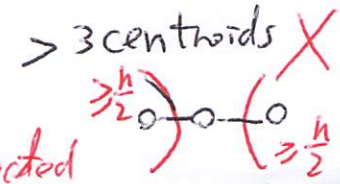
Def. Let $0 < \delta \le \frac{1}{2}$. A connected subgraph S is a δ-separator of T if $|B| \le \delta |V(T)|$ $(= \delta n)$ for every branch B of S. A δ-separator is minimal if any proper subgraph of S is not a δ-separator of T.



A centroid is a minimal $\frac{1}{2}$-separator.





o—o ← two centroids (possible)

← should be connected

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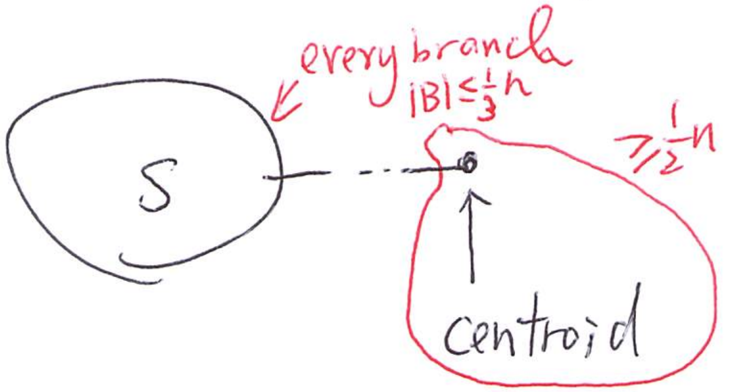
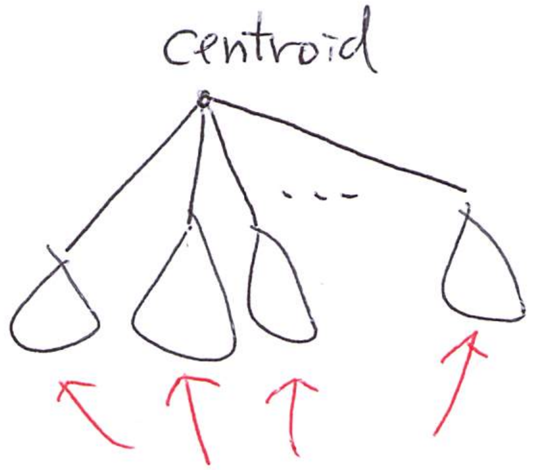
S: a minimal $\frac{1}{3}$-separtor

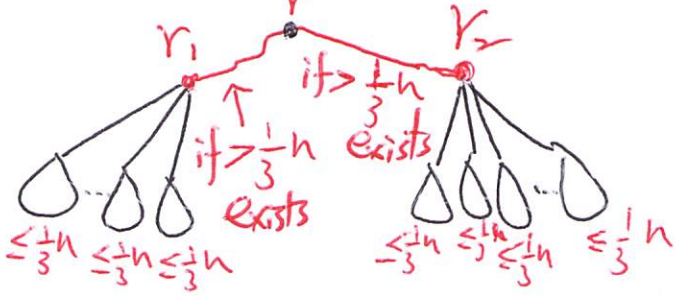


$\Rightarrow$ A centroid must be included in a minimal $\frac{1}{3}$-separator.



At most two branches

$\frac{1}{3}n < |B| \le \frac{1}{2}n$



$SP_T(r_1, r_2)$

A minimal $\frac{1}{3}$-separator

(a path separator)

