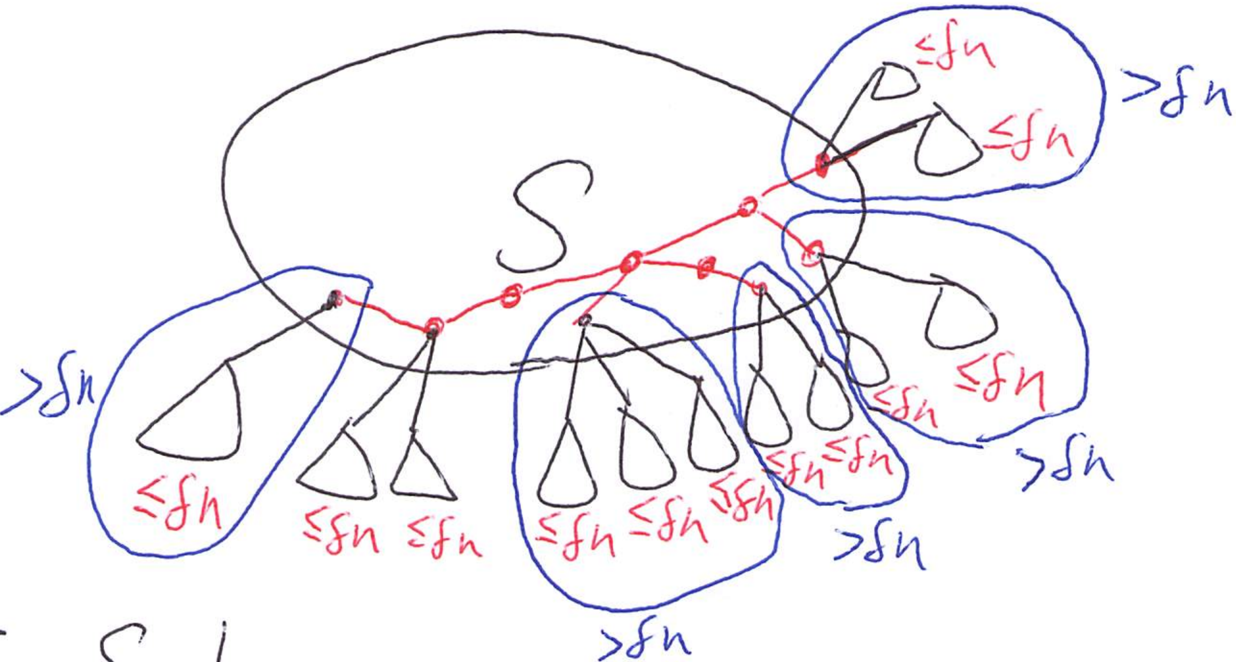
Minimal δ-separators

Kun-Mao Chao@2019
@2021



Let S be a minimal δ-separator of T.

If v is a leaf of S, then $|VB(T,S,v)| > \delta n$, where

branch

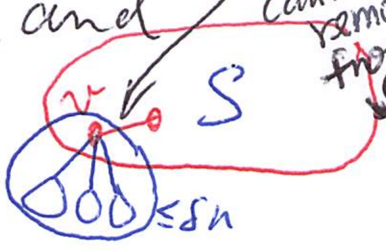
connected to

$$VB(T,S,v) = \{v\} \cup \{u \mid u \in B \in brn(T,S,v)\}.$$

Note:

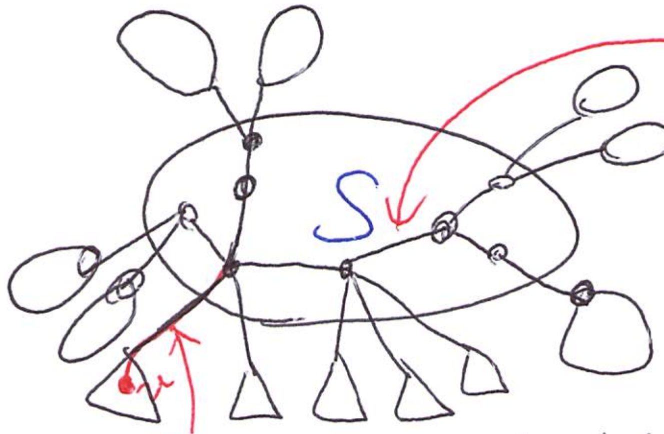
1. Every branch $|B| \le \delta n$ ← δ-separator
2. If v is a leaf of S, and $|VB(T,S,v)| \le \delta n$, then S is not minimal.