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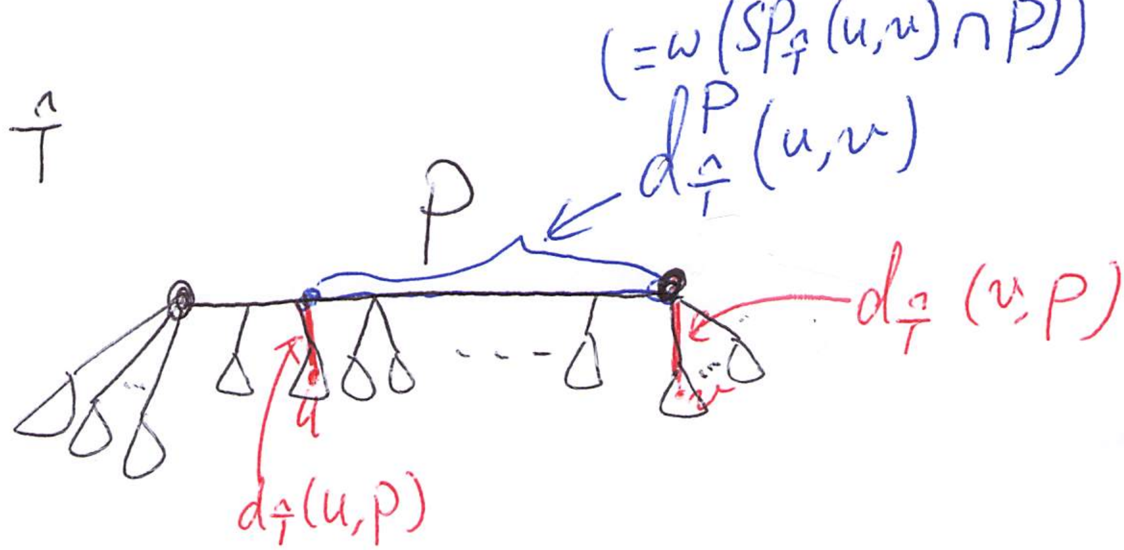
△ If P is a minimal $\frac{1}{3}$-separator of $\hat{T}$, then

$$C(\hat{T}) \geq \frac{4}{3}n \sum_{v} d_{\hat{T}}(v, P) + \frac{4}{9}n^2 \omega(P)$$

pf.



X: the set of the ordered pairs of the vertices not in the same branch.

For $(u,v) \in X$,
$$d_{\hat{T}}(u,v) = d_{\hat{T}}(u,P) + d^{P}_{\hat{T}}(u,v) + d_{\hat{T}}(v,P)$$

$$C(\hat{T}) = \sum_{u,v \in V(\hat{T})} d_{\hat{T}}(u,v)$$

$$\geq \sum_{(u,v) \in X} d_{\hat{T}}(u,v)$$ ← count those pairs not in the same branch

$$= \sum_{(u,v)\in X} \left(d_{\hat{T}}(u,P) + d^{P}_{\hat{T}}(u,v) + d_{\hat{T}}(v,P)\right)$$

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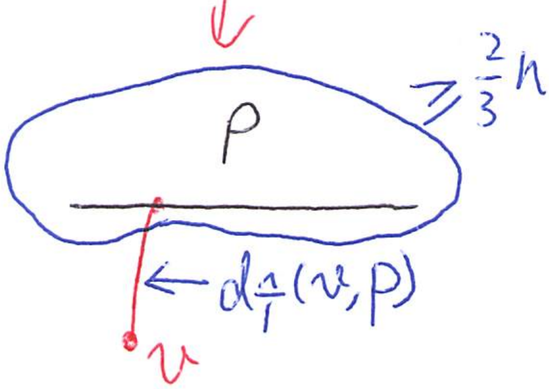
$$C(\hat{T}) \geq \sum_{(u,v)\in X} \left(d_{\hat{T}}(u,P) + d_{\hat{T}}(v,P)\right) + \sum_{(u,v)\in X} d^{P}_{\hat{T}}(u,v)$$

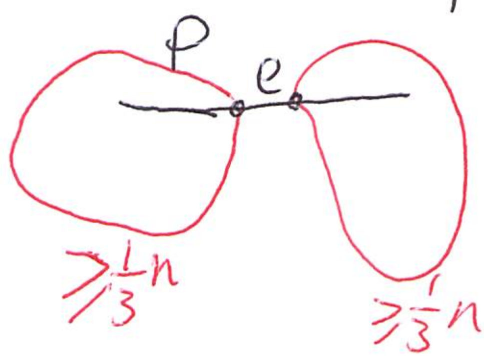
$$\sum_{(u,v)\in X} \left(d_{\hat{T}}(u,P) + d_{\hat{T}}(v,P)\right) \geq 2 \times \frac{2}{3} n \sum_{v} d_{\hat{T}}(v,P)$$



There are at least $\frac{2}{3}n$ vertices not in the same branch of any vertex v.

$$\sum_{(u,v)\in X} d^{P}_{\hat{T}}(u,v) = \sum_{u}\sum_{v} d^{P}_{\hat{T}}(u,v) = \sum_{e\in E(P)} \ell(\hat{T},e)\, w(e) \geq \frac{4}{9} n^2 \sum_{e\in E(P)} w(e) = \frac{4}{9} n^2 w(P)$$

since $\ell(\hat{T},e) \geq \frac{4}{9} n^2$



$$2 \times \frac{1}{3} n \times \left(1 - \frac{1}{3}\right) n = \frac{4}{9} n^2$$

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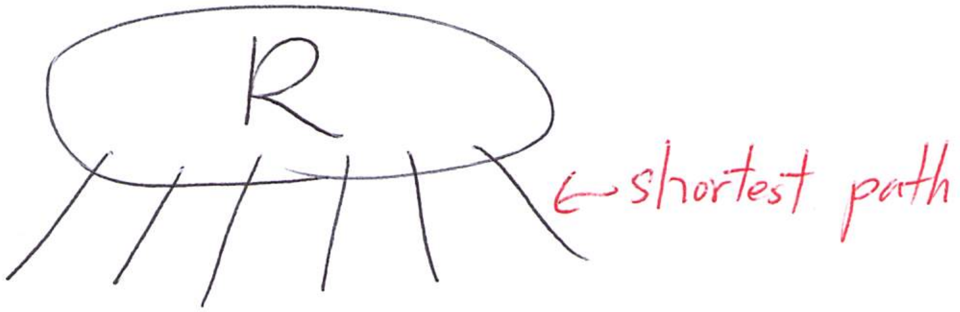
General stars:

Def. Let R be a tree contained in the underlying graph G. A spanning tree is a general star with core R if each vertex is connected to R by a shortest path. Let star(R) denote the set of all general stars with core R.



A shortest-paths tree "rooted" at R.

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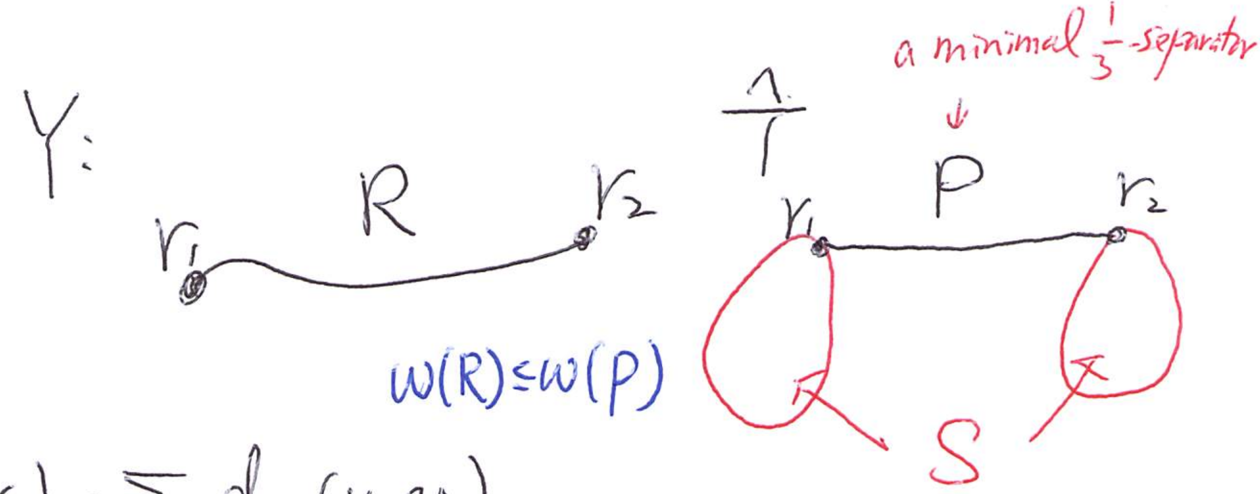
(including the degenerated case $r_1 = r_2$)