This edge can be removed from S.

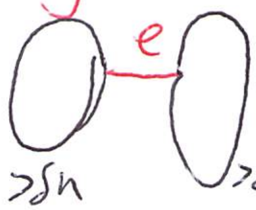


S: a minimal (δ-separator) of $\hat{T}$ Kun-Mao Chao @2019 2021

$\hat{T}$:



For any $e \in S$,



$\Rightarrow \ell(\hat{T}, e) \geq 2\delta(1-\delta)n^2$

vertices not in the same branch

$d_{\hat{T}}(v, S)$ counted at least $(1-\delta)n \times 2$ times.

$$C(\hat{T}) \geq \boxed{2(1-\delta)n} \sum_v d_{\hat{T}}(v, S) + \boxed{2\delta(1-\delta)n^2}\, w(S)$$

$Y \in star(S)$

counted no more than $2n$ times

$d_Y(v, S) = d_G(v, S)$

For any edge $e \in S$, $\ell(Y, e) \leq \frac{1}{2}n^2$

$$C(Y) \leq \boxed{2n} \sum_v d_G(v, S) + \boxed{\tfrac{1}{2}n^2}\, w(S)$$

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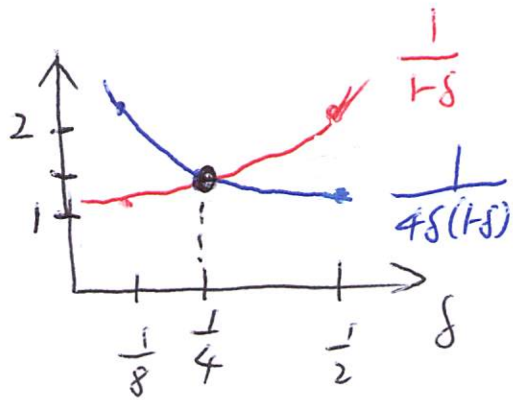
S = minimal δ-separator of $\hat{T}$

$Y \in star(S)$

$$C(Y) \le 2n \sum_v \overset{\le d_{\hat{T}}(v,S)}{\underline{d_G(v,S)}} + \frac{1}{2} n^2 w(S)$$

$$C(\hat{T}) \ge 2(1-\delta) n \sum_v d_{\hat{T}}(v,S) + 2\delta(1-\delta) n^2 w(S)$$

Approximation ratio: $\max\left\{\frac{1}{1-\delta}, \frac{1}{4\delta(1-\delta)}\right\}$



$$\frac{1}{1-\delta} = \frac{1}{4\delta(1-\delta)} \Rightarrow \delta = \frac{1}{4}$$

$$\max\left\{\frac{1}{1-\frac{1}{4}}, \frac{1}{4\times\frac{1}{4}(1-\frac{1}{4})}\right\} = \frac{4}{3} \leftarrow \text{the best bet}$$

How to find

A minimal $\frac{1}{4}$-separator?