△ There exist $r_1, r_2 \in V$ such that if $R = SP_G(r_1, r_2)$ and $Y \in star(R)$, then

$$C(Y) \le 2n \sum_v d_{\hat{T}}(v, P) + \frac{5}{6} n^2 \omega(P).$$

pf.



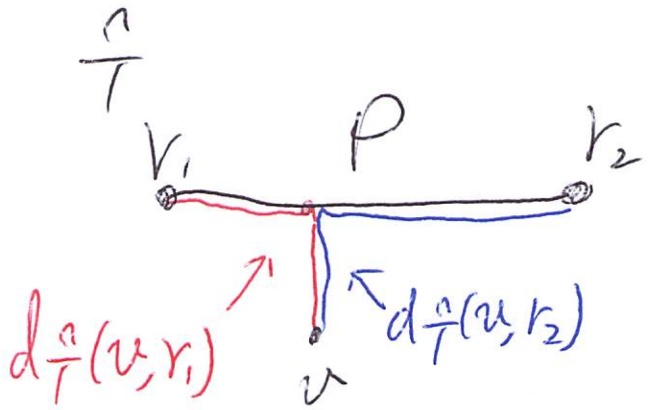
$$\begin{aligned}
C(Y) &= \sum_{u,v} d_Y(u,v) \\
&\le \sum_{u,v} \left(d_Y(u,R) + d_Y^R(u,v) + d_Y(v,R) \right) \\
&= \underbrace{\sum_{u,v} d_Y(u,R)}_{n \sum_u d_Y(u,R)} + \underbrace{\sum_{u,v} d_Y(v,R)}_{n \sum_v d_Y(v,R)} + \sum_{u,v} d_Y^R(u,v) \le \boxed{\frac{n^2}{2}} \omega(R) \\
&\le 2n \sum_v d_Y(v,R) + \frac{1}{2} n^2 \omega(R) \\
&= 2n \sum_v d_G(v,R) + \frac{1}{2} n^2 \omega(R)
\end{aligned}$$

an upper bound of routing load

For $v \in S$, $d_G(v,R) \le \min\{d_G(v,r_1), d_G(v,r_2)\}$
$\le d_{\hat{T}}(v,P)$

For $v \notin S$,

$$d_G(v,R) \leq \min\{d_G(v,r_1), d_G(v,r_2)\}$$
$$\leq (d_G(v,r_1)+d_G(v,r_2))/2$$
$$\leq (d_{\hat{T}}(v,r_1)+d_{\hat{T}}(v,r_2))/2$$
$$= d_{\hat{T}}(v,P)+\tfrac{1}{2}w(P)$$



$\boxed{|S| \geq \tfrac{2}{3}n}$ ← P is a minimal $\frac{1}{3}$-separator.

$$C(Y) \leq 2n\sum_v d_G(v,R) + \tfrac{1}{2}n^2 w(R)$$
$$\leq 2n\sum_v d_{\hat{T}}(v,P) + (2n \times \tfrac{1}{3}n \times \tfrac{1}{2}w(P)) + \tfrac{1}{2}n^2 w(R)$$

$|V-S| \leq \frac{1}{3}n$

$$\leq 2n\sum_v d_{\hat{T}}(v,P) + \tfrac{5}{6}n^2 w(P)$$

$w(P) \geq w(R)$

$$\frac{C(Y)}{C(\hat{T})} \leq \max\left\{\frac{2n}{\frac{4}{3}n}, \frac{\frac{5}{6}n^2}{\frac{4}{9}n^2}\right\}$$
$$= \max\left\{\frac{3}{2}, \frac{15}{8}\right\} = \frac{15}{8}$$

$\frac{15}{8}$-approximation

$\triangle$ $X \in \text{star}(P)$

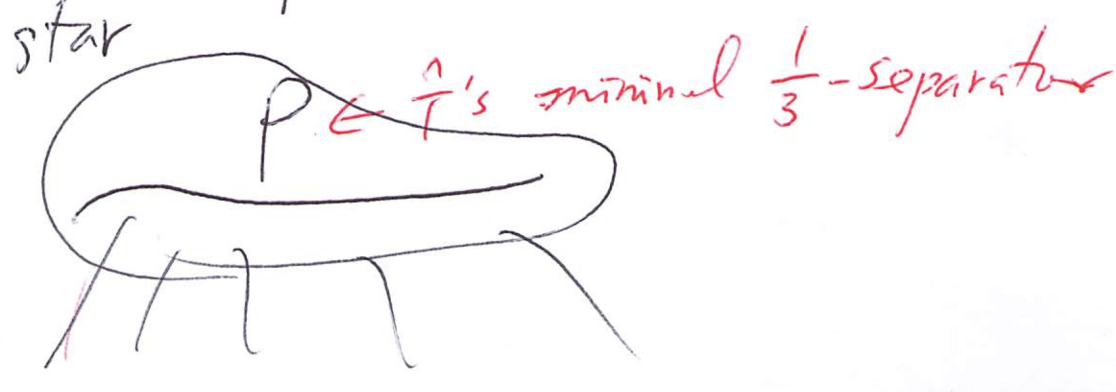
$$C(X) \le 2n \sum_{v} \underbrace{d_G(v, P)}_{\le d_{\hat{T}}(v, P)} + \frac{n^2}{2}\, w(P)$$

$$\frac{C(X)}{C(\hat{T})} \le \max\left\{ \frac{2n}{\frac{4}{3}n}, \frac{\frac{n^2}{2}}{\frac{4}{9}n^2} \right\}$$

$$= \max\left\{ \frac{3}{2}, \frac{9}{8} \right\} = \frac{3}{2}$$

nice, but we don't know P, and we cannot afford to try all possible paths!

X :



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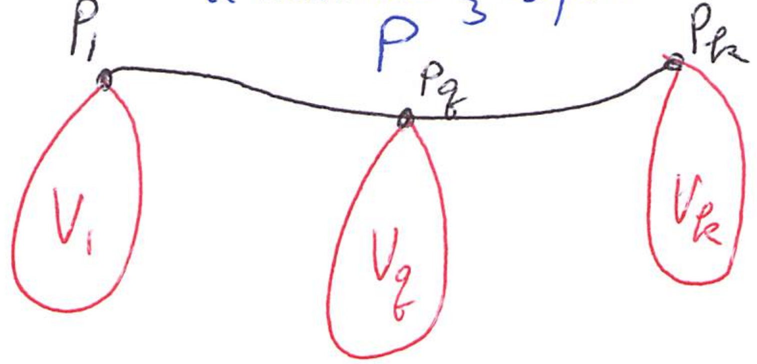
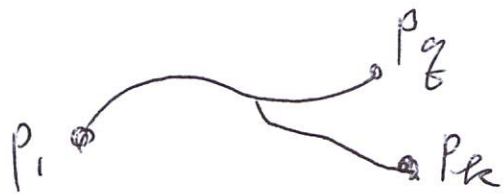
$\hat{T}$