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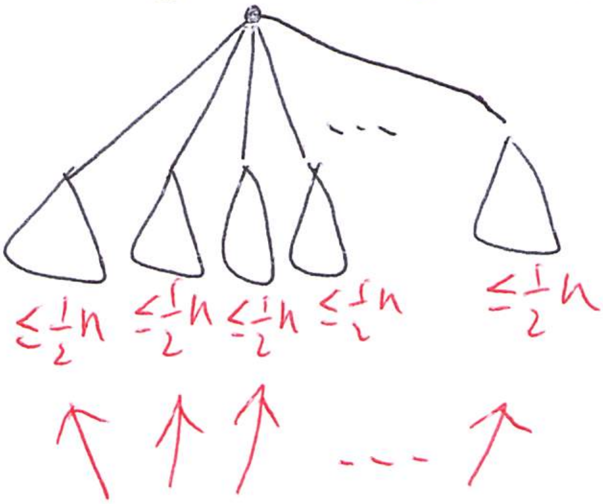
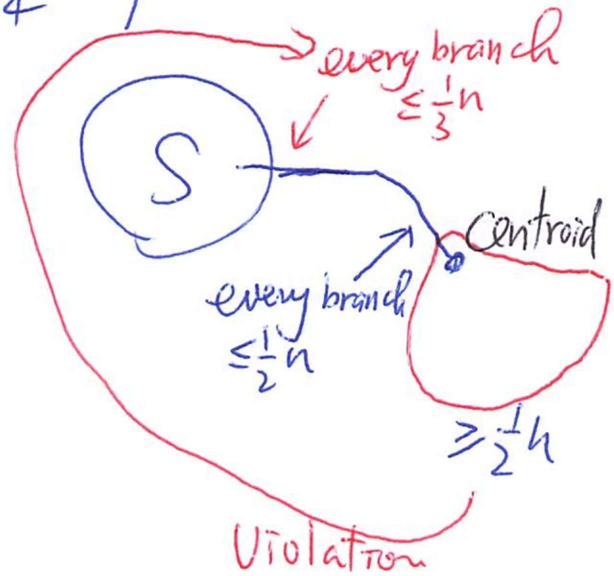
centroid ← A centroid must be included in a minimal $\frac{1}{4}$-separator.

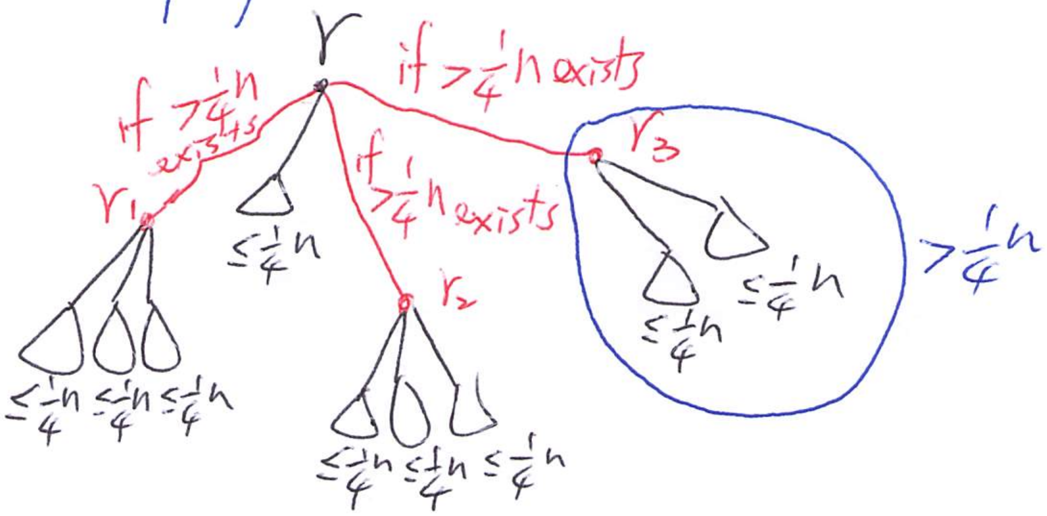


At most three branches

$\frac{1}{4}n < |B| \le \frac{1}{2}n$



A minimal $\frac{1}{4}$-separator:



A fork separator

$\hat{T}$: an optimal MRCT; S: a minimal $\frac{1}{4}$-separator of $\hat{T}$

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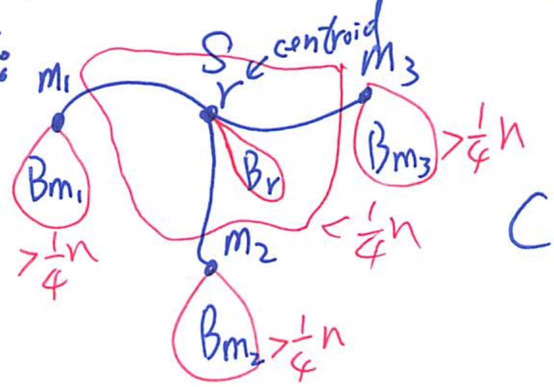
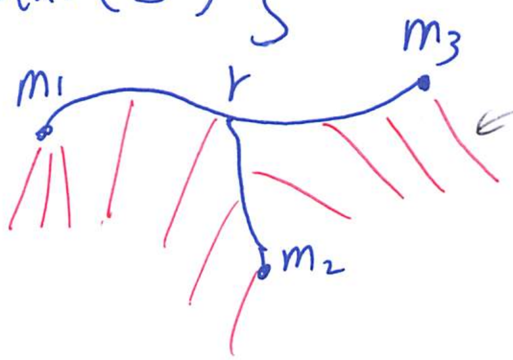
Oct., 2010
Oct., 2011
Mar., 2019

$\hat{T}$:



$$C(\hat{T}) \geq \frac{3}{2} n \sum_v d_{\hat{T}}(v,S) + \frac{3}{8} n^2 w(S)$$

$Y \in star(S)$



$\leftarrow d_Y(v,S) = d_G(v,S)$

$$C(Y) \leq 2n \sum_v d_Y(v,S) + \frac{1}{2} n^2 w(S)$$

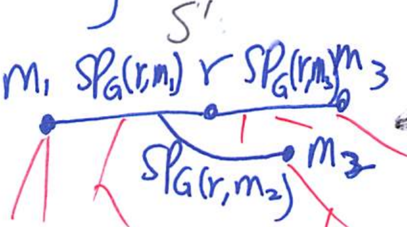
(Y → G)

$$\leq 2n \sum_v d_{\hat{T}}(v,S) + \frac{1}{2} n^2 w(S)$$

$$\frac{C(Y)}{C(\hat{T})} \leq \max\left\{ \frac{2n}{\frac{3}{2}n}, \frac{\frac{1}{2}n^2}{\frac{3}{8}n^2} \right\} = \frac{4}{3}.$$

The thing is that we don't have "S"!

Y'



$\leftarrow d_{Y'}(v,S') = d_G(v,S')$ $\quad w(S') \leq w(S)$

$$C(Y') \leq 2n \sum_v d_G(v,S') + \frac{1}{2} n^2 w(S')$$

($\# > \frac{3}{4}n$) For B_{m_1}, B_{m_2}, B_{m_3}, and B_r, $d_G(v,S') \leq \min\{d_G(v,m_1), d_G(v,m_2), d_G(v,m_3), d_G(v,r)\} \leq d_{\hat{T}}(v,S)$

($\# < \frac{1}{4}n$) For others, $d_G(v,S') \leq d_{\hat{T}}(v,S) + \frac{1}{2} w(S)$

$$C(Y') \leq 2n \sum_v d_G(v,S') + \frac{1}{2} n^2 w(S)$$

$\times \frac{1}{4}n \times \frac{1}{2}w(S)$

$$\leq 2n \sum_v d_{\hat{T}}(v,S) + \frac{3}{4} n^2 w(S) \leq 2C(\hat{T})$$

↑ !