a minimal $\frac{1}{3}$-separator $= (p_1, p_2, \cdots, p_q, \cdots, p_k)$

centroid of $\hat{T}$



$R = SP_G(p_1, p_q) \cup SP_G(p_q, p_k)$ & cycles removed



$w(R) \leq w(P)$

$Y \in star(R)$

$$C(Y) \leq 2n \sum_v d_G(v, R) + \frac{1}{2} n^2 w(R)$$

$v \in V_1 \cup V_q \cup V_k$

$$d_G(v, R) \leq \min\{d_G(v, p_1), d_G(v, p_q), d_G(v, p_k)\}$$
$$\leq \min\{d_{\hat{T}}(v, p_1), d_{\hat{T}}(v, p_q), d_{\hat{T}}(v, p_k)\}$$
$$= d_{\hat{T}}(v, P)$$

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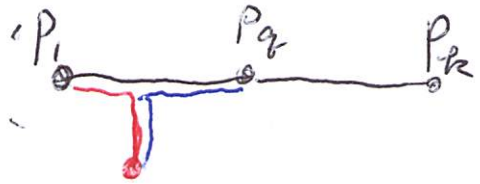
For $v \in \bigcup_{1 < i < q} V_i$, $d_G(v, R) \le \min\{d_G(v, P_1), d_G(v, P_q)\}$

$$\le \left(d_G(v, P_1) + d_G(v, P_q)\right)/2$$

$$\le \left(d_{\hat{T}}(v, P_1) + d_{\hat{T}}(v, P_q)\right)/2$$

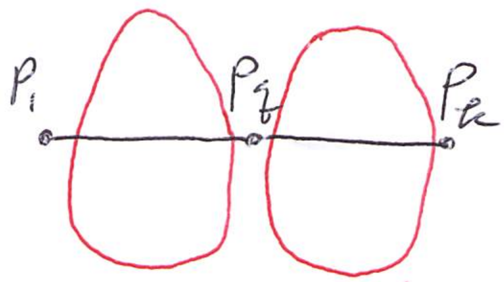
$$= d_{\hat{T}}(v, P) + \frac{1}{2} d_{\hat{T}}(P_1, P_q)$$

$\hat{T}$



For $v \in \bigcup_{q < i < k} V_i$, $d_G(v, R) \le d_{\hat{T}}(v, P) + \frac{1}{2} d_{\hat{T}}(P_q, P_k)$

$\hat{T}$



$\le \frac{1}{2}n - \frac{1}{3}n$ $\quad \le \frac{1}{6}n$

$= \frac{1}{6}n$

$$\boxed{\begin{aligned} & d_{\hat{T}}(P_1, P_q) \\ + \; & d_{\hat{T}}(P_q, P_k) = w(P) \end{aligned}}$$

$w(R) \le w(P)$

$$C(Y) \le 2n \sum_v d_G(v, R) + \frac{1}{2} n^2 w(R)$$

$$\le 2n \sum_v d_{\hat{T}}(v, P) + \left(2n \times \frac{1}{6} n \times \frac{1}{2} d_{\hat{T}}(P_1, P_q)\right)$$

$$+ \left(2n \times \frac{1}{6} n \times \frac{1}{2} d_{\hat{T}}(P_q, P_k)\right) + \frac{1}{2} n^2 w(P)$$

$$= 2n \sum_v d_{\hat{T}}(v, P) + \left(\frac{1}{6} + \frac{1}{2}\right) n^2 w(P)$$

$$= 2n \sum_v d_{\hat{T}}(v, P) + \frac{2}{3} n^2 w(P)$$

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$$\frac{c(Y)}{c(\hat{T})} \le \max\left\{\frac{2n}{\frac{4}{3}n}, \frac{\frac{2}{3}n^2}{\frac{4}{9}n^2}\right\}$$

$$= \max\left\{\frac{3}{2}, \frac{3}{2}\right\} = \frac{3}{2}$$

$\frac{3}{2}$ - approximation!

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