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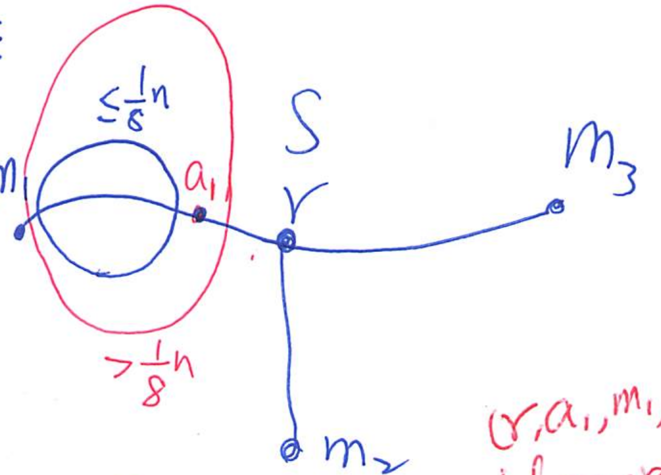
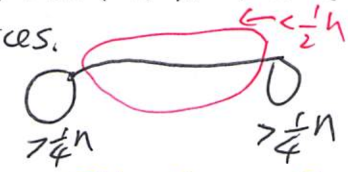
$\hat{T}$:



For degenerated cases, you might need a few more vertices.



(r, a_1, m_1, m_2, m_3)
Constructed in order

Y_1: S_1

$Y_1 \in star(S_1)$

$d_{Y_1}(v, S_1) = d_G(v, S_1)$

$$C(Y_1) \le 2n \sum_v d_G(v, S_1) + \frac{1}{2}n^2 w(S_1)$$

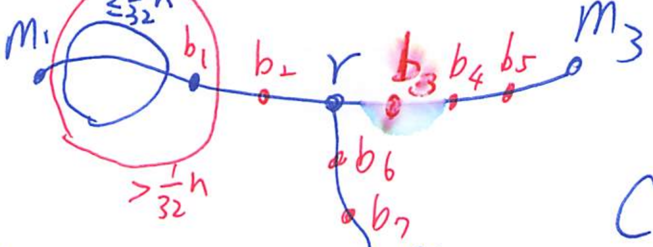
$\times \frac{1}{8}n \times \frac{1}{2}w(S)$

$$\le 2n \sum_v d_{\hat{T}}(v, S) + \frac{5}{8}n^2 w(S)$$

$$\frac{C(Y_1)}{C(\hat{T})} \le \max\left\{\frac{2n}{\frac{3}{2}n}, \frac{\frac{5}{8}n^2}{\frac{3}{8}n^2}\right\} = \frac{5}{3}$$

S

$\hat{T}$



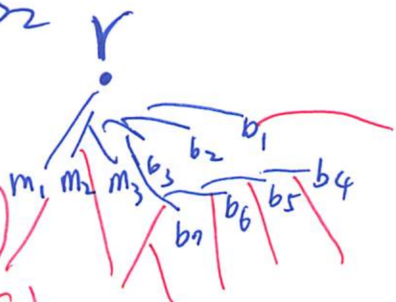
$$C(Y_2) \le 2n \sum_v d_G(v, S_2) + \frac{1}{2}n^2 w(S_2)$$

$\times \frac{1}{32}n \times \frac{1}{2}w(S)$

$$\le 2n \sum_v d_{\hat{T}}(v, S) + \frac{17}{32}n^2 w(S)$$

Y_2: S_2

$Y_2 \in star(S_2)$



$d_{Y_2}(v, S_2) = d_G(v, S_2)$

$$\frac{C(Y_2)}{C(\hat{T})} \le \max\left\{\frac{2n}{\frac{3}{2}n}, \frac{\frac{17}{32}n^2}{\frac{3}{8}n^2}\right\} = \frac{17}{12}$$

$\Delta = \frac{1}{32} \Rightarrow \frac{4}{3} + \frac{8}{3} \times \frac{1}{32} = \frac{17}{12}$

$\Delta = \frac{1}{8} \Rightarrow \frac{4}{3} + \frac{8}{3} \times \frac{1}{8} = \frac{5}{3}$

$\Delta = \frac{1}{10000}$
$\Rightarrow \frac{4}{3} + \frac{8}{3} \times \frac{1}{10000} \approx \frac{4}{3}$

$$\frac{\frac{1}{2}n^2 + \Delta n^2}{\frac{3}{8}n^2} = \frac{4}{3} + \boxed{\frac{8}{3}\Delta}$$

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